ABBOTSLEIGH

**AUGUST 2010**
**YEAR 12**
**ASSESSMENT 4**
**TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION**

# Mathematics Extension 2

## General Instructions

- Reading time – 5 minutes.
- Working time – 3 hours.
- Write using blue or black pen.
- Board-approved calculators may be used.
- A table of standard integrals is provided.
- All necessary working should be shown in every question.

## Total marks – 120

- Attempt Questions 1-8.
- All questions are of equal value.

# Outcomes assessed

## HSC course

- **E1** appreciates the creativity, power and usefulness of mathematics to solve a broad range of problems
- **E2** chooses appropriate strategies to construct arguments and proofs in both concrete and abstract settings
- **E3** uses the relationship between algebraic and geometric representations of complex numbers and of conic sections
- **E4** uses efficient techniques for the algebraic manipulation required in dealing with questions such as those involving conic sections and polynomials
- **E5** uses ideas and techniques from calculus to solve problems in mechanics involving resolution of forces and resisted motion
- **E6** combines the ideas of algebra and calculus to determine the important features of the graphs of a wide variety of functions
- **E7** uses the techniques of slicing and cylindrical shells to determine volumes
- **E8** applies further techniques of integration, including partial fractions, integration by parts and recurrence formulae, to problems
- **E9** communicates abstract ideas and relationships using appropriate notation and logical argument

Harder applications of the Extension 1 Mathematics course are included in this course. Thus the Outcomes from the Extension 1 Mathematics course are included.

### From the Extension 1 Mathematics Course

## Preliminary course

- **PE1** appreciates the role of mathematics in the solution of practical problems
- **PE2** uses multi-step deductive reasoning in a variety of contexts
- **PE3** solves problems involving inequalities, polynomials, circle geometry and parametric representations
- **PE4** uses the parametric representation together with differentiation to identify geometric properties of parabolas
- **PE5** determines derivatives that require the application of more than one rule of differentiation
- **PE6** makes comprehensive use of mathematical language, diagrams and notation for communicating in a wide variety of situations

## HSC course

- **HE1** appreciates interrelationships between ideas drawn from different areas of mathematics
- **HE2** uses inductive reasoning in the construction of proofs
- **HE3** uses a variety of strategies to investigate mathematical models of situations involving projectiles, simple harmonic motion or exponential growth and decay
- **HE4** uses the relationship between functions, inverse functions and their derivatives
- **HE5** applies the chain rule to problems including those involving velocity and acceleration as functions of displacement
- **HE6** determines integrals by reduction to a standard form through a given substitution
- **HE7** evaluates mathematical solutions to problems and communicates them in an appropriate form

**Total marks – 120**
**Attempt Questions 1-8**
**All questions are of equal value**

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

---

**QUESTION 1** (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find $\int \frac{x^2}{\left(1+x^3\right)^2}\,dx$. **2**

(b) Find $\int \frac{x^2+4}{x^2+1}\,dx$. **2**

(c) Use integration by parts to evaluate $\int_0^1 x\,e^{-3x}\,dx$. **3**

(d) (i) Find real numbers $a$, $b$ and $c$ such that

$$\frac{x}{(x-1)^2(x-2)} \equiv \frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{c}{x-2}.$$ **2**

(ii) Evaluate $\int \frac{x}{(x-1)^2(x-2)}\,dx$. **2**

(e) Use the substitution $x = \sin\theta$ to evaluate $\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{x^2}{\sqrt{1-x^2}}\,dx$. **4**

**Marks**

**QUESTION 2** (15 marks) Use a SEPARATE writing booklet.

(a) Let $z = 3 - i$ and $w = 2 + i$. Express the following in the form $x + iy$, where $x$ and $y$ are real numbers:

(i) $\dfrac{z}{w}$ **2**

(ii) $\overline{-2iz}$ **2**

(b) Let $z = \dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i$.

(i) Express $z$ in modulus-argument form. **2**

(ii) Show that $z^6 = 1$. **2**

(iii) Hence, or otherwise, graph all the roots of $z^6 - 1 = 0$ on an Argand diagram. **2**

(c) The complex numbers $\alpha, \beta, \gamma$ and $\delta$ are represented on an Argand diagram by the points $A, B, C$ and $D$ respectively.

(i) Describe the point that represents $\dfrac{1}{2}(\alpha + \gamma)$. **1**

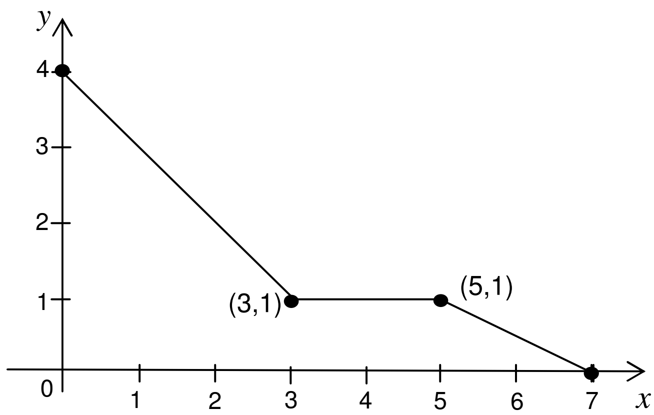
(ii) Deduce that if $\alpha + \gamma = \beta + \delta$ then $ABCD$ is a parallelogram. **2**

(d) Let $z = x + iy$. Find the points of intersection of the curves given by:

$|z - i| = 1$ and $\text{Re}(z) = \text{Im}(z)$. **2**

**QUESTION 3** (15 marks) Use a SEPARATE writing booklet.

(a) The diagram below shows the graph of the function $y = f(x)$.




Draw separate one-third page sketches of the graphs of the following:

(i) $y = f(|x|)$ **2**

(ii) $y = f(2-x)$ **2**

(iii) $y = \log_e f(x)$. **2**

(b) Sketch the graph of $y = \dfrac{1}{x(x-2)}$, without the use of calculus. **3**

(c) (i) Find the value of $g$ for which $P(x) = 9x^4 - 25x^2 + 10gx - g^2$ is divisible by both $x-1$ and $x+2$. **3**

(ii) With this value of $g$, solve the equation $9x^4 - 25x^2 + 10gx - g^2 = 0$. **3**

**Marks**

**QUESTION 4** (15 marks) Use a SEPARATE writing booklet.

(a) The area bounded by the curve $y = x^2 + 2$ and the line $y = 4 - x$ is rotated about the line $y = 1$.

(i) Find the points of intersection of the two curves. **2**

(ii) By considering slices perpendicular to the $x$ axis, show that the area, $A(x)$ of a typical slice is given by:

$$A(x) = \pi\left(8 - 6x - x^2 - x^4\right).$$ **2**

(iii) Find the volume of the solid formed. **2**

(b) Show that for all real $x$, $0 < \dfrac{1}{x^2 + 2x + 2} \leq 1$. **3**

(c) (i) If $I_n = \int x^3 (\log_e x)^n \, dx$, show that $I_n = \dfrac{x^4}{4}(\log_e x)^n - \dfrac{n}{4} I_{n-1}$ . **3**

(ii) Hence, or otherwise, evaluate $\int_1^2 x^3 (\log_e x)^2 \, dx$. **3**

**Marks**

**QUESTION 5** (15 marks) Use a SEPARATE writing booklet.

(a) (i) Factorise the polynomial $z^3 - 1$ over the rational field. **1**

(ii) If $w$ is a complex root of 1, show that $1 + w + w^2 = 0$. **1**

(iii) Hence, or otherwise, simplify $\left(1+w^2\right)\left(1+w^4\right)\left(1+w^8\right)\left(1+w^{10}\right)$. **2**

(b) Prove that if $a \neq c$ there are always two real values of $k$ which will make $ax^2 + 2bx + c + k\left(x^2+1\right)$ a perfect square. **3**

(c) The points $P\left(cp, \dfrac{c}{p}\right)$ and $Q\left(cq, \dfrac{c}{q}\right)$ are two variable points on the hyperbola $xy = c^2$ which move so that the points $P$, $Q$ and $S\left(c\sqrt{2}, c\sqrt{2}\right)$ are always collinear. The tangents to the hyperbola at $P$ and $Q$ meet at the point $R$.

(i) Show that the equation of the chord $PQ$ is $x + pqy = c\left(p+q\right)$ **2**

(ii) Hence show that $p + q = \sqrt{2}\left(1+pq\right)$. **1**

(iii) Show that $R$ is the point $\left(\dfrac{2cpq}{p+q}, \dfrac{2c}{p+q}\right)$. You may assume that the tangent at any point $T\left(ct, \dfrac{c}{t}\right)$ has equation $x + t^2 y = 2ct$. (Do NOT prove this) **3**

(iv) Hence find the equation of the locus of $R$. **2**

**Marks**

**QUESTION 6** (15 marks) Use a SEPARATE writing booklet.

(a) (i) Prove that if $x$ and $y$ are positive numbers then $(x+y)^2 \geq 4xy$. **2**

(ii) Deduce that if $a, b, c$ and $d$ are positive numbers then

$$\frac{1}{4}(a+b+c+d)^2 \geq ac+ad+bc+bd.$$ **2**

(b) Scientists use a pressure gauge which measures depth as it sinks towards the ocean floor. The gauge of mass 2 kg is released from rest at the ocean's surface. As it sinks in a vertical line, the water exerts a resistance to its motion of $4v$ Newtons, where $v$ ms$^{-1}$ is the velocity of the gauge.

Let $x$ be the displacement of the ball measured vertically downwards from the ocean's surface, $t$ be the time in seconds elapsed after the gauge is released, and $g$ be the constant acceleration due to gravity.

(i) Show that $\dfrac{d^2x}{dt^2} = g - 2v$. **2**

(ii) Hence show that $t = \dfrac{1}{2}\log_e\left(\dfrac{g}{g-2v}\right)$. **3**

(iii) Show that $v = \dfrac{g}{2}\left(1-e^{-2t}\right)$. **2**

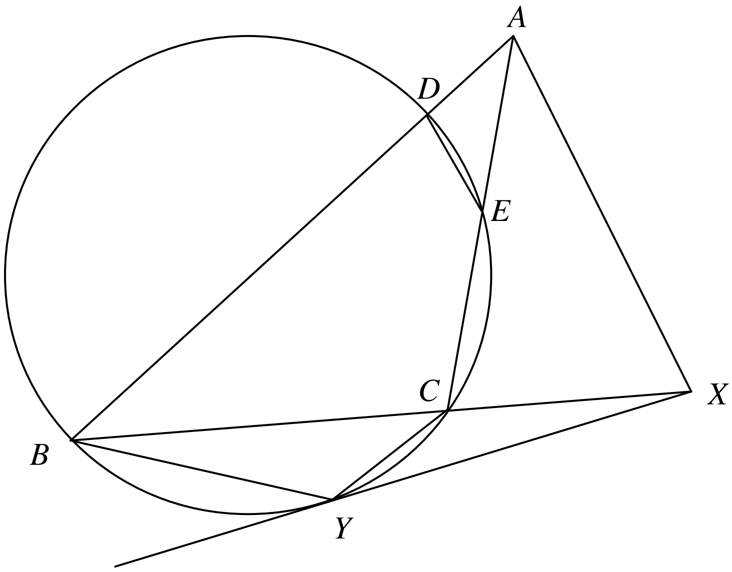
(iv) Write down the limiting (terminal) velocity of the gauge. **1**

(v) At a particular location, the gauge takes 180 seconds to hit the ocean floor. Using $g = 10\,\text{ms}^{-2}$, calculate the depth of the ocean at that location, giving your answer correct to the nearest metre. **3**

**Marks**

**QUESTION 7** (15 marks) Use a SEPARATE writing booklet.

(a) In the diagram $XY$ is a tangent to the circle and $XY = XA$.




(i) Show that $\Delta XCY \parallel\!| \Delta XBY$. **2**

(ii) Hence explain why $\dfrac{XY}{BX} = \dfrac{CX}{XY}$. **1**

(iii) Show that $\Delta AXC \parallel\!| \Delta AXB$. **3**

(iv) Prove that $DE \parallel AX$. **2**

(b) Consider the function $y = f(x)$ in the interval $1 \le x \le n$.

(i) Sketch a possible graph of $y = f(x)$ given $f(x) \ge 0$ and $f''(x) < 0$. **1**

(ii) Show, by comparing the area under the curve $y = f(x)$ between $x = 1$ and $x = n$, with the area of a region found using repeated applications of the Trapezoidal Rule, each of width 1 unit, that

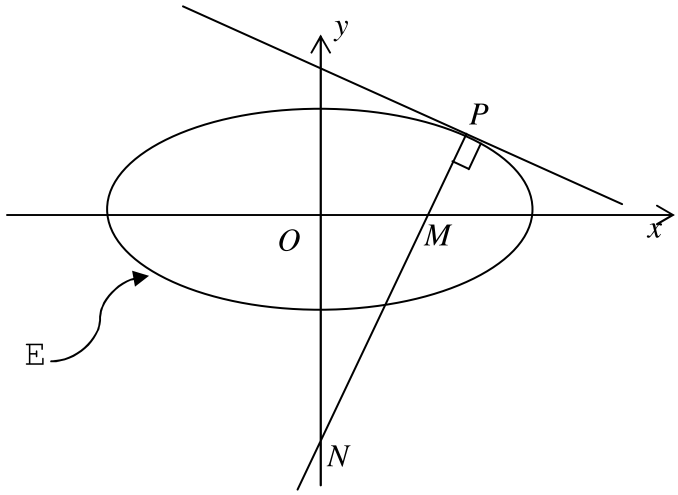
$$\int_1^n f(x)\,dx > \frac{1}{2}f(1) + \frac{1}{2}f(n) + \sum_{r=2}^{n-1} f(r).$$ **2**

(iii) By taking $f(x) = \log_e x$ in the inequality from (b) part (ii) above, deduce that if $n$ is a positive integer, then

$$n! < n^{n+\frac{1}{2}} e^{-n+1}.$$ **4**

**QUESTION 8** (15 marks) Use a SEPARATE writing booklet. **Marks**

(a)




The ellipse E has equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. The point $P(a\cos\theta, b\sin\theta)$ lies on the ellipse.

(i) Show that the equation of the normal to the ellipse at $P$ is
$y-b\sin\theta=\frac{a\sin\theta}{b\cos\theta}(x-a\cos\theta)$. **2**

(ii) The normal at $P$ meets the $x$ axis at $M$ and the $y$ axis at $N$ as shown in the diagram above.
Prove that $\frac{PM}{PN}=1-e^2$ where $e$ is the eccentricity of E. **3**

(b) If $A(x)=\frac{1}{2}+\frac{1}{3}\binom{n}{1}x+\frac{1}{4}\binom{n}{2}x^2+.................+\frac{1}{n+2}x^n$,

(i) Show that $\frac{d}{dx}\left\{x^2A(x)\right\}=x(1+x)^n$. **3**

(ii) Show that $x(1+x)^n=(1+x)^{n+1}-(1+x)^n$. **1**

(iii) Hence show that $x^2A(x)=\frac{(1+x)^{n+2}-1}{n+2}-\frac{(1+x)^{n+1}-1}{n+1}$. **3**

(iv) Deduce that $\sum_{r=0}^{n}\frac{1}{r+2}\binom{n}{r}=\frac{n\cdot 2^{n+1}+1}{(n+1)(n+2)}$. **3**

**End of paper**