

2010 HSC exam - Mathematics Extension 2

Question 8 Solution

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The Basel Problem: Prove that $\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{k^2} = \frac{\pi^2}{6}$.

Note: This was first proved by Euler. However this HSC exam does not follow Euler's method. It follows a method of Yoshio Matsuoka, Kagoshima University, Japan. This was published in 1961 in

Matsuoka, Y., *An Elementary Proof of the Formula $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$* , The American Mathematical Monthly, **Vol. 68, No. 5 (May, 1961)**, pp. 485-487.

and can be downloaded at

<http://www.angelfire.com/ab7/fourunit/matsuoka.pdf>

Proof: Let $A_n = \int_0^{\pi/2} \cos^{2n} x \, dx$ & $B_n = \int_0^{\pi/2} x^2 \cos^{2n} x \, dx$ where n is an integer, $n \geq 0$.

$$\begin{aligned} \text{Then } \int_0^{\pi/2} \cos^{2n} x \, dx &= \int_0^{\pi/2} \cos^{2n-1} x \cos x \, dx \\ &= \int_0^{\pi/2} \cos^{2n-1} x \frac{d}{dx} \sin x \, dx \\ &= [\sin x \cos^{2n-1} x]_0^{\pi/2} - \int_0^{\pi/2} \sin x \frac{d}{dx} \cos^{2n-1} x \, dx \\ &= 0 - 0 - \int_0^{\pi/2} \sin x (2n-1) \cos^{2n-2} x (-\sin x) \, dx \\ &= (2n-1) \int_0^{\pi/2} \cos^{2n-2} x (1 - \cos^2 x) \, dx \\ &= (2n-1) \int_0^{\pi/2} \cos^{2n-2} x \, dx - (2n-1) \int_0^{\pi/2} \cos^{2n} x \, dx \end{aligned}$$

$$\therefore 2n \int_0^{\pi/2} \cos^{2n} x \, dx = (2n-1) \int_0^{\pi/2} \cos^{2n-2} x \, dx \text{ and}$$

$$n \int_0^{\pi/2} \cos^{2n} x \cos^2 x \, dx = \frac{2n-1}{2} \int_0^{\pi/2} \cos^{2(n-1)} x \, dx \text{ \& \; } \therefore nA_n = \frac{2n-1}{2} A_{n-1} \text{ for } n \geq 1.$$

$$\begin{aligned} \text{Also, } A_n &= \int_0^{\pi/2} \cos^{2n} x \cdot \frac{dx}{dx} \, dx \\ &= [x \cos^{2n} x]_0^{\pi/2} - \int_0^{\pi/2} x \cdot 2n \cos^{2n-1} x (-\sin x) \, dx \\ &= 0 - 0 + 2n \int_0^{\pi/2} x \sin x \cos^{2n-1} x \, dx \\ &= 2n \int_0^{\pi/2} x \sin x \cos^{2n-1} x \, dx, \quad n \geq 1 \end{aligned}$$

$$\begin{aligned}
\text{Hence, } A_n &= 2n \int_0^{\pi/2} \sin x \cos^{2n-1} x \frac{d}{dx} \left(\frac{x^2}{2} \right) dx \\
&= 2n \left(\left[\frac{x^2}{2} \sin x \cos^{2n-1} x \right]_0^{\pi/2} - \int_0^{\pi/2} \frac{x^2}{2} \frac{d}{dx} (\sin x \cos^{2n-1} x) dx \right) \\
&= 2n \left(0 - 0 - \int_0^{\pi/2} \frac{x^2}{2} ((2n-1) \sin x \cos^{2n-2} x (-\sin x) + \cos^{2n-1} x \cos x) dx \right) \\
&= n \int_0^{\pi/2} x^2 ((2n-1) \sin^2 x \cos^{2n-2} x - \cos^{2n} x) dx \\
&= n \int_0^{\pi/2} x^2 ((2n-1)(1 - \cos^2 x) \cos^{2n-2} x - \cos^{2n} x) dx \\
&= n \int_0^{\pi/2} x^2 ((2n-1) \cos^{2n-2} x - (2n-1) \cos^{2n} x - \cos^{2n} x) dx \\
&= n \int_0^{\pi/2} x^2 ((2n-1) \cos^{2n-2} x - 2n \cos^{2n} x) dx \\
&= n(2n-1) \int_0^{\pi/2} x^2 \cos^{2n-2} x dx - 2n^2 \int_0^{\pi/2} x^2 \cos^{2n} x dx \\
&= n(2n-1)B_{n-1} - 2n^2 B_n
\end{aligned}$$

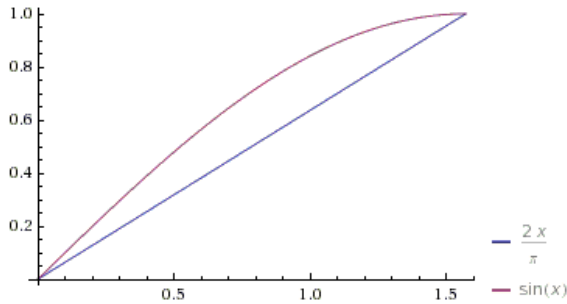
$$\therefore \frac{A_n}{n^2} = \frac{2n-1}{n} B_{n-1} - 2B_n, \quad n \geq 1$$

$$\begin{aligned}
\therefore \frac{1}{n^2} &= \frac{2n-1}{nA_n} B_{n-1} - 2 \frac{B_n}{A_n} \\
&= \frac{2n-1}{n \cdot \frac{2n-1}{2^n} A_{n-1}} B_{n-1} - 2 \frac{B_n}{A_n} \\
&= 2 \frac{B_{n-1}}{A_{n-1}} - 2 \frac{B_n}{A_n} \\
&= 2 \left(\frac{B_{n-1}}{A_{n-1}} - \frac{B_n}{A_n} \right), \quad n \geq 1
\end{aligned}$$

$$\text{So } \sum_{k=1}^n \frac{1}{k^2} = 2 \left(\frac{B_0}{A_0} - \frac{B_1}{A_1} \right) + 2 \left(\frac{B_1}{A_1} - \frac{B_2}{A_2} \right) + \dots + 2 \left(\frac{B_{n-1}}{A_{n-1}} - \frac{B_n}{A_n} \right) = 2 \frac{B_0}{A_0} - 2 \frac{B_n}{A_n}.$$

$$\text{But } \frac{2B_0}{A_0} = \frac{2 \int_0^{\pi/2} x^2 dx}{\int_0^{\pi/2} dx} = \frac{2 \left[\frac{x^3}{3} \right]_0^{\pi/2}}{\left[x \right]_0^{\pi/2}} = \frac{2 \cdot \frac{\pi^3}{24} - 0}{\frac{\pi}{2} - 0} = \frac{\pi^2}{6} \text{ and } \therefore \sum_{k=1}^n \frac{1}{k^2} = \frac{\pi^2}{6} - 2 \frac{B_n}{A_n}.$$

Behold:



$\sin x \geq \frac{2}{\pi}x$ for $0 \leq x \leq \frac{\pi}{2}$ whereupon
 $\sin^2 x \geq \frac{4}{\pi^2}x^2$ and $1 - \cos^2 x \geq \frac{4}{\pi^2}x^2$ and so
 $\cos^2 x \leq 1 - \frac{4}{\pi^2}x^2$ & $\cos^{2n} x \leq \left(1 - \frac{4}{\pi^2}x^2\right)^n$.
So $B_n \leq \int_0^{\pi/2} x^2 \left(1 - \frac{4x^2}{\pi^2}\right)^n dx$.

Noting that

$$\begin{aligned}
&\frac{\pi^2}{8(n+1)} \int_0^{\pi/2} \left(1 - \frac{4x^2}{\pi^2}\right)^{n+1} dx \\
&= \frac{\pi^2}{8(n+1)} \int_0^{\pi/2} \left(1 - \frac{4x^2}{\pi^2}\right)^{n+1} \frac{dx}{dx} dx \\
&= \frac{\pi^2}{8(n+1)} \left[x \left(1 - \frac{4x^2}{\pi^2}\right)^{n+1} \right]_0^{\pi/2} - \frac{\pi^2}{8(n+1)} \int_0^{\pi/2} x \cdot (n+1) \left(1 - \frac{4x^2}{\pi^2}\right)^n \left(\frac{-8x}{\pi^2}\right) dx \\
&= 0 - 0 + \int_0^{\pi/2} x^2 \left(1 - \frac{4x^2}{\pi^2}\right)^n dx
\end{aligned}$$

$$\& \therefore \int_0^{\pi/2} x^2 (1 - \frac{4x^2}{\pi^2})^n dx = \frac{\pi^2}{8(n+1)} \int_0^{\pi/2} (1 - \frac{4x^2}{\pi^2})^{n+1} dx$$

$$\text{we now have that } B_n \leq \frac{\pi^2}{8(n+1)} \int_0^{\pi/2} (1 - \frac{4x^2}{\pi^2})^{n+1} dx$$

$$\text{Let } x = \frac{\pi}{2} \sin t \therefore dx = \frac{\pi}{2} \cos t dt.$$

$$\text{When } x = 0, t = 0 \ \& \ \text{when } x = \frac{\pi}{2}, t = \frac{\pi}{2}.$$

$$1 - \frac{4x^2}{\pi^2} = 1 - \frac{4}{\pi^2} \cdot \frac{\pi^2}{4} \sin^2 t = \cos^2 t.$$

$$\text{For } 0 \leq t \leq \frac{\pi}{2}, 0 \leq \cos^3 t \leq 1.$$

$$\begin{aligned} \therefore B_n &\leq \frac{\pi^2}{8(n+1)} \int_0^{\pi/2} \cos^{2n+2} t \cdot \frac{\pi}{2} \cos t dt \\ &= \frac{\pi^3}{16(n+1)} \int_0^{\pi/2} \cos^{2n+3} t dt \\ &= \frac{\pi^3}{16(n+1)} \int_0^{\pi/2} \cos^{2n} t \cdot \cos^3 t dt \\ &\leq \frac{\pi^3}{16(n+1)} \int_0^{\pi/2} \cos^{2n} t dt \\ &= \frac{\pi^3}{16(n+1)} A_n \\ \therefore B_n &\leq \frac{\pi^3}{16(n+1)} \int_0^{\pi/2} \cos^{2n+3} t dt \leq \frac{\pi^3}{16(n+1)} A_n. \end{aligned}$$

$$\text{Noting } A_n > 0 \ \& \ B_n > 0, 0 < 2 \frac{B_n}{A_n} \leq 2 \cdot \frac{\pi^3}{16(n+1)} = \frac{\pi^3}{8(n+1)}$$

$$\therefore -\frac{\pi^3}{8(n+1)} \leq -2 \frac{B_n}{A_n} < 0$$

$$\therefore \frac{\pi^2}{6} - \frac{\pi^3}{8(n+1)} \leq \frac{\pi^2}{6} - 2 \frac{B_n}{A_n} = \sum_{k=1}^n \frac{1}{k^2} < \frac{\pi^2}{6}$$

$$\therefore \frac{\pi^2}{6} = \lim_{n \rightarrow \infty} \left(\frac{\pi^2}{6} - \frac{\pi^3}{8(n+1)} \right) \leq \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{k^2} \leq \frac{\pi^2}{6}$$

$$\therefore \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{k^2} = \frac{\pi^2}{6} \quad \square$$