Student Name/Number: ______________________

KINROSS WOLAROI SCHOOL

2023

Year 12 Mathematics Extension 2

Trial HSC Examination

Teacher Setting Paper: Mr Bowman
Head of Department: Mr Doyle

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- NESA approved calculator may be used
- Write your answers for Section I on the multiple-choice answer sheet provided
- A reference sheet is provided at the back of this paper
- In Questions 11-16, show relevant mathematical reasoning and/or calculations.

Total marks – 100

Section I – Multiple-Choice

10 marks

Attempt Questions 1-10

Allow 15 minutes for this section

Section II – Free Response

90 marks

Attempt questions 11 - 16

Allow 2 hours and 45 minutes for this section

This examination paper does not necessarily reflect the content or format of the Higher School Certificate Examination in this subject.

Student Name/Number:______________________________

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1 - 10.

QUESTION 1

Which of the following is an expression for $\int \frac{x\,dx}{\sqrt[3]{x^2+1}}$?

(A) $\frac{3}{2}\sqrt[3]{(x^2+1)^2}+C$

(B) $\frac{1}{2}\sqrt[3]{(x^2+1)^2}+C$

(C) $\frac{3}{4}\sqrt[3]{(x^2+1)^2}+C$

(D) $\frac{3}{2}\sqrt[3]{(x^2+1)^2}+C$

QUESTION 2

In an Argand diagram the points $A(-3,2)$ and $B(5,-4)$ lie at opposite ends of a diameter of a circle. What is the equation of the circle?

(A) $|z-1+i|=10$

(B) $|z-1+i|=5$

(C) $|z+1-i|=5$

(D) $|z+1-i|=10$

QUESTION 3

What is the size of the acute angle θ between the vectors $\tilde{\boldsymbol{a}}=2\boldsymbol{i}-\boldsymbol{j}-\boldsymbol{k}$ and $\tilde{\boldsymbol{b}}=2\boldsymbol{i}-2\boldsymbol{k}$?

(A) $\theta=\frac{\pi}{6}$

(B) $\theta=\frac{\pi}{5}$

(C) $\theta=\frac{\pi}{4}$

(D) $\theta=\frac{\pi}{3}$

QUESTION 4

What is the magnitude of the vector $\boldsymbol{i} - 3\boldsymbol{j} + 5\boldsymbol{k}$?

(A) $\sqrt{17}$

(B) $\sqrt{35}$

(C) 17

(D) 35

QUESTION 5

Which of the following is an expression for

$$\int \frac{1}{x^2 - \sqrt{3}\,x + 1}\,dx \quad ?$$

(A) $\tan^{-1}\left(x - \sqrt{3}\right) + C$

(B) $2\tan^{-1}\left(x - \sqrt{3}\right) + C$

(C) $\tan^{-1}\left(2x - \sqrt{3}\right) + C$

(D) $2\tan^{-1}\left(2x - \sqrt{3}\right) + C$

QUESTION 6

A particle moves in simple harmonic motion such that $v^2 + 9x^2 = k$.
What is the period of the particle's motion?

(A) $\frac{2\pi}{k}$

(B) 3π

(C) $\frac{3k}{2\pi}$

(D) $\frac{2\pi}{3}$

QUESTION 7

Consider the statement $P(x)\ is\ odd\ \Rightarrow P'(x)\ is\ even$ where $P(x)$ is a non-zero polynomial.
Which of the following is correct?

(A) The contrapositive statement is false and the converse statement is false.

(B) The contrapositive statement is true and the converse statement is false.

(C) The contrapositive statement is false and the converse statement is true.

(D) The contrapositive statement is true and the converse statement is true.

QUESTION 8

What are the values of real numbers p and q such that $2-i$ is a root of the equation $z^3+pz+q=0$?

(A) $p=-11$ and $q=-20$

(B) $p=-11$ and $q=20$

(C) $p=11$ and $q=-20$

(D) $p=11$ and $q=20$

QUESTION 9

A particle has initial velocity $2\boldsymbol{i}\ ms^{-1}$. At time t seconds it has acceleration

$$\ddot{x}=(2t-3)\boldsymbol{i}+\frac{\pi}{2}\cos\left(\frac{\pi}{2}t\right)\boldsymbol{j}\ \ ms^{-2}.$$

When is the particle at rest?

(A) Never

(B) At time $t=1$ second only

(C) At time $t=2$ seconds only

(D) At times $t=1$ and $t=2$ seconds.

QUESTION 10

What is the value of

$$\int_{-k}^{k} \{f(x) - f(-x)\}\ dx \ \ ?$$

(A) 0

(B) $\int_0^k f(x)\ dx$

(C) $2\int_0^k f(x)\ dx$

(D) $4\int_0^k f(x)\ dx$

END OF SECTION I

Student Name/Number:______________________________

Section II

90 marks
Attempt Questions 11-16
Allow about 2 hours 45 minutes for this section

Answer each question in the booklets provided. Extra writing booklets are available.

Marks

QUESTION 11 (15 marks) **Start a new writing booklet**

(a) Express $\frac{1+2i}{2+i}$ in the form $a+ib$ where a and b are real. **2**

(b) Use the substitution $t=\tan\frac{x}{2}$ to find **2**

$$\int \frac{1}{1+sinx}\,dx$$

(c) The complex numbers $z_1=-1+i$ and z_2 are such that $|z_1z_2|=\sqrt{6}$ and $\arg(z_1z_2)=\frac{7\pi}{12}$.

(i) Express z_1 in the form $re^{i\theta}$ where $r>0$ and $-\pi<\theta\leq\pi$. **1**

(ii) Find z_2 in the form $a+ib$ where a,b are real. **2**

(d) A particle is moving along the x-axis. At time t seconds it has displacement x metres from the origin O, velocity v ms^{-1} and acceleration a ms^{-2} where $a=12-4x$. Initially the particle is at rest 5 metres to the right of O.

(i) Use integration to show that $v^2=-4x^2+24x-20$ **2**

(ii) Find the range of possible values of x. **2**

(e) (i) On an Argand diagram shade the region containing all points representing complex numbers z that satisfy both **2**

$$|z-2i|\leq 2 \text{ and } \frac{\pi}{6}\leq \arg z\leq\frac{\pi}{4}.$$

(ii) Find in simplest exact form the area of the shaded region. **2**

Student Name/Number:______________________________

QUESTION 12 (15 marks) **Start a new writing booklet** **Marks**

(a) (i) Show that $(2^p - 1)\left(1 + 2^p + 2^{2p} + \cdots + 2^{(q-1)p}\right) = 2^{pq} - 1$ for positive integers p, q. **1**

(ii) Hence, use the contrapositive to prove the statement **2**

$$If\ 2^n - 1\ is\ prime, then\ n\ is\ prime$$

(b) Use the substitution $x = \tan^2\theta,\ \ 0 \le \theta \le \frac{\pi}{2}$ to evaluate in simplest exact form **4**

$$\int_{\mathbf{0}}^{\mathbf{1}} \frac{\sqrt{x}}{(1+x)^3}\ dx$$

(c) Evaluate **4**

$$\int_0^{\pi} x^2 \sin x\ dx$$

(d) A particle is moving in simple harmonic motion along the x-axis with amplitude $a = 3$ metres. At time t seconds it has displacement x metres from the origin O and velocity v ms^{-1} given by $v^2 = n^2(a^2 - (x-c)^2)$ for some constants $n > 0, c > 0$. **4**

The particle has speed $2\sqrt{5}$ ms^{-1} when it is at *O* and speed 6ms^{-1} when it is 2 metres to the right of *O*.

Find the centre and period of the motion.

Student Name/Number:______________________________

QUESTION 13 (15 marks) **Start a new writing booklet** **Marks**

(a) The point A has position vector $\overrightarrow{OA} = 2\boldsymbol{i} + 6\boldsymbol{j} - 3\boldsymbol{k}$ relative to an origin O. **2**
Find a unit vector parallel to $\overrightarrow{OA}$.

(b) (i) Write the expansion of $(1 + ia)^4$ in ascending powers of a. **1**

(ii) Hence find the values of a such that $(1 + ia)^4$ is real. **2**

(c) A particle is moving in simple harmonic motion with acceleration given by:

$$\ddot{x} = -12sin2t.$$

Initially the particle is at the origin and has a positive velocity of 6 m/s.

(i) Find the equation of the particle's velocity. **2**

(ii) Show that $\ddot{x} = -4x$ **3**

(d) By completing the square, find **3**

$$\int \frac{1}{9x^2 + 6x + 5}\, dx.$$

(e) Show that $z = 2i, w = \sqrt{3} - i$ and $v = -\sqrt{3} - i$ are vertices of an equilateral triangle. **2**

Student Name/Number:______________________________

QUESTION 14 (15 marks) **Start a new writing booklet** **Marks**

(a) Show that **3**

$$\ln(1+x) > \frac{2x}{x+2} \text{ for } x > 0$$

(b) A particle is projected from point O at ground level with initial speed V and angle of projection θ.

It just clears two poles of height h metres at distances of b and c metres from the point of projection.

You may assume the Cartesian equation for the path, i.e.

$$y = x\tan\theta - \frac{gx^2}{2V^2}\sec^2\theta$$

(i) Show that $V^2 = \frac{(b+c)g\sec^2\theta}{2\tan\theta}$ **3**

(ii) Hence or otherwise, show that $tan\theta = \frac{h(b+c)}{bc}$ **2**

(iii) Find, in terms of h, b and c, the greatest height the particle reaches. **2**

(c) Solve the equation $z^2 = |z|^2 - 4$ **3**

(d) Let a, b and c be real numbers such that $a > b > c > 1$.

(i) Show that $a^{a-b}b^{b-c} > c^{a-c}$. **1**

(ii) Hence show that $a^a b^b c^c > a^b b^c c^a$. **1**

Student Name/Number:______________________________

QUESTION 15 (15 marks) **Start a new writing booklet** **Marks**

(a) If $z = \cos\theta + i\sin\theta$, show that **3**

$$\frac{1}{1+z} = \frac{1}{2}\left(1 - i\tan\frac{\theta}{2}\right)$$

(b) The roots $\alpha, \beta, \gamma, \delta$ of the equation $z^4 + az^3 + bz^2 + cz + d = 0$ $(a, b, c, d$ real) are represented by the vertices of square $PQRS$ in an Argand diagram. Each of the four quadrants contains exactly one vertex of the square, and one of the roots of the equation is $1 + 2i$. Find the values of a and d. **3**

(c) A particle of mass m kg is projected vertically upwards with speed $2g$ ms^{-1} under gravity in a medium in which the resistance to motion has magnitude $\dfrac{mv^2}{g}$ newtons where the speed of the particle is v ms^{-1} and the acceleration due to gravity is g ms^{-2}.

The particle reaches a maximum height of $\frac{1}{2}gln5$ metres before falling vertically downwards back to its starting point.

During its descent, at time t seconds the particle has fallen x metres, has velocity v ms^{-1} and acceleration a ms^{-2} given by

$$a = \frac{g^2 - v^2}{g}.$$

(i) Show that during its descent, **3**

$$x = \frac{g}{2}ln\left(\frac{g^2}{g^2 - v^2}\right).$$

(ii) Find in simplest exact form the speed with which the particle returns to its starting point. **1**

(d) Find the maximum value of $|z|$ if $|z + 1 - i| = 2$ **2**

(e) Evaluate, to two decimal places: **3**

$$\int_3^5 \frac{2x-1}{3x+2}\ dx$$

Student Name/Number:______________________________

QUESTION 16 (15 marks) **Start a new writing booklet** **Marks**

(a) With respect to a fixed origin O, the lines L_1 and L_2 have vector equations $\boldsymbol{r_1} = (-9 + 2\lambda)\boldsymbol{i} + \lambda\boldsymbol{j} + (10 - \lambda)\boldsymbol{k}$ and $\boldsymbol{r_2} = (3 + 3\mu)\boldsymbol{i} + (1 - \mu)\boldsymbol{j} + (17 + 5\mu)\boldsymbol{k}$ respectively where λ, μ are scalar parameters. The point A with position vector $5\boldsymbol{i} + 7\boldsymbol{j} + 3\boldsymbol{k}$ lies on L_1. The point B is the reflection of the point A in the line L_2.

(i) Find the position vector of the point of intersection P of the lines L_1 and L_2. **2**

(ii) Show that L_1 and L_2 are perpendicular to each other. **2**

(iii) Find the position vector of the point B. **2**

(b) (i) Find real numbers A, B, C and D such that **2**

$$\frac{5x^3 - 3x^2 + 2x - 1}{x^2(x^2 + 1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 1}$$

(ii) Hence find **2**

$$\int \frac{5x^3 - 3x^2 + 2x - 1}{x^2(x^2 + 1)}\ dx.$$

(c) The displacement of a particle at time t is x, measured from a fixed point O where a, c are positive constants. If **5**

$$\frac{dx}{dt} = a(c^2 - x^2)$$

and $x = 0$ when $t = 0$, prove that

$$x = \frac{c(e^{2act} - 1)}{e^{2act} + 1}.$$

If $x = 3$ when $t = 1$, and $x = \frac{75}{17}$ when $t = 2$, show that $c = 5$ and evaluate a.

END OF PAPER.

Student Name/Number:______________________________

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MULTIPLE-CHOICE ANSWER SHEET

For multiple choice questions, choose the best answer A, B, C or D and fill in the correct circle.

1. Ⓐ Ⓑ Ⓒ Ⓓ
2. Ⓐ Ⓑ Ⓒ Ⓓ
3. Ⓐ Ⓑ Ⓒ Ⓓ
4. Ⓐ Ⓑ Ⓒ Ⓓ
5. Ⓐ Ⓑ Ⓒ Ⓓ
6. Ⓐ Ⓑ Ⓒ Ⓓ
7. Ⓐ Ⓑ Ⓒ Ⓓ
8. Ⓐ Ⓑ Ⓒ Ⓓ
9. Ⓐ Ⓑ Ⓒ Ⓓ
10. Ⓐ Ⓑ Ⓒ Ⓓ