Killara HIGH SCHOOL
MATHEMATICS DEPARTMENT

Student Number

2023 **YEAR 12**

Mathematics Extension 2

Trial HSC Examination

Date: Monday 7th August, 2023

Q	Marks
MC	/10
11	/14
12	/14
13	/14
14	/16
15	/18
16	/14
Total	**/100**

General Instructions:

- Reading time – 10 minutes
- Working time – 3 hours
- Write using blue or black pen
- NESA approved calculators may be used
- Show relevant mathematical reasoning and/or calculations
- No white-out may be used

Total Marks: 100

Section I - 10 marks

- Allow about 15 minutes for this section

Section II - 90 marks

- Allow about 2 hours and 45 minutes for this section

This question paper must not be removed from the examination room.

This assessment task constitutes 40% of the course.

Section I

10 marks

Allow about 15 minutes for this section

Use the multiple-choice sheet for Questions 1–10

---

1 During an election, politician A declares that creating more jobs will reduce crime. Politican B declares instead that reducing crime will create jobs. Politician B's statement is the _________ of politician A's statement.

(A) Negation

(B) Inverse

(C) Converse

(D) Contrapositive

2 The contrapositive of the statemement "If xy and $x - y$ are even, then both x and y are even" is best given by:

(A) If either x or y are odd, then either xy or $x - y$ are odd

(B) If both x and y are odd, then both xy and $x - y$ are odd

(C) If either xy or $x - y$ are odd, then either x or y are odd

(D) If both xy and $x - y$ are odd, then both x and y are odd

3 The angle between the diagonals of a cube is:

(A) $\cos^{-1}\frac{1}{9}$

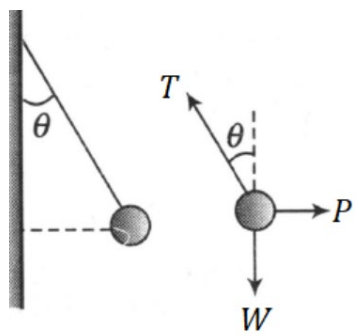
(B) $\cos^{-1}\frac{1}{3}$

(C) $\cos^{-1}\frac{1}{\sqrt{3}}$

(D) $\cos^{-1}\frac{\sqrt{3}}{2}$

4 A metal sphere is hung by a string fixed to a wall. The sphere is pushed away from the wall by a stick. The forces acting on the sphere are shown in the second diagram.

Which of the following statements is incorrect?



(A) $P = W\tan\theta$

(B) $\vec{T}+\vec{P}+\vec{W}=0$

(C) $T^2 = P^2 + W^2$

(D) $\vec{T}=\vec{P}+\vec{W}$

5 The complex numbers $z = x + iy$ which satisfy the equation $\left|\frac{z-3i}{z+3i}\right| = 1$ lie on

(A) circle with centre $(0,0)$ and radius 3

(B) a circle passing through the origin

(C) the straight line $y = 3$

(D) the x-axis

6 The roots of the equation $z^n = (z+1)^n$

(A) are collinear

(B) are vertices of a regular polygon

(C) lie on a circle

(D) lie on a parabola with vertex $\left(-\frac{1}{2}, 0\right)$

7 If $\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}$ are three vectors of which every pair is non-collinear. If $\underset{\sim}{a} + \underset{\sim}{b}$ and $\underset{\sim}{b} + \underset{\sim}{c}$ are collinear with vectors $\underset{\sim}{c}$ and $\underset{\sim}{a}$ respectively, then

(A) $\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}$ is a null vector

(B) $\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}$ is a unit vector

(C) $\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}$ is a vector of magnitude 2 unitsx

(D) $\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}$ is a vector of magnitude 3 units

8 If $\int f(x)dx = F(x)$, then $\int x^3\, f(x^2)dx$ is equal to

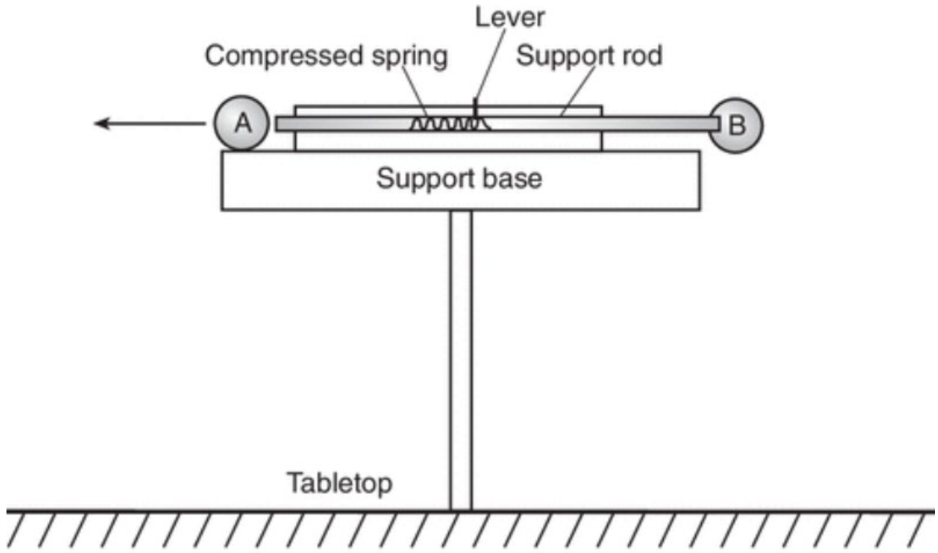
(A) $\frac{1}{2}\left[x^2 F(x) - \frac{1}{2}\int \left(F(x)\right)^2 dx\right]$

(B) $\frac{1}{2}\left[x^3 F(x^2) - 3\int x^2 F(x^2)dx\right]$

(C) $\frac{1}{2}\left[x^2 \left(F(x)\right)^2 - \int \left(F(x)\right)^2 dx\right]$

(D) $\frac{1}{2}\left[x^2 F(x^2) - \int F(x^2)d(x^2)\right]$

9 The diagram below represents a setup for demonstrating motion.



When the lever is released, the support rod withdraws from ball B, allowing it to fall. At the same instant the rod contacts ball A, propelling it horizontally to the left.

Which statement describes the motion that is observed after the lever is released and the balls fall? [Neglect friction.]

(A) Ball A travels at constant velocity.

(B) Ball A hits the tabletop at the same time as ball B

(C) Ball B hits the tabletop before ball A

(D) Ball B travels with an increasing acceleration

10 Unit vectors $\vec{a}$ and $\vec{b}$ are inclined at and angle 2θ such that $|\vec{a} - \vec{b}| < 1$ and $0 \leq \theta \leq \pi$, then θ lies in the interval:

(A) $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$

(B) $\left[\frac{\pi}{6}, \pi\right]$

(C) $\left[\frac{\pi}{2}, \frac{5\pi}{6}\right]$

(D) $\left[\frac{5\pi}{6}, \pi\right]$

End of Section I

Section II

90 marks

Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

In Questions 11-16, your response should include relevant mathematical reasoning and/or calculations.

Question 11 (14 Marks) Use the Question 11 Writing Booklet.

(a) The complex numbers z_1, z_2 and z_3 are such that $z_1 = 3 - i\sqrt{3}, z_2 = \frac{1}{2}e^{i\frac{2\pi}{5}}$ and

$z_3 = z_1 z_2$.

(i) Find exactly the modulus and argument of z_3. **3**

(ii) Sketch an Argand diagram showing z_1, z_2 and z_3. **2**

You may use the polar axes on the sheet provided.

(iii) Find the smallest positive integer value of n for which ${z_3}^n$ is purely imaginary. **3**

State the modulus of ${z_3}^n$ in this case, giving answer in surd form.

(b) Use partial fractions to find **4**

$$\int \frac{(2x^2 + 5x + 9)}{(x-1)(x^2 + 2x + 5)} dx$$

Question 11 continues on the next page

(c) Find: **2**

$$\int_{0}^{\frac{\pi}{3}} \frac{d\theta}{1+\sin\theta}$$

End of Question 11

Question 12 (14 Marks) Use the Question 12 Writing Booklet.

(a) Prove that if a, b are integers such that 7 divides $a + b$ and $a^2 + b^2$, then 7 divides both a and b. **2**

(b) Show that $x \geq \ln(1 + x)$ for all $x > -1$, stating clearly when the equality holds. **2**

(c) Find **3**

$$\int \frac{\sqrt{1 + x^2}}{x^4} dx$$

(d) Prove that for $\forall a, b, c \in \mathbb{Z}^+$, where a, b and c form a Pythagorean triple (that is, $a^2 + b^2 = c^2$), that a, b, and c cannot all be odd numbers. **3**

(e) In an engine, the piston undergoes vertical simple harmonic motion with amplitude 7 cm. A washer of mass m kg rests on top of the piston and moves with it. At optimal speeds the washer stays in contact with the piston. The motor speed is slowly increased. **4**

Find the frequency of the piston at which the washer no longer stays in contact with the piston.

End of question 12

Question 13 (14 Marks) Use the Question 13 Writing Booklet.

(a) (i) It is given that $-1+2i$ is a root of the equation, **1**

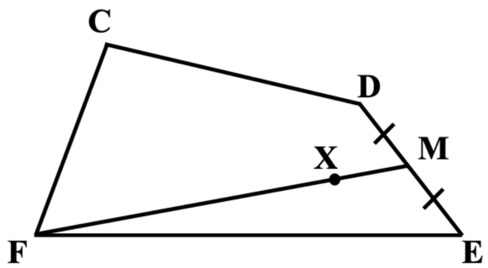
$$z^3+2(1+i)z^2+(5+4i)z+10i=0.$$

Explain why $-1-2i$ may not be a root.

(ii) Solve the equation $z^3+2(1+i)z^2+(5+4i)z+10i=0$, giving your answers in the form $a+ib$, where a and b are exact values. **4**

(iii) Hence solve $iz^3+2(1+i)z^2+(4-5i)z-10i=0$. **2**

(b) In the diagram below C, D, E and F are points in a plane. $\overrightarrow{CD}=\boldsymbol{a}$, $\overrightarrow{DE}=\boldsymbol{b}$ and $\overrightarrow{FC}=\underset{\sim}{a}-\underset{\sim}{b}$. M is the midpoint of DE. X is the point on FM such that $FX:XM=n:1$.



(i) Express $\overrightarrow{FE}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$. **1**

(ii) Given that CXE is a straight line, find the value of n. **4**

(iii) Find the point P where $\overrightarrow{CD}$ and $\overrightarrow{FM}$ intersect. **2**

End of question 13

Question 14 (16 Marks) Use the Question 14 Writing Booklet.

(a) Consider the sequence of real numbers $x_1 \geq x_2 \geq x_3 \geq \cdots \geq x_n$ and $y_1 \geq y_2 \geq y_3 \geq \cdots \geq y_n$.

Prove that, if $z_1, z_2, z_3, \ldots, z_n$ be any permutation of the numbers $y_1, y_2, y_3, \ldots, y_n$, then **4**

$$\sum_{i=1}^{n}(x_i - y_i)^2 \leq \sum_{i=1}^{n}(x_i - z_i)^2$$

(b) (i) For $a, b > 0$, prove that **2**

$$\frac{a}{b} + \frac{b}{a} \geq 2$$

(ii) Let $a_1, a_2, a_3, \ldots\ a_n$ be positive real numbers such that $a_1 a_2 a_3 \ldots a_n = 1$. **2**

Prove that,

$$(1 + a_1)(1 + a_2)(1 + a_3) \ldots.\ (1 + a_n) \geq 2^n$$

(iii) Prove that for $a, b, c, d > 0$, **2**

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{d} + \frac{d^2}{a} \geq a + b + c + d$$

Question 14 continues on the next page

(c) Let:

$$I_n = \int \text{cosec}^n x \qquad n \in \mathbb{Z}$$

(i) Prove that, for $n \geq 2$ **3**

$$I_n = \frac{n-2}{n-1} I_{n-2} - \frac{\text{cosec}^{\,n-2} x \cot x}{n-1}$$

(ii) Hence, show that **3**

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \text{cosec}^6 x \, dx = \frac{56}{135}\sqrt{3}$$

End of question 14

Question 15 (18 Marks) Use the Question 15 Writing Booklet.

(a) A gas company has plans to install a pipeline from a gas field to a storage facility. One part of the route for the pipeline must pass under a river. This part of the pipeline is in a straight line between two points, P and Q.

Points are defined relative to an origin $(0, 0, 0)$ at the gas field. The x-, y- and z-axes are in the directions east, north and vertically upwards respectively, with units in metres. P and Q has position vectors,

$$\overrightarrow{OP} = \begin{pmatrix} 1136 \\ 92 \\ p \end{pmatrix} \text{ and } \overrightarrow{OQ} = \begin{pmatrix} 200 \\ 20 \\ -15 \end{pmatrix}$$

(i) The length of the pipeline PQ is 939 metres. Given that the level of P is below that of Q, find the value of p. **2**

(ii) A thin layer of rock lies below the ground. This layer is modelled as a plane. Three points in this plane are $A(400,\ 600, -20)$, $B(500,\ 200, -70)$ and $C(600, -340, -50)$. **3**

Find the normal vector $\underset{\sim}{n}$, perpendicular to $\overrightarrow{AB}$ and $\overrightarrow{BC}$.

(iii) Hence, find the point at which the pipeline meets the rock. **3**

(iv) Find the angle that the pipeline between the points P and Q makes with the horizontal. **2**

Question 15 continues on the next page

(b) Consider the function $f(x) = \sin x \log_e(x+n)$.

(i) Using integration by parts, show that **3**

$$\int_0^{2\pi} \sin x \log_e(x+n)dx = -\log_e\left(1+\frac{2\pi}{n}\right) + \int_0^{2\pi} \frac{\sin x}{(x+n)^2}dx$$

(ii) Prove that $\left|\int_0^{2\pi} \frac{\sin x}{(x+n)^2}dx\right| < \frac{2\pi}{n^2}$ **3**

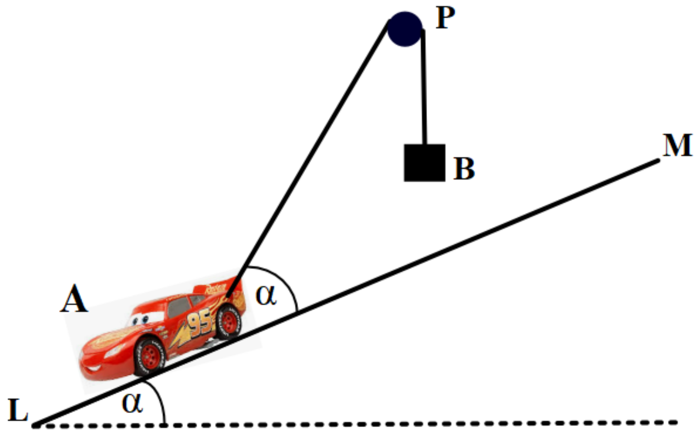
(iii) Deduce that as n $\to \infty$, **3**

$$\frac{\int_0^{2\pi} \sin x \log_e(1+x)dx}{-\frac{2\pi}{n}} \to 1$$

End of question 15

Question 16 (14 Marks) Use the Question 16 Writing Booklet.

(a) **5**



At a racecourse, a model car weighing $2m$ kilograms is held in place on a ramp by a hanging mass of m kilograms. The two bodies A and B of masses $2m$ and m kilograms respectively are attached to the ends of a light inextensible string. The string passes over a smooth pulley P. The car rests in equilibrium on a rough ramp LM.

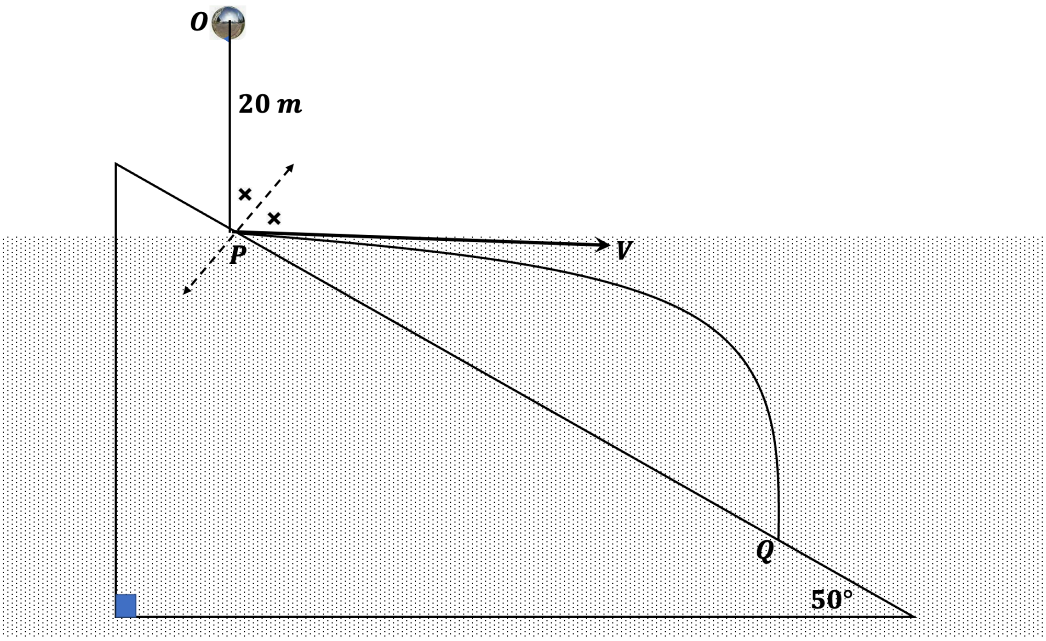
The rough ramp LM makes an angle α to the horizontal and, the rope attached to the car A makes an equal angle of α to the ramp. The body B hangs vertically below P.

Find the range of values of α for which the car A will not slip down the ramp or lose contact with the ramp.

Question 16 continues on the next page

(b) The diagram below shows a smooth platform inclined at an angle of $50°$ to the horizontal, partially immersed in a medium. A smooth ball falls freely from O and strikes the platform at the point P, 20 metres vertically below it as shown in the diagram (air resistance is negligible).

The ball then bounces off the platform with velocity of $V\ ms^{-1}$ and strikes it again at the point Q. As it bounces and enters the medium, the ball experiences the effect of gravity and a resistance of $0.4V$ per unit mass in both horizontal and vertical directions.



(The acceleration due to gravity is $9.8\ \mathrm{ms}^{-2}$).

(i) Show that the ball has a speed of $2\sqrt{10g}$ as it strikes the platform just above the medium. **2**

(ii) Verify that the ball will strike the platform again at Q after 3.32 seconds. **4**

(iii) Calculate the velocity and angle of impact at Q. **2**

End of Examination

2023 Ext 2 Trial Marking guidelines

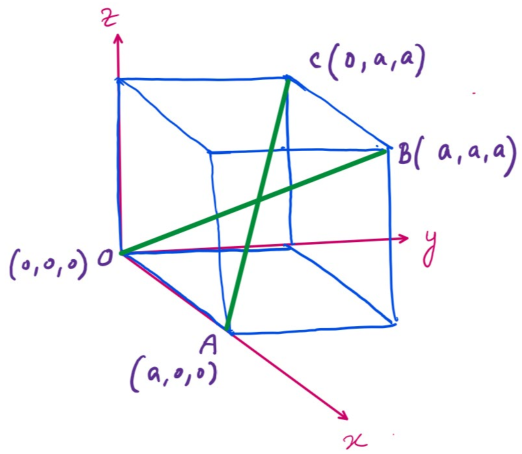
Question 1

Converse (C)

Question 2

(C)

Question 3



Direction cosines of $OB = \left(\frac{a-0}{\sqrt{a^2+a^2+a^2}}, \frac{a-0}{\sqrt{a^2+a^2+a^2}}, \frac{a-0}{\sqrt{a^2+a^2+a^2}}\right)$

$= \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

Direction cosines of $AC = \left(\frac{0-a}{\sqrt{a^2+a^2+a^2}}, \frac{a-0}{\sqrt{a^2+a^2+a^2}}, \frac{a-0}{\sqrt{a^2+a^2+a^2}}\right)$

$= \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$\overrightarrow{OB}.\overrightarrow{AC} = |\overrightarrow{OB}|\,|\overrightarrow{AC}|\cos\theta$

$-\frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1 \times 1\cos\theta$

$\cos\theta = \frac{1}{3}$

$\theta = \cos^{-1}\frac{1}{3}$

Question 4

As the metal sphere is in equilibrium under the effect of the three forces, $\vec{T} + \vec{P} + \vec{W} = 0$.

From the figure, $T\cos\theta = W$ (1) and $T\sin\theta = P$. (2)

From (1) and (2) Then $P = W\tan\theta$ and $T^2 = P^2 + W^2$

Hence, Option D is incorrect.

Question 5

A

$$\left|\frac{z-3i}{z+3i}\right| = 1$$

Then, $\quad |z-3i| = |z+3i|$

Interpreting the meaning we get the $|PA| = |PB|$

Then P is on the perpendicular bisector of the line joining $A(3i)$ and $B(-3i)$

Hence, P lies on the x-axis.

Question 6

Multiple choice work

$$\left|\frac{z+1}{z}\right|^n = 1$$

For multiple choice, let $n = 1$

$|z+1| = |z|$

Using symmetry, $z = -\frac{1}{2}$

That is $x = -\frac{1}{2}$

Equation of the locus is $2x + 1 = 0$

Which is linear.

Therefore the roots are collinear. (A)

Or

Let $z = x + iy$ and solve algebraically.

Question 7

$\underset{\sim}{a} + \underset{\sim}{b}$ is collinear with $\underset{\sim}{c}$, then $\underset{\sim}{a} + \underset{\sim}{b} = \lambda c$

$\Rightarrow \underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c} = \lambda \underset{\sim}{c} + \underset{\sim}{c} = \underset{\sim}{c}(1+\lambda)$

Given $\underset{\sim}{b} + \underset{\sim}{c}$ is collinear with $\underset{\sim}{a}$, then $\underset{\sim}{b} + \underset{\sim}{c} = \mu \underset{\sim}{a}$

$\Rightarrow \underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c} = \underset{\sim}{a} + \mu \underset{\sim}{a} = \underset{\sim}{a}(1+\mu)$

Then, by equating,

$\underset{\sim}{c}(1+\lambda) = \underset{\sim}{a}(1+\mu)$

But $\underset{\sim}{a}$ and $\underset{\sim}{c}$ are not collinear.

So, $1+\lambda = 1+\mu = 0 \Longrightarrow \lambda = \mu = -1$

Then $\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c} = \underset{\sim}{0}$

$\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}$ is a null vector (A)

Question 8

$\int f(x)dx = F(x)$, then $\int x^3 f(x^2)dx$ is equal to

$$\int x^3 f(x^2)dx = \int x^2 \,.\, \frac{1}{2}\big(2x\, f(x^2)\big)dx$$

$$= \frac{1}{2}\left[x^2 F(x^2) - \int 2x\, F(x^2)dx\right]$$

$= \frac{1}{2}[x^2 F(x^2) - \int F(x^2)d(x^2)]$ (D)

Question 9

(A) V_x is constant, but not V_y $(Not\ A)$

(B) $y = -\frac{gt^2}{2} + h$ in both cases, hence both A nd B takes the same time

(C) Not C (using B)

(D) y – acceleration is constant which is the only force acting on the body, so, not D

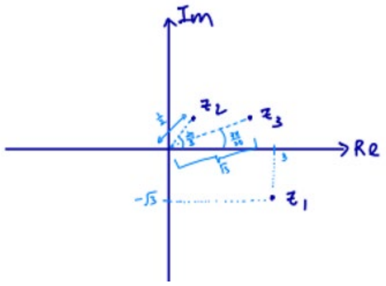
Answer B

Question 10

(D)

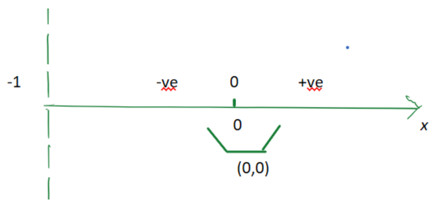
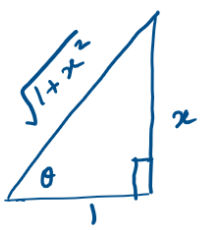
Question 11 (12 marks)

11 a(i)	$\lvert z_3\rvert = \lvert z_1\rvert\lvert z_2\rvert$ $\frac{1}{2} \times \sqrt{12} = \sqrt{3}$ $\arg z_3 = \arg z_2 + \arg z_1$ $\arg z_1 = -\frac{\pi}{6}$ $= \frac{2\pi}{5} - \frac{\pi}{6} = \frac{7\pi}{30}$	1 mark: correctly calculates the modulus. 1 mark: calculates $\arg z_1$ and $\arg z_2$ 1 mark: correctly calculates $\arg z_3$	

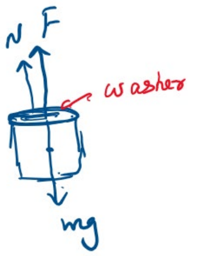
a(ii)			
a(iii)	$z_3{}^n = \left(\sqrt{3}e^{\frac{7\pi}{30}i}\right)^n$ $= \sqrt{3}^n e^{\frac{7n\pi}{30}i}$ $z_3{}^n$ is purely imaginary, hence real part equals zero. $\text{Hence}, \cos\frac{7n\pi}{30} = 0 = \cos\frac{(2k+1)\pi}{2}, \quad k \in R$ $\frac{7n\pi}{30} = \frac{(2k+1)\pi}{2}$ $n = \frac{15(2k+1)}{7}$ When $k = 3,\ n = 15$ $\therefore\ \text{modulus:}\ \left(\sqrt{3}\right)^{15} = 3^7\sqrt{3}$ $m = 2187$	*correctly writes $z_3{}^n$ * equates the real parts and solves for n * substitutes the least value (k=3) to give the least integer value of n (n=15) and gives the correct value of m 3 marks: All three parts correct 2 marks: dot points 1 and 2 correct and attempts to find the least value of k to make n an integer. 1 mark: correctly writes $z_3{}^n$ Or Attempts to solve for n from their $z_3{}^n$ in terms of k, and makes significant way.	

11 (b)	$\int \frac{(2x^2+5x+9)}{(x-1)(x^2+2x+5)}dx$ $\frac{(2x^2+5x+9)}{(x-1)(x^2+2x+5)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+2x+5}$ Let $x = 1$ $A = 2$ $2x^2+5x+9 = A(x^2+2x+5)$ $+ \quad (Bx+C)(x-1)$ Comparing x^2 term, $A + B = 2 \rightarrow \quad B = 0$ Comparing constants, $9 = 5A - C \ \rightarrow \quad C = 1$ $\int \frac{(2x^2+5x+9)}{(x-1)(x^2+2x+5)}dx$ $= \int \frac{2}{x-1} + \frac{1}{x^2+2x+5}dx$ $= \int \frac{2}{x-1} + \frac{1}{(x+1)^2+4}dx$ $= 2\ln\|x-1\| + \frac{1}{2}\tan^{-1}\frac{x+1}{2} + C$	1 mark: Separates the integrand into appropriate general forms of partial fractions and evaluates at least one of the pronumerals 1 mark: evaluates the pronumerals correctly. 2 marks: correctly integrates. 1 mark: correctly integrates $\frac{2}{x-1}$ and attempts to integrate the quadratic denominator	
11 (c)	$\int_0^{\frac{\pi}{3}} \frac{d\theta}{1+\sin\theta}$ Let $t = \tan\frac{\theta}{2}, \quad then \ \ d\theta = \frac{2dt}{1+t^2}$ When $\theta = 0, t = 0; \ \ \theta = \frac{\pi}{3}, \ y = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$ $\int_0^{\frac{1}{\sqrt{3}}} \frac{\frac{2dt}{1+t^2}}{1+\frac{2t}{1+t^2}} = \int_0^{\frac{1}{\sqrt{3}}} \frac{2dt}{1+t^2+2t}$ $\int_0^{\frac{1}{\sqrt{3}}} \frac{2dt}{(1+t)^2} = \left[-\frac{2}{1+t}\right]_0^{\frac{1}{\sqrt{3}}}$ $= -\frac{2}{1+\frac{1}{\sqrt{3}}} + 2$ $= 2 - \frac{2\sqrt{3}}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1}$ $= 2 - 3 + \sqrt{3}$ $= \sqrt{3} - 1$	1 mark: correctly converts the integrand and the limits in terms of t 1 mark: correctly integrates and evaluates	

Question 12

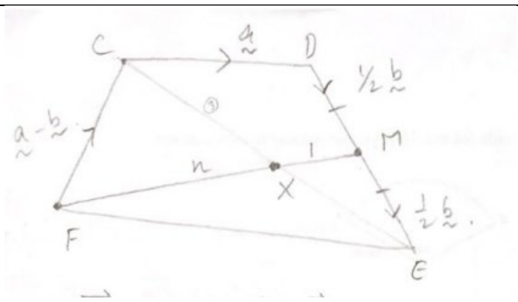
12(a)	Since 7 divides $a + b$, it divides, $(a+b)^2 = a^2 + 2ab + b^2$ Hence 7 also divides the difference of this and $a^2 + b^2$, which is $2ab$ But 7 does not divide 2, so it must divide ab Since 7 is a prime, it must divide either a or b. But if it divides a, it divides b as well (since it divides $a + b$); similarly, it 7 divides b, it also divides a. So it divides both a and b	1 mark for substantive progress towards solution by finding $2ab$ is divisible by 7. 2 marks complete and logical solution	
12(b)	We need to prove that $x \geq \ln(1+x)$ for $\forall x > -1$ Consider the function $f(x) = x - \ln(1+x)$ $f'(x) = 1 - \dfrac{1}{1+x}$ $f'(x) = 0 \Leftrightarrow \quad x = 0$  Thus, $f(x)$ has an absolute minimum of 0 at $x = 0\ for\ x > -1$, with equality iff $x = 0$.	1 mark: correctly proves *f*(*x*) has a minimum value at $x = 0$! mark: explains that f(x) has the minimum value 0 and also equality holds iff x = 0 (must give clear working and explanation)	
12(c)	$\displaystyle\int \frac{\sqrt{1+x^2}}{x^4}\,dx$ Let $x = \tan\theta$ Then, $dx = \sec^2\theta\, d\theta$ $\sqrt{1+x^2} = \sec\theta$ $\displaystyle\int \frac{\sqrt{1+x^2}}{x^4}\,dx = \int \frac{\sec\theta \sec^2\theta\, d\theta}{\tan^4\theta}$ $\displaystyle= \int \frac{\cos\theta}{\sin^4\theta}\,d\theta$ $\displaystyle= -\frac{1}{3\sin^3\theta} + C$ 	Uses the correct substitution and transforms the integrand. 1 mark: correctly integrates. 1 mark: gives the solution in terms of x	

	$= -\dfrac{1}{3\left(\dfrac{x}{\sqrt{1+x^2}}\right)^3} + C$ $= -\dfrac{(1+x^2)^{\frac{3}{2}}}{3x^3} + C$		
12 (d)	for $\forall a, b, c \in \mathbb{Z}^+$, that form Pythagorean triplet (that is, $a^2 + b^2 = c^2$), a, b, and c cannot all be odd numbers. We will prove this statement by contradiction. Suppose a and b are both odd. Then $a = 2s + 1$ and $b = 2t + 1$, $s, t \in \mathbb{Z}^+$ $c^2 = a^2 + b^2$ $= (2s+1)^2 + (2t+1)^2$ $= 4s^2 + 4s + 1 + 4t^2 + 4t + 1$ $4(s^2 + s + t^2 + t) + 2$ This means that c^2 is even and so c is even, say $c = 2u$ and therefore $c^2 = 4u^2$. Putting this together with the previous equation, $4u^2 = 4(s^2 + s + t^2 + t) + 2$ $2u^2 = 2(s^2 + s + t^2 + t) + 1$ This is impossible as LHS is even, and RHS is odd. Hence, the contradiction is false, and the statement is correct.	3 marks	
12 (e)	The displacement $x = a\sin(\omega t + \phi)$ $\dot{x} = \omega\, a\cos(\omega t + \phi)$ $\ddot{x} = -\omega^2\, a\sin(\omega t + \phi)$ Amplitude = 7 cm = 0.07 m $F + N = mg$ As the washer does not stay in contact with the piston, at some frequency, the normal force on the washer equals zero. $F_{max} = mg$ Maximum acceleration = $-\omega^2 a = g$ $\omega^2 = \dfrac{g}{a}$	1 mark: develops the equations of motion and states the maximum acceleration. 1 mark: converts amplitude to metres and writes the forces acting on the washer of mass m. 1 mark: calculates ω	

	$\omega = \sqrt{\frac{g}{a}} = \sqrt{\frac{10}{0.07}} = \sqrt{\frac{1000}{7}} = 10\sqrt{\frac{10}{7}}$ Frequency of motion $f = \frac{1}{T}$ $T = \frac{2\pi}{\omega} = \frac{2\pi\sqrt{7}}{10\sqrt{10}} = \frac{\pi\sqrt{7}}{5\sqrt{10}}$ $f = \frac{1}{T} = \frac{5\sqrt{10}}{\pi\sqrt{7}}$ hertz  $F_{max} = ma\omega^2$	1 mark: calculates the frequency.	

Question 13

13(a) (i)	Since the equation does not have all real coefficients, the conjugate root theorem does not apply. Hence, $-1-2i$ may not be a root.	1 mark: correct explanation	
13a (ii)	Let $z^3 + 2(1+i)z^2 + (5+4i)z + 10i \cong (z+1-2i)(z^2+Az+B)$ Comparing constants, $10i = (1-2i)B$ Then, $B = \frac{10i}{1-2i} = \frac{10i(1+2i)}{5} = -4+2i$ Comparing z^2 term, $2(1+i) = A+1-2i$ Then $A = 1+4i$ Thus, the polynomial equation is $(z+1-2i)(z^2+(1+4i)z+(-4+2i)) = 0$ 2 **marks** Solving the quadratic, $z = \frac{-1-4i \pm \sqrt{(1+4i)^2 - 4(-4+2i)}}{2}$ $z = \frac{-1-4i \pm \sqrt{1}}{2}$	2 marks: factorises the polynomial into linear and quadratic factors 2 marks: correctly solves the quadratic equation and gives all the three roots. (Award 1 mark: if minor error in calculations) Students may choose to use the sum of roots, product of roots methos. This would leave you having to find the square root.	

	$z = -2i, \quad \frac{-2-4i}{2} = -1-2i$ Roots are $z = -2i, \quad -1+2i, \quad -1-2i$		
13a (iii)	$z^3 + 2(1+i)z^2 + (5+4i)z + 10i = 0$ If we replace, z with iz, $(iz)^3 + 2(1+i)(iz)^2 + (5+4i)(iz) + 10i = 0$ $-iz^3 - 2(1+i)z^2 + (-4+5i)z + 10i = 0$ $iz^3 + 2(1+i)z^2 + (4-5i)z - 10i = 0$ Then, the roots are $iz = -2i, \quad -1+2i, \quad -1-2i$ $z = -2, \frac{-1+2i}{i}, \frac{-1-2i}{i}$ $z = -2, \quad 2+i, -2+i$	1 mark: substitutes z with iz to get the necessary equation 1 mark: converts the solutions using the transformation	
13 (b)	 Method 1 $\overrightarrow{FE} = \overrightarrow{FC} + \overrightarrow{CD} + \overrightarrow{DE}$ $= \underset{\sim}{a} - \underset{\sim}{b} + \underset{\sim}{a} + \underset{\sim}{b} = 2\underset{\sim}{a} \qquad 1\, mark$ Without losing laws of generality, let us keep F as the origin. Position vector of C is $\underset{\sim}{a} - \underset{\sim}{b}$ $\overrightarrow{CE} = \underset{\sim}{a} + \underset{\sim}{b}$ Thus vector equation of the line CX is $\overrightarrow{FC} + s\,\overrightarrow{CE}$, where $s \in R$ Thus $\overrightarrow{CX} = \underset{\sim}{a} - \underset{\sim}{b} + s\left(\underset{\sim}{a} + \underset{\sim}{b}\right) \qquad 1\, mark$ $\overrightarrow{FM} = \overrightarrow{FC} + \overrightarrow{CM} = \underset{\sim}{a} - \underset{\sim}{b} + \underset{\sim}{a} + \frac{1}{2}\underset{\sim}{b}$ $= 2\underset{\sim}{a} - \frac{1}{2}\underset{\sim}{b}$ Thus vector equation of the line $\overrightarrow{FX} = t\,\overrightarrow{FM}$, $t \in R$	1 mark: correct expression for $\overrightarrow{FE}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{\boldsymbol{b}}$ (ii) 1 mark: finds the correct equation of the line CX 1 mark: correct equation FX 2 marks: gets the correct simultaneous equations and solves for s and t	

	$= t\left(2\underset{\sim}{a} - \frac{1}{2}\underset{\sim}{b}\right) \quad 1\ mark$ Finding point of intersection of the lines, $\underset{\sim}{a} - \underset{\sim}{b} + s\left(\underset{\sim}{a} + \underset{\sim}{b}\right) = t\left(2\underset{\sim}{a} - \frac{1}{2}\underset{\sim}{b}\right)$ $(1+s)\underset{\sim}{a} + (s-1)\underset{\sim}{b} = 2t\underset{\sim}{a} - \frac{1}{2}t\underset{\sim}{b}$ Compare $\underset{\sim}{a}$ and $\underset{\sim}{b}$ coefficients, $1 + s = 2t$ $s - 1 = -\frac{1}{2}t$ Solving simultaneously, $s = 2t - 1$ $2t - 2 = -\frac{1}{2}t$ $\frac{5}{2}t = 2$ $t = \frac{4}{5}$ Then, $s = \frac{3}{5}$ 2 marks $\overrightarrow{FX} = t\left(2\underset{\sim}{a} - \frac{1}{2}\underset{\sim}{b}\right)$ $\lvert\overrightarrow{FX}\rvert = t\left\lvert\left(2\underset{\sim}{a} - \frac{1}{2}\underset{\sim}{b}\right)\right\rvert$ from F and $(1-t)$ multiples from M. Hence, $\frac{n}{1} = \frac{t}{1-t}$ $n = 4 \quad 1\ mark$ Method 2 $\overrightarrow{FM} = 2\underset{\sim}{a} - \frac{1}{2}\underset{\sim}{b}$ $\overrightarrow{EX} = \overrightarrow{EM} + \overrightarrow{MK}$ $\overrightarrow{EX} = \frac{-1}{2}\underset{\sim}{b} + \frac{1}{n+1}\overrightarrow{MF}$ $= -\frac{1}{2}\underset{\sim}{b} - \frac{1}{n+1}\left(2\underset{\sim}{a} - \frac{1}{2}\underset{\sim}{b}\right)$ $= -\frac{1}{2}\underset{\sim}{b} - \frac{2}{n+1}\underset{\sim}{a} + \frac{1}{2(n+1)}\underset{\sim}{b}$ $= -\frac{2}{n+1}\underset{\sim}{a} + \frac{-n-1+1}{2(n+1)}\underset{\sim}{b}$ $= -\frac{2}{n+1}\underset{\sim}{a} + \frac{-n}{2(n+1)}\underset{\sim}{b}$ $\overrightarrow{EX} = -\frac{1}{n+1}\left(2\underset{\sim}{a} + \frac{n}{2}\underset{\sim}{b}\right) \quad (1)$	(Award 1 mark: gets the correct simultaneous equation and attempts to solve for s and t. (minor error) 1 mark: explains and correctly solves for n	

	CXE is a straight line. $\overrightarrow{EX} = \lambda\,\overrightarrow{EC}$ $\overrightarrow{EX} = -\lambda\left(\underset{\sim}{a} + \underset{\sim}{b}\right) \quad (2)$ Equating (1) and (2), $-\dfrac{1}{n+1}\left(2\underset{\sim}{a} + \dfrac{n}{2}\underset{\sim}{b}\right) = -\lambda\left(\underset{\sim}{a} + \underset{\sim}{b}\right)$ Comparing coefficients of $\underset{\sim}{a}$ and $\underset{\sim}{b}$, $\dfrac{2}{n+1} = \dfrac{n}{2(n+1)}$ $n^2 + n = 4n + 4$ $n^2 - 3n - 4 = 0$ $(n-1)(n+1) = 0$ $n = 4, \qquad n \neq -1$ Hence, $n = 4$		

Question 14

14a(i)	We need to prove $\sum_{i=1}^{n}\left(x_i - y_i\right)^2 \leq \sum_{i=1}^{n}(x_i - z_i)^2$ Expanding both sides, $\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2 - 2\sum_{i=1}^{n} x_i y_i$ $\leq \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} z_i^2 - 2\sum_{i=1}^{n} x_i z_i$ But, $\sum_{i=1}^{n} y_i^2 = \sum_{i=1}^{n} z_i^2$ z_i s Are only permutations of y_i s. Thus, it is enough to prove that $\sum_{i=1}^{n} x_i y_i \geq \sum_{i=1}^{n} x_i z_i$ (2 marks) Consider the pairing $x_1 \to y_1$, $x_2 \to y_2$, ... $x_n \to y_n$. By switching around some of the y values, we have obtained the pairing $x_1 \to z_1$, $x_2 \to z_2$, ... $x_n \to z_n$. Without loss of generality, Suppose that we switch around two y- values y_m and y_n where $y_m > y_n$.	Award 2 marks if $\sum_{i=1}^{n} x_i y_i \geq \sum_{i=1}^{n} x_i z_i$ And gives a verbal justification. (Award 1 mark: if any expands the sequence, attempts to simplify with some error 2 marks: $\sum_{i=1}^{n} x_i y_i \geq \sum_{i=1}^{n} x_i z_i$ Proves the result with full working and logical explanation. 1 mark: Attempts to prove after making the	

<table>
<tr>
<td></td>
<td>
The switching of numbers will only affect the sum of products (sum of squares will remain unaltered.

By switching y_m and y_n, the sum of products will increase by

$$x_m y_n + x_n y_m - x_m y_m - x_n y_n$$
$$= x_n(y_m - y_n) - x_m(y_m - y_n)$$
$= (x_n - x_m)(y_m - y_n) < 0$ as $y_m > y_n$ and $x_n < x_m$

Thus $x_m y_n + x_n y_m < x_m y_m + x_n y_n$

LHS is where the permutation has been applied, so let us call $y_n \rightarrow z_m, \ y_m \rightarrow z_n$

Thus $x_m z_m + x_n z_n < x_m y_m + x_n y_n$
Equality holds when $x_m = x_n$ and $y_m = y_n$

Generalising this result we have proved that
$$\sum_{i=1}^{n} x_i y_i \geq \sum_{i=1}^{n} x_i z_i$$
Note: This means that the largest of the sum product will be, when the largest of one sequence is paired with the largest of the other .
And hence,
$$\sum_{i=1}^{n} (x_i - y_i)^2 \leq \sum_{i=1}^{n} (x_i - z_i)^2$$
</td>
<td>mapping and converting to z.</td>
<td></td>
</tr>
<tr>
<td></td>
<td colspan="2">
Method 2 : Using contradiction

First part the same as method 1.

We need to prove,
$$\sum_{i=1}^{n} (x_i - y_i)^2 \leq \sum_{i=1}^{n} (x_i - z_i)^2$$
Expanding both sides,
$$\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2 - 2\sum_{i=1}^{n} x_i y_i$$
$$\leq \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} z_i^2 - 2\sum_{i=1}^{n} x_i z_i$$
But,
$$\sum_{i=1}^{n} y_i^2 = \sum_{i=1}^{n} z_i^2$$
z_i s Are only permutations of y_i s.

Thus, it is enough to prove that
</td>
<td>
Award 2 marks if
$$\sum_{i=1}^{n} x_i y_i \geq \sum_{i=1}^{n} x_i z_i$$
And gives a verbal justification.

(Award 1 mark: if any expands the sequence, attempts to simplify with some error

2 marks: correct Proof by contradiction

Award 1 mark: writes correct the contradiction statement, and then some minor error
</td>
</tr>
</table>

	$$\sum_{i=1}^{n} x_i y_i \geq \sum_{i=1}^{n} x_i z_i$$ We will now prove that the left-hand side of the inequality is the greatest sum reached out of all possible values of $\sum_{i=1}^{n} x_i z_i$. Obviously, if $x_1 = x_2 = x_3 = \cdots = x_n$ or $y_1 = y_2 = \cdots = y_n$, the inequality is true. Now, assume, for contradiction, that neither of those conditions are true and that there exists some order of z_i s that are not ordered in the form, $z_1 \geq z_2 \geq z_3 \geq \cdots . \geq z_n$ such that $\sum_{i=1}^{n} x_i z_i$ is at a maximum out of all possible permutations and is greater than the sum $\sum_{i=1}^{n} x_i y_i$. This necessarily means that in the sum $\sum_{i=1}^{n} x_i z_i$ there exists two terms $x_p z_m$ and $x_q z_n$ such that $x_p > x_q$ and $z_m < z_n$. $$x_p z_n + x_q z_m - (x_p z_m + x_q z_n) = (x_p - x_q)(z_n - z_m) > 0$$ which means if we make the terms $x_p z_n$ and $x_q z_m$ instead of the original $x_p z_m$ and $x_q z_n$, we can achieve a higher sum. However, this is impossible, since we assumed we had the highest sum. Thus, the inequality $$\sum_{i=1}^{n} x_i y_i \geq \sum_{i=1}^{n} x_i z_i$$ is proved, which is equivalent to what we wanted to prove.	

14b(i)	For $a, b > 0$, $$\left(\sqrt{a} - \sqrt{b}\right)^2 \geq 0$$ $$a + b - 2\sqrt{ab} \geq 0$$ Then, $a + b \geq 2ab$ Replace, $$a \to \frac{a}{b} \text{ and } \mathrm{b} \to \frac{\mathrm{b}}{\mathrm{a}}$$ Thus, $$\frac{a}{b} + \frac{b}{a} \geq 2\sqrt{\frac{a}{b} \cdot \frac{b}{a}} = 2$$	2 marks: proves the AM – GM inequality and then applies to prove the result. 1 mark: Applies AM-GM inequality to get the result.	
14b (ii)	$$a_1, a_2, a_3, \ldots \ a_n > 0$$ Using AM-GM inequality $$1 + a_1 \geq 2\sqrt{a_1}$$ $$1 + a_2 \geq 2\sqrt{a_2}$$ $$\vdots$$ $$1 + a_n \geq 2\sqrt{a_n}$$ Multiplying the inequalities, $$(1 + a_1)(1 + a_2)(1 + a_3) \ldots . \ (1 + a_n) \geq 2^n\sqrt{a_1 a_2 a_3 \ldots a_n}$$ $$a_1 a_2 a_3 \ldots a_n = 1$$ Thus, $$(1 + a_1)(1 + a_2)(1 + a_3) \ldots . \ (1 + a_n) \geq 2$$	2 marks: correct proof 1 mark: Minor error in the proof	
14b	Let $a \leq b \leq c \leq d$		

(iii)	$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{d}+\frac{d^2}{a} \geq \frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}+\frac{d^2}{d}$ $\geq \frac{a^2}{b}+\frac{b^2}{a}+\frac{c^2}{c}+\frac{d^2}{d}$ $\geq \frac{a^2}{a}+\frac{b^2}{b}+\frac{c^2}{c}+\frac{d^2}{d}$ $\geq a+b+c+d$	2 marks: correct proof with all logical steps 1 mark: makes reasonable rearrangements at least once.	
14 (c)(i)	$I_n = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \operatorname{cosec}^n x\, dx$ $I_n = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \operatorname{cosec}^{n-2} x \operatorname{cosec}^2 x\, dx$ $u = \operatorname{cosec}^{n-2} x,\ \ v' = cosec^2 x$ $I_n = cosec^{n-2}x\,(-\cot x) -$ $(n-2)\int cosec^{n-3}x\,(-cosec\,x \cot x)(-\cot x)\,dx$ $= cosec^{n-2}x\,(-\cot x) -$ $(n-2)\int cosec^{n-2}x\,(\cot^2 x)dx$ $= cosec^{n-2}x\,(-\cot x) -$ $(n-2)\int cosec^{n-2}x\,(\operatorname{cosec}^2 x - 1)dx$ $= cosec^{n-2}x\,(-\cot x) -$ $(n-2)\int (cosec^n x - cosec^{n-2}x)dx$ $I_n = cosec^{n-2}x\,(-\cot x) - (n-2)I_n + (n-2)I_{n-2}$ $(n-1)I_n = -cosec^{n-2}x \cot x \mp (n-2)I_{n-2}$ Hence, $I_n = \frac{n-2}{n-1}I_{n-2} - \frac{\operatorname{cosec}^{n-2}x \cot x}{n-1}$	1 mark: Splits the integral and begins the process of integration by parts. 1 mark: converts $\cot^2 x$ into $cosec^2 x$ 1 mark: expresses the integrals as I_n and I_{n-2} and completes the proof	
14c (ii)	$I_2 = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \operatorname{cosec}^2 x\, dx = -[\cot x]_{\frac{\pi}{3}}^{\frac{\pi}{2}} = \frac{\sqrt{3}}{3}$ $I_4 = = \frac{2}{3}I_2 - \left[\frac{\operatorname{cosec}^2 x \cot x}{3}\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$ $= \frac{2}{9}\sqrt{3} - \frac{1}{3}\left(0 - \frac{4}{3}\times\frac{\sqrt{3}}{3}\right)$ $= \frac{2}{9}\sqrt{3} + \frac{4\sqrt{3}}{27} = \frac{10}{27}\sqrt{3}$	3 marks: A fully correct method using the reduction formula correctly to reach the value for I_6 (Substitutions must be shown for the non-zero terms)	

	$$I_6 = = \frac{4}{5}I_4 - \left[\frac{\text{cosec}^4 x \cot x}{5}\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$ $$= \frac{4}{5}\left(\frac{4}{27}\sqrt{3} + \frac{2}{9}\sqrt{3}\right) + \frac{16}{135}\sqrt{3}$$ $$= \frac{56}{135}\sqrt{3}$$	2 marks: Uses the reductio formula correctly to find I_4 in terms of I_2 (need not evaluate yet) 1 mark: Begins the process of application of reduction to find I_6 in terms of I_4	

Question 15

15a(i)	$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$ $$= \begin{pmatrix} 200 \\ 20 \\ -15 \end{pmatrix} - \begin{pmatrix} 1136 \\ 92 \\ p \end{pmatrix} = \begin{pmatrix} -936 \\ -72 \\ p+15 \end{pmatrix}$$ $$936^2 + 72^2 + (p+15)^2 = 939^2$$ $$936^2 + 72^2 + p^2 + 30p + 225 = 939^2$$ $$441 = 225 + 30p + p^2$$ $$p^2 + 30p - 216 = 0$$ $$p = 6, \quad p = -36$$ Point P is below R. Then, $P = -36$	1 mark: Finds $\overrightarrow{PQ}$ and uses $\lvert\overrightarrow{PQ}\rvert = 939$ 1 mark: correctly solves for p and chooses the correct value, citing reason.	
15a (ii)	Let $\overrightarrow{OA} = \begin{pmatrix} 400 \\ 600 \\ -20 \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} 500 \\ 200 \\ -70 \end{pmatrix}$ $$\overrightarrow{OC} = \begin{pmatrix} 600 \\ -340 \\ -50 \end{pmatrix}$$ Find the vectors AB and BC parallel to the plane. $\overrightarrow{AB} = \begin{pmatrix} 100 \\ -400 \\ -50 \end{pmatrix} = -50\begin{pmatrix} -2 \\ 8 \\ 1 \end{pmatrix}$, , $\overrightarrow{BC} = \begin{pmatrix} 100 \\ -540 \\ 20 \end{pmatrix} = 20\begin{pmatrix} 5 \\ -27 \\ 1 \end{pmatrix}$,	1 mark: Finds $\overrightarrow{AB}$ and $\overrightarrow{BC}$, and states that the normal to the plane is the perpendicular to the two vectors in the plane. 1 mark: States $\overrightarrow{AB}.\underset{\sim}{n} = 0$ and $\overrightarrow{BC}.\underset{\sim}{n} = 0$ and attempts to find $\underset{\sim}{n}$ 1 mark: correct calculations and gives	

<table>
<tr><td></td><td>Vectors AB and BC are in the same plane.
Finding the vector perpendicular
to $\overrightarrow{AB}$ and $\overrightarrow{BC}$.
Let $\underset{\sim}{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be the normal to the vectors (normal to the plane).
Hence, $\overrightarrow{AB}.\underset{\sim}{n} = 0$ and $\overrightarrow{BC}.\underset{\sim}{n} = 0$
Hence,
$-2a + 8b + c = 0$
$5a - 27b + c = 0$
$3a = 15b \quad \rightarrow \quad a = 5b$
$c = 2(5b) - 8b = 2b$

Hence, $\underset{\sim}{n} = \begin{pmatrix} 5b \\ b \\ 2b \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$</td><td>the correct normal vector.</td><td></td></tr>
<tr><td>15a (iii)</td><td>Hence, find the coordinates of the point where the pipeline meets the rock.

$\overrightarrow{PQ} = \begin{pmatrix} -936 \\ -72 \\ 21 \end{pmatrix} = -3\begin{pmatrix} 312 \\ 24 \\ -7 \end{pmatrix}$
Equation of line PQ is $\underset{\sim}{r} = \begin{pmatrix} 200 \\ 20 \\ -15 \end{pmatrix} + \lambda \begin{pmatrix} 312 \\ 24 \\ -7 \end{pmatrix}$, where $\lambda \in \mathbb{R}$

For some value of λ, $\overrightarrow{PQ}$ meets the plane.

Then,

$\overrightarrow{PQ}.\ \underset{\sim}{n} = \overrightarrow{(OA)}\,.\underset{\sim}{n}$
$\begin{pmatrix} 200 + 312\lambda \\ 20 + 24\lambda \\ -15 - 7\lambda \end{pmatrix} . \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 400 \\ 600 \\ -20 \end{pmatrix} . \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$
$990 + 1570\lambda = 2560$
$\lambda = 1$

Then, the point of intersection of the pipeline with the rock is
$\begin{pmatrix} 200 + 312\lambda \\ 20 + 24\lambda \\ -15 - 7\lambda \end{pmatrix} = \begin{pmatrix} 512 \\ 44 \\ -22 \end{pmatrix}$ for $\lambda = 1$.

Coordinate $(512, 44, -22)$</td><td>1 mark: Writes the equation of the line PQ in vector form in terms of λ

1 mark: states $\overrightarrow{PQ}.\ \underset{\sim}{n} = \overrightarrow{(OA)}\,.\underset{\sim}{n}$ or otherwise, finds the value of λ

1 mark: finds the coordinates of the point of intersection</td><td></td></tr>
<tr><td>15a (iv)</td><td>Let the angle with the horizontal be θ

Project $\overrightarrow{PQ}$ with XY plane,</td><td></td><td></td></tr>
</table>

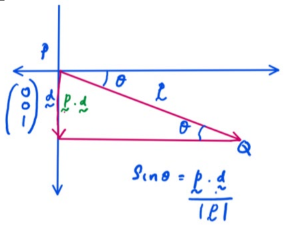
<table>
<tr>
<td></td>
<td>

$$\sin\theta = \frac{\left|\begin{pmatrix}312\\24\\-7\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}\right|}{\left|\begin{pmatrix}312\\24\\-7\end{pmatrix}\right|}$$

$$\sin\theta = \frac{7}{313}$$

$$\theta = 1.28^\circ \approx 1.3^\circ$$

</td>
<td>



2 marks: correct answer from correct working

1 mark: draws a diagram and attempts to find angle with the horizontal

</td>
<td></td>
</tr>
</table>

<table>
<tr><td></td><td></td><td></td><td></td></tr>
<tr>
<td>15(b)
(i)</td>
<td colspan="2">

Let

$$I = \int_0^{2\pi} \sin x \log_e(1+x)\,dx$$

$$= [\log(x+n)\times(-\cos x)]_0^{2\pi} - \int_0^{2\pi}\frac{-\cos x}{x+n}dx$$

$$= \log(2\pi+n)\times -1 - \log n\times(-1) + \int_0^{2\pi}\frac{\cos x}{x+n}dx \qquad \mathbf{1\ mark}$$

$$= -\log\left(n\left(\frac{2\pi}{n}+1\right)\right) + \log n + \int_0^{2\pi}\frac{\cos x}{x+n}dx$$

$$= -\log n - \log\left(1+\frac{2\pi}{n}\right) + \log n + \int_0^{2\pi}\frac{\cos x}{x+n}dx$$

$$= -\log\left(1+\frac{2\pi}{n}\right) + \int_0^{2\pi}\frac{1}{x+n}\cos x\,dx \qquad \mathbf{1\ mark}$$

Integrating by parts again,

$$= -\log\left(1+\frac{2\pi}{n}\right) + \left[\frac{1}{x+n}\times\sin x\right]_0^{2\pi} - \int_0^{2\pi}\frac{\sin x}{(x+n)^2}dx$$

$$= -\log\left(1+\frac{2\pi}{n}\right) + 0 - \int_0^{2\pi}\frac{\sin x}{(x+n)^2}dx$$

$$= -\log\left(1+\frac{2\pi}{n}\right) - \int_0^{2\pi}\frac{\sin x}{(x+n)^2}dx \qquad \mathbf{1\ mark}$$

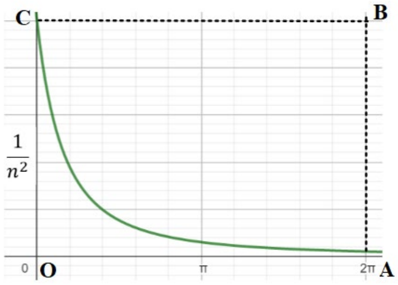
</td>
<td>

1 mark: Applies integration by parts correctly

1 mark: Applies log rules to simplify the expression correctly

1 mark: correctly applies integration by parts to prove the result

</td>
</tr>
<tr>
<td>15(b)
(ii)</td>
<td>We need to prove that</td>
<td></td>
<td></td>
</tr>
</table>

	$\left\lvert \int_0^{2\pi} \frac{\sin x}{(x+n)^2}\,dx \right\rvert < \frac{2\pi}{n^2}$ We have $-1 \le \sin x \le 1 \qquad \forall x \in \mathcal{R}$ Thus, $-\frac{1}{(x+n)^2} \le \frac{\sin x}{(x+n)^2} \le \frac{1}{(x+n)^2} \qquad as$ $(x+n)^2 > 0 \ \ \forall x \in [0, 2\pi] \quad \boldsymbol{1\ mark}$ Hence, $\left\lvert \int_0^{2\pi} \frac{\sin x}{(x+n)^2}\,dx \right\rvert \le \int_0^{2\pi} \frac{1}{(x+n)^2}\,dx$ $g(x) = \frac{1}{(x+n)^2}$ is a decreasing function in $[0, 2\pi]$ Hence, the maximum value of $\frac{1}{(x+n)^2}$ in $[0, 2\pi]$ $occurs\ when\ x = 0$, and equals $\frac{1}{n^2}, n \neq 0$  Hence, $\int_0^{2\pi} \frac{1}{(x+n)^2}\,dx < Area\ OABC = \frac{2\pi}{n^2}$ Hence, $\left\lvert \int_0^{2\pi} \frac{\sin x}{(x+n)^2}\,dx \right\rvert \le \int_0^{2\pi} \frac{1}{(x+n)^2}\,dx < \frac{2\pi}{n^2}$ Hence, $\left\lvert \int_0^{2\pi} \frac{\sin x}{(x+n)^2}\,dx \right\rvert < \frac{2\pi}{n^2}$	1 mark: develops the inequality and hence, the inequality of the area under the curves 1 mark: : reasons to find the maximum value of the function in $[0, 2\pi]$ and attempts to find the maximum area under the curve 1 mark: Explains value of the integral is less than the upper-bound rectangle and derives the result.	
(iii)	Using the results from (i) and (ii), $\int_0^{2\pi} \sin x \log_e(1+x)\,dx = -\log_e\left(1 + \frac{2\pi}{n}\right) + \int_0^{2\pi} \frac{\sin x}{(x+n)^2}\,dx$ $< -\log_e\left(1 + \frac{2\pi}{n}\right) + \frac{2\pi}{n^2}$		2 marks: *Combines the results from 15(b) (i) and (ii)

<table>
<tr>
<td></td>
<td>
Using the result from Q13,
$x \geq \ln(1+x)$ for $\forall x > -1$,

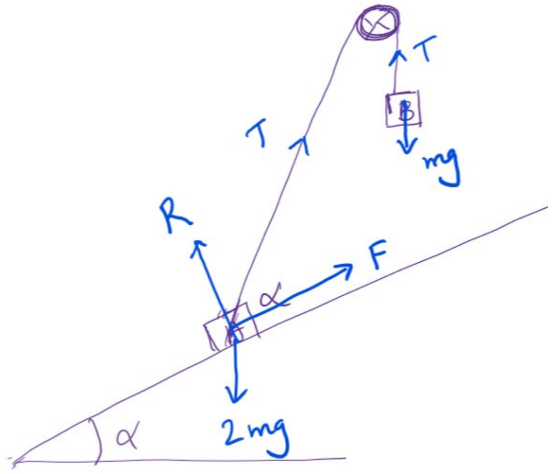
$\log_a\left(1+\frac{2\pi}{n}\right) \leq \frac{2\pi}{n}$
Thus,

$$\int_0^{2\pi} \sin x \log_e(1+x)dx = -\log_e\left(1+\frac{2\pi}{n}\right) + \int_0^{2\pi} \frac{\sin x}{(x+n)^2}dx$$
$$< -\log_e\left(1+\frac{2\pi}{n}\right) + \frac{2\pi}{n^2}$$
$$< -\frac{2\pi}{n} + \frac{2\pi}{n^2}$$
$$< -\frac{2\pi}{n}\left(1-\frac{1}{n}\right)$$
$$As\ n \to \infty, \frac{1}{n} \to 0$$
Thus,
$$\int_0^{2\pi} \sin x \log_e(1+x)\, dx \to -\frac{2\pi}{n}$$
Hence,
$$\int_0^{2\pi} \frac{\sin x \log_e(1+x)dx}{-\frac{2\pi}{n}} \to 1\ \ as\ n \to \infty$$
</td>
<td>
* Uses the result from Q13 to give the largest value of log (1+x)
* Finds the limiting value of the given integral
$$\int_0^{2\pi} \sin x \log_e(1+x)\, dx$$
$$\to -\frac{2\pi}{n}$$
*Hence, derives the result.

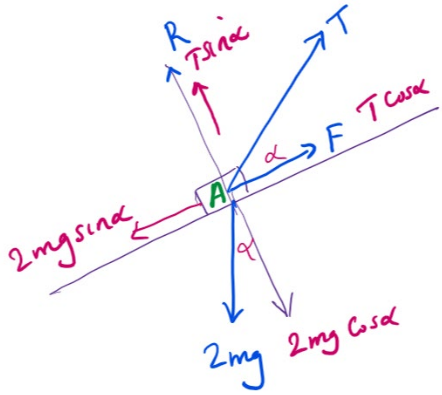
1 mark: Uses the results (i) and (ii), and Q13 and attempts to arrive at the limiting value
</td>
</tr>
</table>

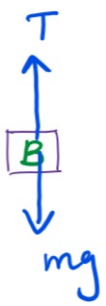
Question 16

<table>
<tr>
<td>16
(a)</td>
<td>



Freebody diagrams are as shown below:





At A, resolving along and perpendicular to the plane,

$$\binom{T\cos\alpha}{T\sin\alpha} + \binom{-2mgsin\alpha}{-2mgcos\alpha} = \binom{F}{0}$$

At B, $T = mg$

$F = T\cos\alpha - 2mg\sin\alpha$
$R = T\sin\alpha - 2mg\cos\alpha$

For the body not to slip down the plane, $F \geq 0$

$F = T\cos\alpha - 2mg\sin\alpha \geq 0$

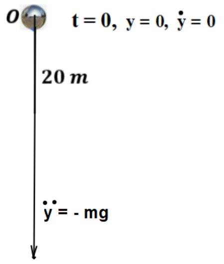
$T = mg$

$2mg\sin\alpha - mg\cos\alpha \leq 0$

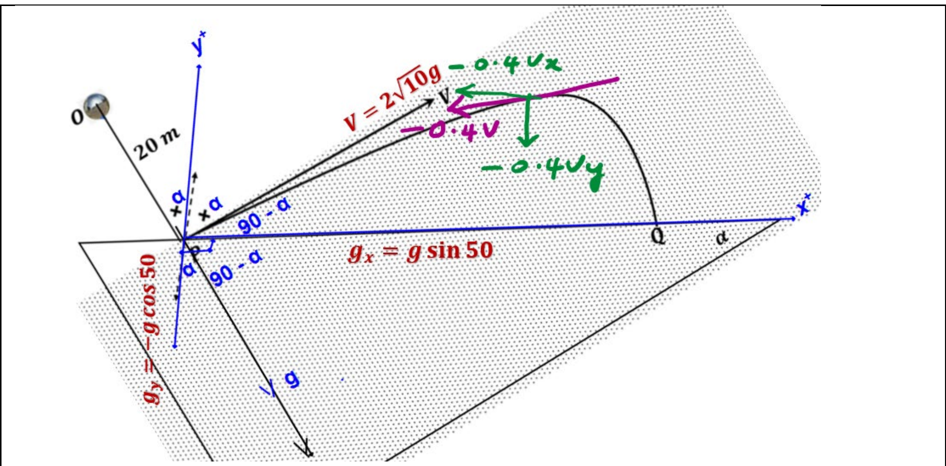
</td>
<td>2 marks: correct free body diagrams for both A and B

1 mark: writes the correct force equations.

1 marks: sets $F \geq 0$ and gives the range of values for α</td>
<td></td>
</tr>
</table>

	$mg > 0, \quad$ hence, $\quad 2\sin\alpha \le \cos\alpha$ $\tan\alpha \le \frac{1}{2} \quad (1)$ $\alpha \le 26.565\ldots \approx 27°$ For the body not to lose contact with the surface, $R = \; T\sin\alpha - 2mg\cos\alpha \ge 0$ $2mg\cos\alpha - mg\sin\alpha \le 0$ $mg\,(2\cos\alpha - \sin\alpha) \le 0$ $mg > 0, \quad \tan\alpha \le 2 \quad (2)$ Using (1) and (2), Hence, $\alpha < 27°$	1 mark: sets $R \ge 0$, solves the trig inequation $\tan\alpha \le 2$ and solves simultaneously with the solution of $\tan\alpha \le \frac{1}{2}$	
16 (b) (i)	 The force equations are: $m\ddot{y} = mg$ $\dot{y} = \int g\,dt = gt + c$ $t = 0, \dot{y} = 0 \Rightarrow \dot{y} = gt$ $y = \int gt\,dt = \frac{1}{2}gt^2 + C$ $t = 0, y = 0, \quad then\ C = 0$ $Thus, \quad y = \frac{1}{2}gt^2$ When, $y = 20,\ t^2 = \frac{2y}{g}$ $t = \sqrt{\frac{40}{g}}, \quad t > 0$ When, $t = \sqrt{\frac{40}{g}}, \quad \dot{y} = g\sqrt{\frac{40}{g}} = \sqrt{40g} = 2\sqrt{10g}$ Thus the speed of impact on the platform is $2\sqrt{10g}$	2 marks: correct answer from correct working 1 mark: sets up the equations of motion, and attempts to find the time of impact.	

16
B(ii)



At $t = 0$,
Resolving the vectors along and perpendicular to the plane,

$$\underset{\sim}{r} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \underset{\sim}{v} = \begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} 2\sqrt{10}g \sin \alpha \\ 2\sqrt{10}g \cos \alpha \end{pmatrix}$$

$$\underset{\sim}{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix} = \begin{pmatrix} g_x \\ g_y \end{pmatrix} + \begin{pmatrix} -0.4V_x \\ -0.4V_y \end{pmatrix}$$

Note $g_y = -9.8 \cos 50$)
When the ball strikes the ground, $a_y = 0$

$$a_y = \frac{dV_y}{dt} = g_y - 0.4V_y$$

$$\int_{u_y}^{V_y} \frac{-0.4dV_y}{g_y - 0.4V_y} = -0.4 \int_0^t dt$$

$$\ln \left| \frac{g_y - 0.4V_y}{g_y - 0.4u_y} \right| = -0.4t$$

$$0.4V_y = g_y - (g_y - 0.4u_y)e^{-0.4t} \qquad \text{1 mark}$$

$$0.4\frac{dy}{dt} = g_y - (g_y - 0.4u_y)e^{-0.4t}$$

$$0.4 \int_0^y dy = \int_0^t g_y - (g_y - 0.4u_y)e^{-0.4t} dt$$

$$0.4\, y = g_y t - (g_y - 0.4u_y)\left(\frac{g_y - o.4u_y}{-0.4}\right)(e^{-0.4t} - 1)$$

$$y = \frac{1}{0.4}(g_y t - (g_y - 0.4u_y)\left(\frac{g_y - o.4u_y}{-0.4}\right)(e^{-0.4t} - 1))$$

Here, $g_y = -9.8 \cos 50$
$u_y = 2\sqrt{10 \times 9.8} \cos 50$ and substitute $t = 3.32\ sec$

Substituting the values,

2 marks: resolves the vectors and writes the $\underset{\sim}{r}$, $\underset{\sim}{v}$ and $\underset{\sim}{a}$ equations.

1 mark: correctly derives the vertical component of the velocity

1 mark: correctly derives the expression for V_y and evaluates (verifies)

	$y = 0.037697 \ldots . \approx \; 0$ Hence, the ball hits the plane after 3.32 seconds.		
16 b (iii)	$a_x = \frac{dV_x}{dt} = g_x - 0.4V_x$ And $a_y = \frac{dV_y}{dt} = g_y - 0.4V_y$ Using the results from 16 b(ii), $0.4V_y = g_y - (g_y - 0.4u_y)e^{-0.4t}$ and Similarly, along the plane, $0.4V_x = g_x - (g_x - 0.4u_x)e^{-0.4t}$ Substitute $g_y = -9.8\cos 50$ $u_y = 2\sqrt{10 \times 9.8}\cos 50$ $g_x = 9.8\sin 50$ $u_x = 2\sqrt{10 \times 9.8}\sin 50$ $At\ t = 3.32, \quad V_x = 17.81528 \ldots$ $V_y = -8.20227 \ldots$ Direction $\tan^{-1}\frac{-8.20227}{17.81375} = -28.418.. = 151.581 \ldots$ Then velocity is $V = \sqrt{V_x^2 + V_y^2} = 19.6114 \approx 19.6\ m/s$ at an angle of approximately 152° to the vertical.	*gives the expression for V_x *calculates V_x at t=3.32 *calculates V_y at t = 3.32 *Calculates the resultant velocity *Calculates the angle of impact 2 marks: correct answer from correct working 1 mark: At least three of the aspects are correctly executed	