Moriah College

2023 **TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION**

Mathematics Extension 2

General Instructions

- Reading time - 10 minutes
- Working time - 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I - 10 marks (pages 2-6)

- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II - 90 marks (pages 7-17)

- Attempt Questions 11-16
- Allow about 2 hours and 45 minutes for this section

Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10

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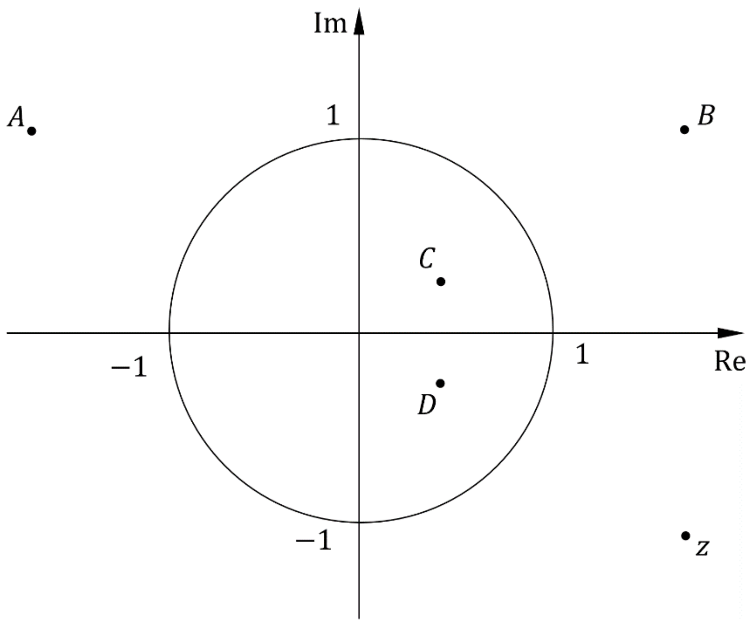
1 The diagram shows the complex number z in the fourth quadrant of the complex plane.

The modulus of z is 2.

Which of the points marked A, B, C or D best shows the position of $\frac{1}{z}$?



A. Point A

B. Point B

C. Point C

D. Point D

2 Which expression is equal to

$$\int \tan^3 x \sec^2 x \, dx?$$

A. $\dfrac{\tan^4 x}{4} + c$

B. $\tan^4 x + c$

C. $\dfrac{\tan^5 x}{5} + \dfrac{\tan^3 x}{3} + c$

D. $\tan^5 x + \tan^3 x + c$

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3 The converse of the statement "If $p = 0$, then $pq = 0$" is?

A. If $p \neq 0$, then $pq = 0$.

B. If $p \neq 0$, then $pq \neq 0$.

C. If $pq = 0$, then $p = 0$.

D. If $pq = 0$, then $p = 0$ or $q = 0$.

4 Which of the following statements is correct?

A. $\forall a, b \in \mathbb{R}$ $\quad \sin a < \sin b \Rightarrow a < b$

B. $\forall a, b \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ $\quad \sin a < \sin b \Rightarrow a < b$

C. $\forall a, b \in \mathbb{R}$ $\quad \cos a < \cos b \Rightarrow a < b$

D. $\forall a, b \in [0, \pi]$ $\quad \cos a < \cos b \Rightarrow a < b$

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5 Each pair of lines given below intersects at $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

Which pair of lines are perpendicular?

A. $\ell_1\colon\ \underset{\sim}{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$ and $\ell_2\colon\ \underset{\sim}{r} = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$

B. $\ell_1\colon\ \underset{\sim}{r} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\ell_2\colon\ \underset{\sim}{r} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}$

C. $\ell_1\colon\ \underset{\sim}{r} = \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}$ and $\ell_2\colon\ \underset{\sim}{r} = \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

D. $\ell_1\colon\ \underset{\sim}{r} = \begin{pmatrix} 3 \\ -2 \\ 8 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 4 \\ -5 \end{pmatrix}$ and $\ell_2\colon\ \underset{\sim}{r} = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$

6

Which of the following statements about inequality proofs is true?

A. If $a > b$ and $c > d$, then $a + c > b + d$.

B. If $a > b$ and $c > d$, then $a - c > b - d$.

C. If $a > b$ and $c > d$, then $ac > bd$.

D. If $a > b$ and $c > d$, then $\frac{a}{c} > \frac{b}{d}$.

7 Which expression is equal to

$$\int\left(x^2\frac{d}{dx}(x^3-1)+(x^3-1)\frac{d}{dx}(x^2)\right)dx\ ?$$

A. $\dfrac{x(x^2-1)}{6}+c$

B. $\dfrac{x^3}{3}\times\dfrac{x^4-x}{4}+c$

C. $x^2(x^3-1)+c$

D. $2x(3x^2-1)+c$

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8 The non-zero complex numbers ω and z are linked by the formula $\omega = z + k\bar{z}$, where k is real and $k \neq 0$.

Which of the following statements is INCORRECT?

A. ω can only be purely imaginary if $k = -1$

B. ω can only be purely real if $k = 1$

C. ω and z must have different moduli

D. ω and z must have different arguments

---

9 Consider the complex number $z = cos\theta + isin\theta$.
Which of the following is the modulus of $z + 1$?

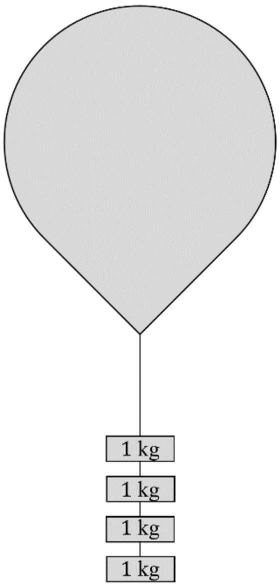
A. $2cos\frac{\theta}{2}$

B. $2cos\frac{\theta}{2}+1$

C. $2cos\theta$

D. $2cos\theta+1$

10 When a large helium balloon is attached to four bricks, each of mass one kilogram, the balloon is neutrally buoyant.
This means that when the balloon and bricks are lifted from the ground and released they are stationary, neither rising nor falling back to the ground.



The string attaching the bricks is cut at some point along its length, and at least one of the bricks falls away, causing the balloon to rise vertically.

The balloon is subject to air resistance of magnitude $0.5v^2$, where v is measured in metres per second, and gravity of $10\ \text{ms}^{-2}$. The balloon and string have negligible mass.

Given that the terminal velocity of the balloon is $2\sqrt{5}$ m/s, how many bricks fell from the balloon when the string was cut?

A. 1

B. 2

C. 3

D. 4

Section II

90 marks
Attempt Questions 11-16
Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (16 marks) Use the Question 11 Writing Booklet

(a) The complex numbers $z = 4e^{\frac{\pi}{3}i}$ and $w = 2e^{\frac{\pi}{6}i}$ are given.

(i) Find $|wz|$. **1**

(ii) Find the value of $\frac{z}{w}$, giving the answer in the form $re^{i\theta}$. **2**

(iii) Hence, or otherwise, find the value of w^2. **1**

(b) (i) Find the values of $a,\ b$ and c given **2**

$$\frac{1}{x(x^2+1)} = \frac{a}{x} + \frac{bx+c}{x^2+1}.$$

(ii) Hence, or otherwise, evaluate **3**

$$\int_1^2 \frac{dx}{x(x^2+1)}.$$

Question 11 continues on page 8

Question 11 (continued)

(c) It is given that the point R is $(2,1,-1)$, $\overrightarrow{RS}=\begin{pmatrix}-4\\-1\\2\end{pmatrix}$ and $\overrightarrow{RT}=3\overrightarrow{RS}$. **2**

Find the coordinates of T.

(d) (i) Find the two square roots of $2i$, giving the answers in the form $x+iy$, where x and y are real numbers. **2**

(ii) Hence, or otherwise, solve $2z^2+2\sqrt{2}z+1-i=0$, leaving answers in the form $x+iy$. **3**

End of Question 11

Question 12 (15 marks) Use the Question 12 Writing Booklet

(a) A particle is in simple harmonic motion about $x = 2$ with amplitude $a = 3$. The particle completes one full cycle every two seconds. Find the maximum acceleration of the particle. **2**

(b) Let z be a complex number.

(i) On an Argand diagram, sketch the curve given by the equation $|z - 2i| = 2$ **1**

(ii) Find the complex number z that solves the equation $|z - 2i| = 2$ and the equation $\arg(z) = \frac{\pi}{4}$ **2**

(c) Use integration by parts to evaluate **3**

$$\int_1^2 xe^{2x}\,dx.$$

(d) Let p and q both be positive integers.
Prove that there does not exist values of p and q such that $4p^2 - q^2 = 25$ **3**

(e) Consider the inequality $\frac{x+y}{2} \geq \sqrt{xy}$ where x and y are non-negative

(i) By writing down and ~~using a contradiction~~, prove the inequality **2**

typo: prove by contrapositive

(ii) A rectangle has dimensions x and y.

Given that the rectangle has perimeter P, and area A, use the inequality to show that $P^2 \geq 16A$ **2**

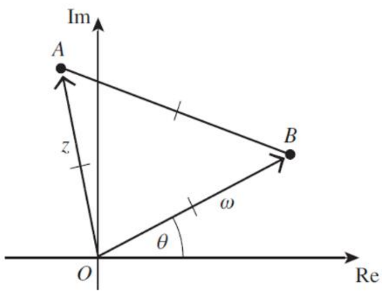
End of Question 12

Question 13 (14 marks) Use the Question 13 Writing Booklet

(a) A particle is initially 2 metres to the **left** of the origin, moving to the **left** at a speed of 3 metres per second. The acceleration of the particle is given by $a = v + v^3$, where v is the velocity of the particle. **3**

Find an expression for x, the displacement of the particle, in terms of v.

(b) The triangle OAB shown below is equilateral.
The points A and B represent the complex numbers z and w respectively.

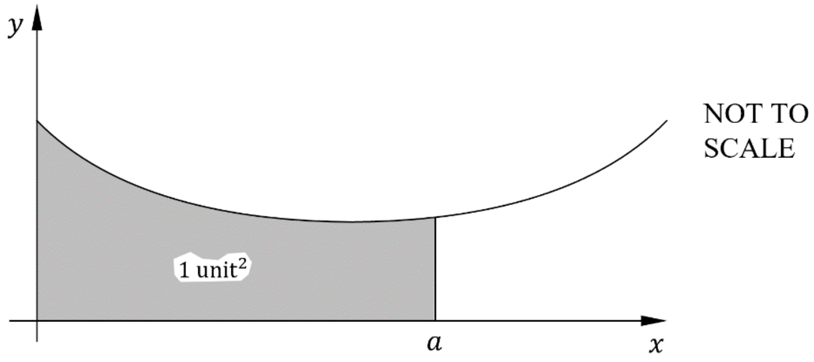


(i) Find z in terms of w **1**

(ii) Hence, find $z^3 + w^3$ **2**

(c) Prove that $\log_2 n$ is irrational when n is an odd integer greater than or equal to 3. **3**

(d) The area shown below is bounded by the curve $y = \frac{1}{1+\sin x}$, the x-axis, the y-axis and the line $x = a$. **5**



By using the substitution $t = tan\frac{x}{2}$, find the value of a if the area is 1 unit^2.

End of Question 13

Question 14 (14 marks) Use the Question 13 Writing Booklet

(a) (i) Find the vector $\overrightarrow{AB}$ joining the points $A\ (1\,,0\,,1\,)$ and $B\ (3\,,1\,,2)$ **1**

(ii) Find the projection of $\overrightarrow{AB}$ onto the vector $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ **2**

(iii) Hence, find the shortest distance from the point $(3\,,1\,,2)$ to the line given by the vector equation **3**

$$\underset{\sim}{r} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

(b) (i) Show that for any integer n, $e^{in\theta} - e^{-in\theta} = 2i\sin(n\theta)$. **2**

(ii) By expanding $\left(e^{i\theta} - e^{-i\theta}\right)^3$, show that **3**

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$

(iii) Consider the polynomial equation $4x^3 - 3x = \frac{1}{\sqrt{2}}$ **3**
Using the substitution $x = \sin\theta$, find the 3 **distinct** real roots of the equation, leaving your answers in trigonometric form.

Question 15 (14 marks) Use the Question 15 Writing Booklet

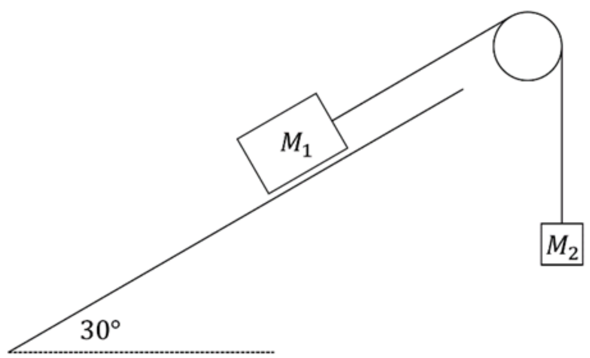
(a) Using the substitution $x = 2 + 2\cos^2\theta$, evaluate the following: **4**

$$\int_2^3 \sqrt{\frac{x-2}{4-x}}\,dx$$

(b) Two masses, M_1 kilograms and M_2 kilograms, are attached by a light inextensible string.

The string is placed over a smooth pulley, and the mass M_1 rests on a smooth frictionless plane inclined at 30 degrees to the horizontal, as shown.

The mass M_2 is suspended vertically by the string.



The mass M_1 accelerates down the inclined plane at 2 m/s^2, and $g = 10$ m/s^2.

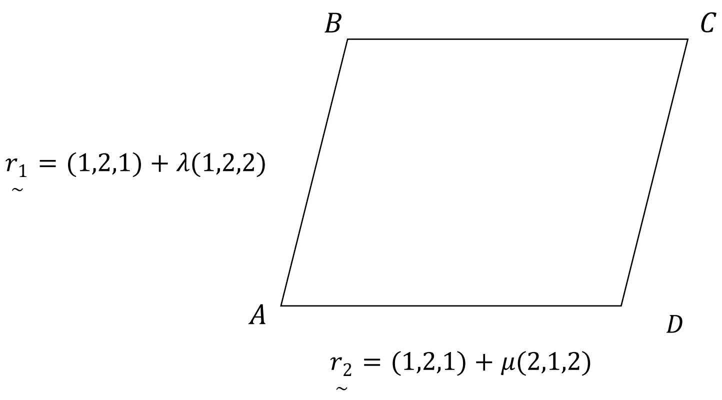
Let the tension in the string be T.

(i) By resolving the forces on mass M_1 down the inclined plane, show that be $T = 3M_1$ **2**

(ii) Find the value of $\frac{M_1}{M_2}$ **2**

(c) The rhombus $ABCD$ is shown below.

The two sides AB and AD are formed by the lines with vector equations $\underset{\sim}{r_1}(1,2,1) + \lambda(1,2,2)$ and $\underset{\sim}{r_2} = (1,2,1) + \mu(2,1,2)$ respectively as shown.



(i) Using the direction vectors, show that $cos\angle BAD = \frac{8}{9}$ **2**

(ii) Explain why $\mu = \lambda$ **2**

(iii) If $\mu = 3$, then find the area of the rhombus, giving your answer in simplest surd form. **2**

Question 16 (17 marks) Use the Question 16 Writing Booklet

(a) Let $f(x) = x^2 + bx + c$, **3**

The solutions to $f(x) = 0$ are $x = 1$ and $x = 3$.

The solutions to $f(x + m) + n = 0$ are $x = \pm 2i$.

Find the values of m and n.

(b) A sequence is defined by the recursive formula

$$T_n = \frac{2n + 1}{2n - 3} \times T_{n-1}$$

where $T_1 = 3$ and $n > 1$.

(i) Use mathematical induction to prove that $T_n = 4n^2 - 1$ **3**

(ii) You are given the formula

$$\sum_{r=1}^{n} r^2 = \frac{n(n + 1)(2n + 1)}{6}$$

You don't need to prove this formula.

Using it, or otherwise, find a factorised expression for the sum of the first n terms of T_n. **2**

(c) A unit mass, P_1, is projected vertically upwards in a medium at 10 m/s. The mass is subject to the force due to gravity (use $g = 10$ m/s^2) and a resistance force of $0.1v^2$ where v is its velocity.

(i) Show that the equation of motion of P_1 is given by **1**

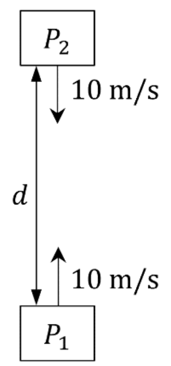
$$\ddot{x} = -\frac{100 + v^2}{10}$$

(ii) Show that the time T taken for P_1 to reach its maximum height is **2**

$$T = \frac{\pi}{4}$$

(iii) A second identical unit mass, P_2, is projected vertically downwards at 10 m/s in the same medium as P_1. **2**
Show that this initial speed of projection of P_2 is in fact its terminal velocity.

(iv) **4**
The two masses are projected at the same time.
Initially the two masses are d metres apart, as shown below.
If the two masses collide at the instant P_1 reaches its maximum height, show that $d = 5\left(\ln 2 + \frac{\pi}{2}\right)$ metres.



End of paper

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MATHEMATICS EXT. 2 TRIALS 2023 SOLUTIONS.

MCQ: C A C B A A C D A A

11. (a) (i) 8 ✓ 1

(ii) $2e^{\frac{\pi}{6}i}$ ✓✓ 2

(iii) $4e^{\frac{\pi}{3}i}$ ✓ 1

b) (i) $\frac{a}{x} + \frac{bx+c}{x^2+1} = \frac{a(x^2+1)+x(bx+c)}{x(x^2+1)} = \frac{(a+b)x^2+cx+a}{x(x^2+1)}$ ✓

$\therefore a = 1 \quad b = -1 \quad c = 0$ ✓ 2

(ii) $\int_1^2 \frac{dx}{x(x^2+1)} = \int_1^2 \left(\frac{1}{x} - \frac{x}{x^2+1}\right) dx$

$= \left[\ln x - \frac{1}{2}\ln|x^2+1|\right]_1^2$ ✓ $= (\ln 2 - \frac{1}{2}\ln 5) - (0 - \frac{1}{2}\ln 2)$

$= \frac{3}{2}\ln 2 - \frac{1}{2}\ln 5$ or $\ln \frac{2\sqrt{2}}{\sqrt{5}}$ ✓ 2

c) $\overrightarrow{RT} = \begin{pmatrix} -12 \\ -3 \\ 2 \end{pmatrix}$ ✓ $\therefore T = \begin{pmatrix} 2-12 \\ 1-3 \\ -1+6 \end{pmatrix} = \begin{pmatrix} -10 \\ -2 \\ 5 \end{pmatrix}$ ✓ 2

d) (i) Since $2i = 2e^{\frac{\pi}{2}i}$, square roots are $\pm\sqrt{2}e^{\frac{\pi}{4}i}$ ✓

$= \pm(1+i)$ ✓ by inspection. 2

[MANY OTHER METHODS]

(ii) By quadratic formula, $z = \frac{-2\sqrt{2} \pm \sqrt{8 - 8(1-i)}}{4}$ ✓

$= \frac{-2\sqrt{2} \pm 2\sqrt{2i}}{4} = \frac{-\sqrt{2} \pm \sqrt{2i}}{2}$ ✓

$= \frac{-\sqrt{2} \pm (1+i)}{2}$ ✓

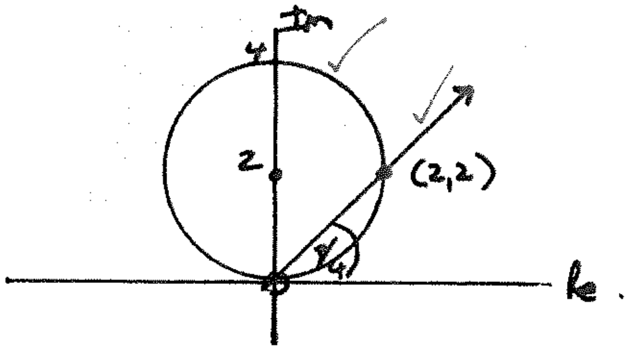
$= \frac{-\sqrt{2}+1+i}{2}$ or $\frac{-\sqrt{2}-1-i}{2}$ 3

12 (a) $x = 3\sin \pi t + 2$

$\dot{x} = 3\pi \cos \pi t$

$\ddot{x} = -3\pi^2 \sin \pi t$ ✓ $\therefore$ max. acceleration is $3\pi^2$ ✓ 2

(b) (i)



(ii) $z = 2 + 2i$ ✓ 3

(c) $\int_1^2 x e^{2x} \, dx$

$u = x \quad u' = 1$ ✓

$v = \frac{1}{2}e^{2x} \quad v' = e^{2x}$

$= \left[\frac{1}{2} x e^{2x}\right]_1^2 - \frac{1}{2}\int_1^2 e^{2x} \, dx$

$= e^4 - \frac{e^2}{2} - \frac{1}{2}\left(\frac{1}{2}\right)\left[e^{2x}\right]_1^2$ ✓

$= e^4 - \frac{e^2}{2} - \frac{e^4}{4} + \frac{e^2}{4} = \frac{3e^4}{4} - \frac{e^2}{4}$ ✓ 3

(d) $4p^2 - q^2 = (2p+q)(2p-q)$ ✓ But $25 = 1 \times 25$ or 5×5 only. ✓

$2p+q$ and $2p-q$ cannot both equal 5 if $q > 0$ ✓

If $2p+q = 25$ and $2p-q = 1$ then $p = 13\frac{1}{2}$ which is not an integer. 3

e) (i) ERROR IN QUESTION. ✓✓ 2

(ii) $A = xy$, $P = 2(x+y)$

$\frac{P}{4} = \frac{x+y}{2} \geq \sqrt{xy} = \sqrt{A}$ ✓

$\therefore P^2 \geq 16A$ ✓ 2

13 (a) $a = v\frac{dv}{dx} = v + v^3$ ✓

$\therefore \frac{dv}{dx} = 1 + v^2$

$\therefore \frac{dx}{dv} = \frac{1}{1+v^2}$ ✓

$\therefore x = \tan^{-1} v + c$

But when $x = -2$, $v = -3$

$\therefore x = \tan^{-1} v - 2 - \tan^{-1}(-3)$ ✓ (or $\tan^{-1} v + 2 + \tan^{-1} 3$) 3

b) (i) $z = w \operatorname{cis}\frac{\pi}{3}$ ✓ (or equivalent)

(ii) $z^3 + w^3 = w^3 \operatorname{cis}\pi + w^3$ ✓

$= w^3(\operatorname{cis}\pi + 1)$

$= w^3(-1+1) = 0$ ✓ 3

c) Suppose $\log_2 n = \frac{a}{b}$, ✓ $a, b \in \mathbb{Z}$.

$\therefore 2^{a/b} = n$

$\therefore 2^a = n^b$ ✓ but 2^a is even, n^b is odd ✓ $\therefore$ contradiction. 3

d) $\int_0^a \frac{dx}{1+\sin x} = \int_0^{\tan a/2} \frac{2dt}{\left(1 + \frac{2t}{1+t^2}\right)(1+t^2)}$ ✓ $t = \tan\frac{x}{2}$, $dx = \frac{2dt}{1+t^2}$

$= 2\int_0^{\tan a/2} \frac{dt}{1+t^2+2t} = 2\int_0^{\tan\frac{a}{2}} \frac{dt}{(1+t)^2}$ ✓

$= -2\left[\frac{1}{1+t}\right]_0^{\tan a/2}$ ✓ $= -2\left(\frac{1}{1+\tan a/2} - 1\right) = 1$ ✓

$\therefore \frac{1}{1+\tan\frac{a}{2}} = \frac{1}{2}$ $\therefore 1 + \tan\frac{a}{2} = 2$

$\therefore \tan\frac{a}{2} = 1$

$\therefore \frac{a}{2} = \frac{\pi}{4}$ $\therefore a = \frac{\pi}{2}$ ✓ 5

QUESTION 14

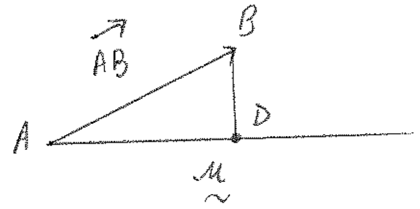
(a) (i) $\overrightarrow{AB} = \begin{pmatrix}3\\1\\2\end{pmatrix} - \begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}2\\1\\1\end{pmatrix}$

(ii) $\text{proj} = \dfrac{\begin{pmatrix}2\\1\\1\end{pmatrix}\begin{pmatrix}2\\-1\\1\end{pmatrix}}{\begin{pmatrix}2\\-1\\1\end{pmatrix}\begin{pmatrix}2\\-1\\1\end{pmatrix}} \begin{pmatrix}2\\-1\\1\end{pmatrix}$ let $\underset{\sim}{u} = \begin{pmatrix}2\\-1\\1\end{pmatrix}$

$= \dfrac{4}{6}\begin{pmatrix}2\\-1\\1\end{pmatrix}$

$= \dfrac{2}{3}\begin{pmatrix}2\\-1\\1\end{pmatrix}$

(iii)



$|\overrightarrow{DB}| = |\overrightarrow{AB} - \text{proj}_{\underset{\sim}{u}} \overrightarrow{AB}|$

$= \left|\begin{pmatrix}2\\1\\1\end{pmatrix} - \dfrac{2}{3}\begin{pmatrix}2\\-1\\1\end{pmatrix}\right|$

$= \left|\begin{matrix}2/3\\5/3\\-1/3\end{matrix}\right|$

$= \dfrac{1}{3}\left|\begin{matrix}2\\5\\-1\end{matrix}\right|$

Shortest distance $= \dfrac{1}{3}\sqrt{30}$

(b) (i) $LHS = \cos(n\theta) + i\sin(n\theta) - (\cos(-n\theta) + i\sin(-n\theta))$

$= \cos(n\theta) + i\sin(n\theta) - \cos(n\theta) + i\sin(n\theta)$

$= 2i\sin(n\theta)$

(ii) from (i) $(e^{i\theta} - e^{-i\theta})^3 = (2i\sin\theta)^3$

$= -8i\sin^3\theta \quad \ldots \text{①}$

using binomial expansion $(e^{i\theta} - e^{-i\theta})^3 = e^{3i\theta} - e^{-3i\theta} - 3e^{i\theta} + 3e^{-i\theta}$

$= 2i\sin 3\theta - 3(2i\sin\theta) \ldots \text{②}$

equating ① and ② $-8i\sin^3\theta = 2i\sin 3\theta - 6i\sin\theta$

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

(iii) $4\sin^3\theta - 3\sin\theta = \frac{1}{\sqrt{2}}$

$-\sin 3\theta = \frac{1}{\sqrt{2}}$

$\sin 3\theta = -\frac{1}{\sqrt{2}}$

$3\theta = \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{13\pi}{4}, \frac{15\pi}{4}, \frac{21\pi}{4}, \frac{23\pi}{4}$

$\theta = \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{15\pi}{12}, \frac{21\pi}{12}, \frac{23\pi}{12}$

$\therefore$ Distinct roots $x = \sin\frac{5\pi}{12}, \sin\frac{13\pi}{12}, \sin\frac{5\pi}{4}$

QUESTION 15

(a) $x = 2 + 2\cos^2\theta$ $x = 2 \to \theta = \frac{\pi}{2}$

$dx = -4\cos\theta\sin\theta$ $x = 3 \quad \theta = \frac{\pi}{4}$

$$I = -4\int_{\frac{\pi}{2}}^{\pi/4} \cos\theta\sin\theta\sqrt{\frac{2\cos^2\theta}{2-2\cos^2\theta}}\, d\theta$$

$$= 4\int_{\pi/4}^{\pi/2} \cos^2\theta \, d\theta$$

(using $\cos 2\theta = 2\cos^2\theta - 1$)

$$= 2\int_{\pi/4}^{\pi/2} 1 + \cos 2\theta \, d\theta$$

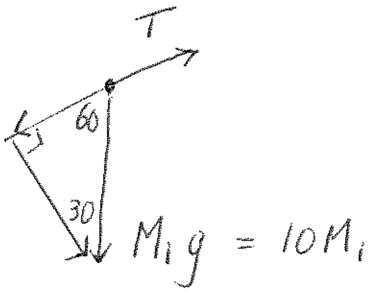
$$= 2\left[\theta + \frac{1}{2}\sin 2\theta\right]_{\pi/4}^{\pi/2}$$

$$= 2\left[\frac{\pi}{2} - \frac{\pi}{4} - \frac{1}{2}\right]$$

$$= \frac{\pi}{2} - 1$$

(b)

(i) $2M_1 = -T + 10M_1 \sin 30$



$2M_1 = -T + 5M_1$

$T = 3M_1$

(ii) <u>Forces on M_2</u>

$2M_2 = -10M_2 + T$

$T = 12M_2$

Now $3M_1 = 12M_2$

$\frac{M_1}{M_2} = 4$

(c) (i) $\cos \angle BAD = \frac{\begin{pmatrix}1\\2\\2\end{pmatrix}\begin{pmatrix}2\\1\\2\end{pmatrix}}{\left|\begin{matrix}1\\2\\2\end{matrix}\right|\left|\begin{matrix}2\\1\\2\end{matrix}\right|}$

$= \frac{8}{\sqrt{9} \cdot \sqrt{9}}$

$= \frac{8}{9}$

ABCD is a rhombus

A is the point $(1, 2, 1)$

$|\vec{AD}| = |\vec{AB}| \rightarrow \left|\lambda\begin{pmatrix}1\\2\\2\end{pmatrix}\right| = \left|\mu\begin{pmatrix}2\\1\\2\end{pmatrix}\right|$

$\lambda = \mu$ as $\left|\begin{matrix}1\\2\\2\end{matrix}\right| = \left|\begin{matrix}2\\1\\2\end{matrix}\right|$

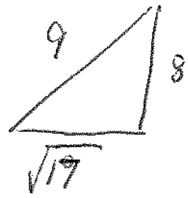
If $\mu = 3$ then $|\vec{AD}| = \left|3\begin{pmatrix}2\\1\\2\end{pmatrix}\right|$

$= 3 \times 3$

$= 9$

Now Area $= 2 \times \frac{1}{2} \times 9 \times 9 \times \sin \angle BAD$

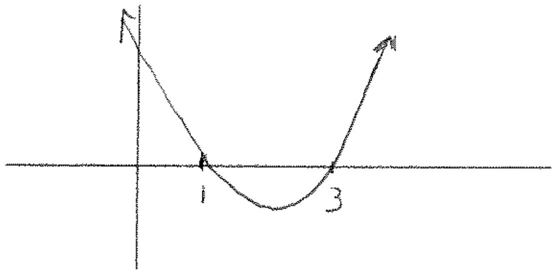
$= 81 \times \frac{\sqrt{17}}{9}$



$= 9\sqrt{17}\ u^2$

QUESTION 16

(a)



$f(x) = (x-2)^2 - 1$

$f(x+m) + n = (x+m-2)^2 + n - 1$

we know $f(x+m)+n = 0$ has roots $x = \pm 2i$

$\therefore \quad (x+m-2)^2 + n - 1 = x^2 + 4$

$\therefore \quad m - 2 = 0$

$m = 2$

$n = 5$

Let $n = 1$

$T_1 = \frac{2+1}{2-3}$ $4(1)^2 - 1 = 3$ which equals T_1 given

Assume $T_k = 4k^2 - 1$

RTP $T_{k+1} = 4(k+1)^2 - 1$

Now $T_{k+1} = \frac{2(k+1)+1}{2(k+1)-3} \cdot T_k$ (by definition)

$= \frac{2k+3}{2k-1} \cdot (4k^2 - 1)$ (by assumption)

$$T_{k+1} = \frac{(2k+3)}{(2k-1)} \cdot (2k+1)(2k-1)$$

$$= 4k^2 + 8k + 3$$

$$= 4k^2 + 8k + 4 - 1$$

$$= 4(k+1)^2 - 1 \quad \text{(as required)}$$

$\therefore$ By MI result is proven

(ii) $S_n = T_1 + T_2 + \dots + T_n$

$$= 3 + 15 + \dots + (4n^2 - 1)$$

$$= 4(1^2) - 1 + 4(2^2) - 1 + 4(3^2) - 1 + \dots + 4(n^2) - 1$$

$$= 4(1^2 + 2^2 + 3^2 + \dots + n^2) - n$$

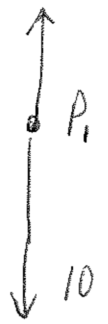
$$= \frac{4n(n+1)(2n+1)}{6} - n$$

$$= \frac{2n(n+1)(2n+1) - 3n}{3}$$

$$= \frac{n[2(n+1)(2n+1) - 3]}{3}$$

$$= \frac{n(4n^2 + 6n - 1)}{3}$$

(c) (i)



$$m\ddot{x} = -mg - 0.1v^2$$

$$m = 1 \quad g = 10$$

$$\ddot{x} = -10 - \frac{v^2}{10}$$

$$\ddot{x} = -\frac{100 + v^2}{10}$$

(ii)

$$\frac{dv}{dt} = -\frac{100 + v^2}{10}$$

$$\int_{10}^{0} \frac{dv}{100+v^2} = -\int_{0}^{T} \frac{dt}{10}$$

$$\frac{1}{10}\left[\tan^{-1}\frac{v}{10}\right]_{10}^{0} = -\frac{1}{10}\left[t\right]_{0}^{T}$$

$$-\tan^{-1} 1 = -T$$

$$T = \frac{\pi}{4}$$

(iii) equation for P_2

$$m\ddot{x} = mg - 0.1v^2 \text{ (taking down as positive)}$$

$$\ddot{x} = 10 - 0.1v^2$$

terminal velocity $\rightarrow$ $\ddot{x} = 0$ $\rightarrow$ $v^2 = 100$

$$v = 10$$

$\therefore$ Initial speed is terminal velocity.

(iv) Particle P_1 $\rightarrow$ we seek $\frac{dx}{dt}$ or $\frac{v\,dv}{dx}$

$$v\frac{dv}{dx} = -\frac{(100+v^2)}{10}$$

$$\int_{10}^{0} \frac{-10v\,dv}{100+v^2} = \int_0^H dx \qquad \text{where } H \text{ is maximum height}$$

$$-5\Big[\ln(100+v^2)\Big]_{10}^{0} = H$$

$$H = 5\ln\left(\frac{200}{100}\right)$$

$$H = 5\ln 2$$

For particle P_2, its velocity is constant

$\therefore$ In $\frac{\pi}{4}$ seconds it travels $\frac{10\pi}{4}$ m.

Particles meet when P_1 travels $5\ln 2$
and P_2 travels $\frac{5\pi}{2}$

$$\therefore \quad d = \left(5\ln 2 + \frac{5\pi}{2}\right) \text{ metres}$$

$$d = 5\left(\ln 2 + \frac{\pi}{2}\right)$$