Ellipse of Best Fit Around the Mona Lisa's Face

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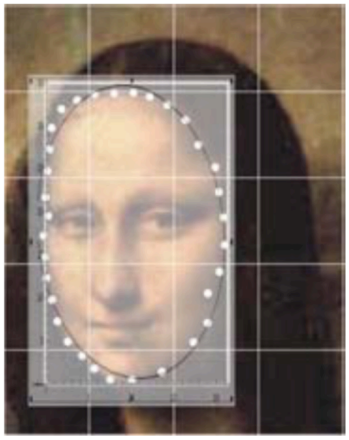
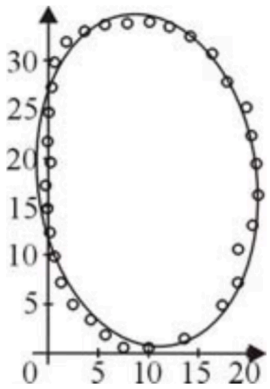
Two months ago I published an article on line of best fit using the Penrose inverse (Buchanan, 2020). For simple 1-1 curves of best fit, a transformation can turn it into a line of best fit. For example an exponential curve of best fit can be turned into a line of best fit by use of a logarithm. However not all curves of best fit can so easily be transformed. Thankfully though, the Penrose inverse method can be generalised for more difficult curves of best fit too and this article is an example of that.

Often artists begin drawing or painting a face by starting with an ellipse. We never see this ellipse in the finished product. However we can get it back by plotting points around the face and then finding the ellipse of best fit.





So for example let $(x_1, y_1), \ldots, (x_{33}, y_{33})$ be the following 33 points around the Mona Lisa's face (and one should not obsess too much about the accuracy of these points - each one is an artistic decision, not a mathematical one):

(14, 2), (17, 6), (18, 7), (18, 11), (20, 13), (21, 17), (20.5, 20), (20, 23), (19, 26), (17, 28), (16, 31), (14, 32), (12, 33), (10, 33.5), (7, 33.3), (6, 33.1), (4, 32.3), (2, 31.7), (1, 30), (0.5, 22.5), (0, 25), (0,22) (0.5, 19.5), (-0.5, 17.5), (0, 15), (0.5, 12.5), (1, 10), (2, 7), (3, 5), (4,3), (6, 2), (7.5, 0.5), (10, 0.5)

We want to find the equation of the ellipse of best fit around her face. So suppose it is of the form $ax^2 + by^2 + cx + dy + exy = 1$.

Where $A = \begin{pmatrix} x_1^2 & y_1^2 & x_1 & y_1 & x_1y_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{33}^2 & y_{33}^2 & x_{33} & y_{33} & x_{33}y_{33} \end{pmatrix}$ we have that $A \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

But because A is not a square matrix we use the Penrose inverse to solve it to get the ellipse of best fit coefficients $\begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = (A^TA)^{-1}A^T \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ where A^T is the transpose of A, i.e., the matrix resulting from swapping rows and columns.

This is quite difficult to do by hand so we use technology to do it. In Mathematica the Penrose inverse $(A^TA)^{-1}A^T$ is called the PseudoInverse.

Then to find a, b, c, d, e we enter in Mathematica

PseudoInverse[$\{\{x_1^2, y_1^2, x_1, y_1, x_1y_1\}, \ldots, \{x_{33}^2, y_{33}^2, x_{33}, y_{33}, x_{33}y_{33}\}\}$].$\{\{1\}, \ldots, \{1\}\}$ as follows:

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PseudoInverse[{{14^2, 2^2, 14, 2, 14*2}, {17^2, 6^2, 17, 6, 17*6}, {18^2, 7^2, 18, 7, 18*7},
{18^2, 11^2, 18, 11, 18*11}, {20^2, 13^2, 20, 13, 20*13}, {21^2, 17^2, 21, 17, 21*17},
{20.5^2, 20^2, 20.5, 20, 20.5*20}, {20^2, 23^2, 20, 23, 20*23},{19^2, 26^2, 19, 26, 19*26},
{17^2, 28^2, 17, 28, 17*28}, {16^2, 31^2, 16, 31, 16*31}, {14^2, 32^2, 14, 32, 14*32},
{12^2, 33^2, 12, 33, 12*33}, {10^2, 33.5^2, 10, 33.5, 10*33.5}, {7^2, 33.3^2, 7, 33.3, 7*33.3},
{6^2, 33.1^2, 6, 33.1, 6*33.1},{4^2, 32.3^2, 4, 32.3, 4*32.3}, {2^2, 31.7^2, 2, 31.7, 2*31.7},
{1^2, 30^2, 1, 30, 1*30}, {0.5^2, 22.5^2, 0.5, 22.5, 0.5*22.5}, {0^2, 25^2, 0, 25, 0*25},
{0^2, 22^2, 0, 22, 0*22}, {0.5^2, 19.5^2, 0.5, 19.5, 0.5*19.5},
{(-0.5)^2, 17.5^2, -0.5, 17.5, -0.5*17.5}, {0^2, 15^2, 0, 15, 0*15},
{0.5^2, 12.5^2, 0.5, 12.5, 0.5*12.5}, {1^2, 10^2, 1, 10, 1*10},
{2^2, 7^2, 2, 7, 2*7}, {3^2, 5^2, 3, 5, 3*5}, {4^2, 3^2, 4, 3, 4*3}, {6^2, 2^2, 6, 2, 6*2},
{7.5^2, 0.5^2, 7.5, 0.5, 7.5*0.5}, {10^2, 0.5^2, 10, 0.5, 10*0.5}}].{{1}, {1}, {1}, {1},
{1}, {1}, {1}, {1}, {1}, {1}, {1}, {1}, {1}, {1}, {1}, {1}, {1}, {1}, {1}, {1}, {1},
{1}, {1}, {1}, {1}, {1}, {1}, {1}, {1}, {1}, {1}, {1}}
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The result is {{-0.00769248}, {-0.00294774}, {0.166871}, {0.112133}, {-0.000796667}} whereupon the equation of the ellipse of best fit for Mona Lisa's face is

$$-0.00769248x^2 - 0.00294774y^2 + 0.166871x + 0.112133y - 0.000796667xy = 1$$

Reference

Buchanan, D., Alternative Solutions to Cambridge Extension 1 Year 12 Exercise 15E, 2020, online at http://www.angelfire.com/ab7/fourunit/15Esol.pdf.