2005 Mathematics Extension 2 HSC Examination Solutions*

By Derek Buchanan

1. (a) Letting $u = \sin\theta,\ du = \cos\theta\, d\theta$
$\Rightarrow \int \frac{\cos\theta}{\sin^5\theta}\, d\theta = \int u^{-5}\, du = \frac{u^{-4}}{-4} + C = -\frac{1}{4\sin^4\theta} + C.$

(b) (i) $5x \equiv a(x+2) + b(x-3) \Rightarrow$ if $x = 3,\ 5a = 5(3) = 15 \therefore a = 3$ & if $x = -2, -5b = -10 \therefore b = 2$

(ii) Hence $\int \frac{5x}{x^2-x-6}\, dx = \int \frac{5x}{(x-3)(x+2)}\, dx = \int \left(\frac{3}{x-3} + \frac{2}{x+2}\right) dx$
$= 3\ln(x-3) + 2\ln(x+2) + C$

(c) $\int_1^e x^7 \ln x\, dx = \int_1^e \ln x \frac{d}{dx}\left(\frac{x^8}{8}\right) dx = [\frac{1}{8}x^8 \ln x]_1^e - \int_1^e \frac{x^8}{8}\frac{d}{dx}(\ln x)\, dx$
$= \frac{1}{8}e^8 \ln e - \frac{1}{8}(1^8)\ln 1 - \int_1^e \frac{x^8}{8}\frac{1}{x}\, dx = \frac{e^8}{8} - \int_1^8 \frac{x^7}{8}\, dx = \frac{e^8}{8} - [\frac{x^8}{64}]_1^e$
$= \frac{e^8}{8} - (\frac{e^8}{64} - \frac{1^8}{64}) = \frac{7e^8+1}{64}.$

(d) $\int \frac{dx}{\sqrt{4x^2-1}} = \frac{1}{2}\int \frac{dx}{\sqrt{x^2 - \left(\frac{1}{2}\right)^2}} = \frac{1}{2}\ln\left(x + \sqrt{x^2 - (\frac{1}{2})^2}\ \right) + C_1$
$= \frac{1}{2}\ln\left(2x + \sqrt{4x^2-1}\ \right) + C_2$ (where $C_2 = C_1 - \frac{1}{2}\ln 2$)

(e) (i) $t = \tan\frac{\theta}{2} \Rightarrow \frac{dt}{d\theta} = \frac{1}{2}\sec^2\frac{\theta}{2} = \frac{1}{2}(1 + \tan^2\frac{\theta}{2}) = \frac{1}{2}(1+t^2)$

(ii) $\sin\theta = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}} = \frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} = \frac{2t}{1+t^2}$

(iii) Letting $t = \tan\frac{\theta}{2},\ d\theta = \frac{2dt}{1+t^2}$ (from **(i)**) and $\operatorname{cosec}\theta = \frac{1+t^2}{2t}$ (from **(ii)**)
$\Rightarrow \int \operatorname{cosec}\theta\, d\theta = \int \frac{1+t^2}{2t}\cdot\frac{2\,dt}{1+t^2} = \int \frac{dt}{t} = \ln t + C = \ln\tan\frac{\theta}{2} + C$

2. (a) (i) $2z + iw = 2(3+i) + i(1-i) = 6 + 2i + i + 1 = 7 + 3i$

(ii) $\bar{z}w = (\overline{3+i})(1-i) = (3-i)(1-i) = 3 - 1 - i - 3i = 2 - 4i$

(iii) $\frac{6}{w} = \frac{6}{1-i} = \frac{6(1+i)}{(1-i)(1+i)} = \frac{6+6i}{1+1} = 3 + 3i$

(b) (i) $\beta = 1 - i\sqrt{3} = \sqrt{1^2 + \sqrt{3}^{\,2}}\, \text{cis}\,(-\tan^{-1}\frac{\sqrt{3}}{1}) = 2\text{cis}\,(-\frac{\pi}{3})$

(ii) $\beta^5 = (2\text{cis}\,(-\frac{\pi}{3}))^5 = 2^5\text{cis}\,(-\frac{5\pi}{3}) = 32\text{cis}\,\frac{\pi}{3}$

*The question paper is available at
http://www.boardofstudies.nsw.edu.au/hsc_exams/hsc2005exams/pdf_doc/maths_ext2_05.pdf
and there are extensive free resources available for the course at
http://www.fourunitmaths.cjb.net
such as over 100 trial papers, comprehensive notes and assignments.

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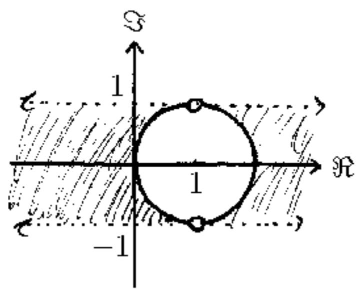
(iii) $\beta^5 = 32cis\,\frac{\pi}{3} = 32(\frac{1}{2} + i\frac{\sqrt{3}}{2}) = 16 + i16\sqrt{3}$

(c) $z - \bar{z} = 2i\Im(z) \therefore |z - \bar{z}| < 2 \Rightarrow -1 < \Im(z) < 1$

$\{z \in \mathbb{C} : |z - \bar{z}| < 2 \wedge |z - 1| \geq 1\}$

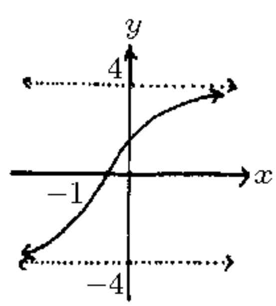


(d) **(i)** $\angle POQ$ is bisected by ℓ, so
$\arg(z_1) + \arg(z_2) = \alpha - \frac{1}{2}\angle POQ + \alpha + \frac{1}{2}\angle POQ = 2\alpha$

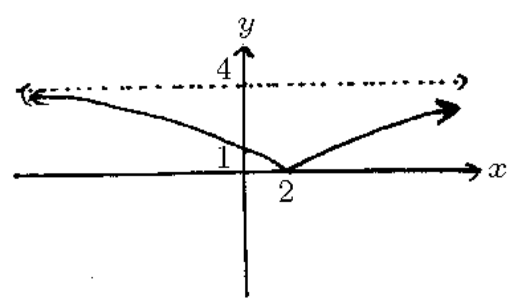
(ii) $z_1 z_2 = |z_1||z_2|cis\,(\arg(z_1))cis\,(\arg(z_2)) = |z_1||z_1|cis\,(\arg(z_1) + \arg(z_2))$
$= |z_1|^2 cis\,2\alpha$

(iii) $\alpha = \frac{\pi}{4}$ and R represents $z_1 z_2 \Rightarrow$ as z_1 varies, since z_1 can't be 0, $|z_1|^2 > 0$ and also $cis\,2\alpha = cis\,\frac{\pi}{2} = i \therefore$ the locus of R is $\{iy : y > 0\}$, i.e., the positive part if the imaginary axis.

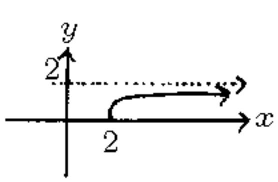
3. **(a)** **(i)** $y = f(x + 3)$



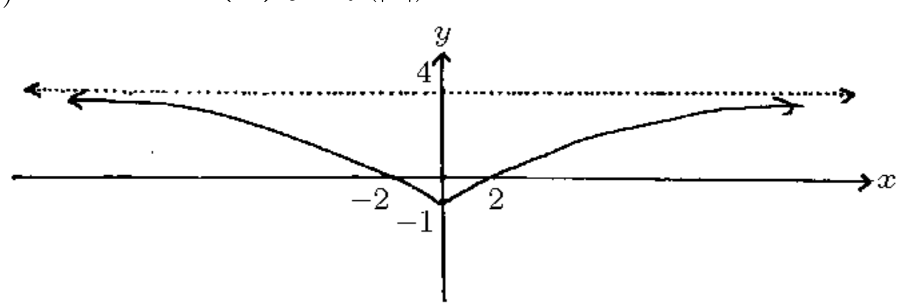
(ii) $y = |f(x)|$



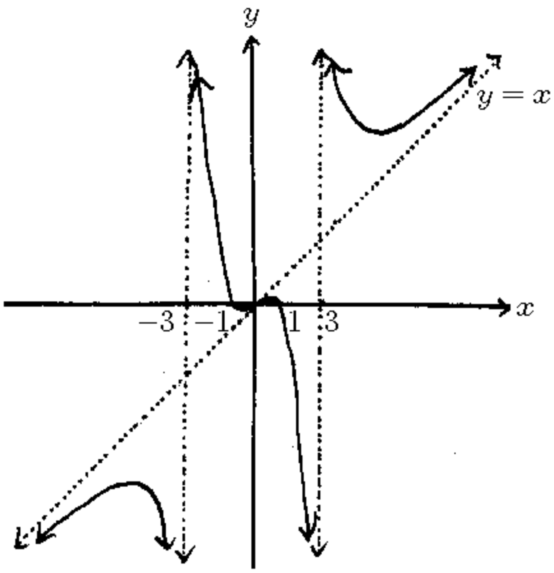
(iii) $y = \sqrt{f(x)}$



(iv) $y = f(|x|)$



(b) $y = x + \frac{8x}{x^2-9} = \frac{x(x+1)(x-1)}{(x+3)(x-3)} \Rightarrow$ asymptotes are $y = x$, $x = \pm 3$ and the x-intercepts are at $x = \pm 1$, 0 and the y-intercept is at $y = 0$. As $x \to \pm\infty, y \to x^{\pm}$. As $x \to -3^{\pm}, y \to \pm\infty$. As $x \to 3^{\pm}, y \to \pm\infty$.



(c) $3x^2 - 4y - 4xy' + 3y^2y' = 3x^2 - 4y + y'(3y^2 - 4x) = 0 \therefore 3(2)^2 - 4(1) + y'(2).(3(1)^2 - 4(2)) = 8 - 5y'(2) = 0 \therefore y'(2) = \frac{8}{5} \therefore$ the normal is $y - 1 = \frac{-1}{y'(2)}(x - 2) = -\frac{5}{8}x + \frac{5}{4} \therefore y = -\frac{5}{8}x + \frac{9}{4}$

(d) Horizontally, $N\sin\theta = \frac{mv^2}{r}$ and vertically, $N\cos\theta = mg$. $\therefore N = \sqrt{N^2} = \sqrt{(N\sin\theta)^2 + (N\cos\theta)^2} = \sqrt{(\frac{mv^2}{r})^2 + (mg)^2} = m\sqrt{g^2 + \frac{v^4}{r^2}}$

4. (a) (i) $V = 2\pi\int_0^N xy\,dx = 2\pi\int_0^N xe^{-x^2}\,dx = -\pi\int_0^N -2xe^{-x^2}\,dx$
$= -\pi\int_0^N \left(\frac{d}{dx}(-x^2)\right)e^{-x^2}\,dx = -\pi[e^{-x^2}]_0^N = -\pi(e^{-N^2} - e^0)$
$= \pi(1 - e^{-N^2})$

(ii) $\lim_{N\to\infty} V = \pi$

(b) (i) $\alpha + \beta + \gamma + \delta = -p$, $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -r$

(ii) $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = (\alpha + \beta + \gamma + \delta)^2 - 2(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta) = (-p)^2 - 2q = p^2 - 2q$

(iii) If $P(x) = x^4 - 3x^3 + 5x^2 + 7x - 8$, from **(ii)**, $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = (-3)^2 - 2(5) = -1 < 0$ and so $P(x) = 0$ cannot have four real roots for otherwise $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$ would have to be > 0.

(iv) $P(0) = -8 < 0 < 2 = P(1)$ and $P(x)$ is continuous $\forall x \in \mathbb{R} \Rightarrow P(x) = 0$ has a root between 0 and 1. But the coefficients are real, so it has 2 real quadratic factors, one of which has 2 real linear factors, and by

(iii), the other has 2 complex factors whose zeros are a conjugate pair. Hence $x^4 - 3x^3 + 5x^2 + 7x - 8 = 0$ has exactly two real roots. Also, if $\theta = \left(\sqrt[3]{1727 + 3\sqrt{331449}} + \sqrt[3]{1727 - 3\sqrt{331449}} + 5\right)/3$ then by the quartic formula, these two real roots are

$$x = \left(3 - \sqrt{4\theta - 11} \pm \sqrt{8\sqrt{\theta^2 + 32} - 6\sqrt{4\theta - 11} - 4\theta - 2}\ \right)/4$$

$$\approx 0.828794716... \text{ or } -1.2727703...$$

(c) **(i)** If $x_1 \neq 0,\ a^2y_1(0) - b^2x_1(-b) = (a^2 - b^2)x_1y_1 \Rightarrow y_1 = \frac{b^3}{a^2-b^2}$ but if $x_1 = 0,\ y_1 = \pm b$.

(ii) If $y_1 = \frac{b^3}{a^2-b^2} \leq b,\ b^2 \leq a^2 - b^2, \therefore \frac{b^2}{a^2} \leq \frac{1}{2} \therefore e = \sqrt{1 - \frac{b^2}{a^2}} \geq \sqrt{1 - \frac{1}{2}}$. So $e \geq \frac{1}{\sqrt{2}}$.

5. **(a)** **(i)** Area $(\triangle ABC) = \frac{1}{2}bc = \frac{1}{2}ad \Rightarrow b^2c^2 = (bc)^2 = (ad)^2 = d^2a^2$ $= d^2(b^2 + c^2)$ by Pythagoras' Theorem.

(ii) $\tan\alpha = \frac{h}{AB},\ \tan\beta = \frac{h}{AC},\ \tan\gamma = \frac{h}{AP} \Rightarrow$ from **(i)**, $AB^2.AC^2 = AP^2(AB^2 + AC^2) \therefore \frac{h^2}{\tan^2\alpha} \cdot \frac{h^2}{\tan^2\beta} = \frac{h^2}{\tan^2\gamma}\left(\frac{h^2}{\tan^2\alpha} + \frac{h^2}{\tan^2\beta}\right)$ $\Rightarrow \tan^2\gamma = \tan^2\alpha + \tan^2\beta$

(b) **(i)** The five possibilities are:
FMMMMM; MFMMMM; MMFMMM; MMMFMM; MMMMFM

(ii) $2!(\binom{4}{0} + \binom{5}{1} + \binom{6}{2} + \binom{7}{3} + \binom{8}{4}) = 252$

(c) **(i)** If $y = f(x),\ \int_0^b x\,dy + \int_0^a y\,dx = \int_0^b f^{-1}(y)\,dy + \int_0^a f(x)\,dx$ $= \int_0^b f^{-1}(x)\,dx + \int_0^a f(x)\,dx = ab \therefore \int_0^a f(x)\,dx = ab - \int_0^b f^{-1}(x)\,dx$.

(ii) $f(x) = \sin^{-1}(\frac{x}{4}) \Rightarrow f^{-1}(x) = 4\sin x$. So **(i)** and $\sin^{-1}(\frac{2}{4}) = \frac{\pi}{6} \Rightarrow$ $\int_0^2 \sin^{-1}(\frac{x}{4})\,dx = 2(\frac{\pi}{6}) - \int_0^{\pi/6} 4\sin x\,dx = \frac{\pi}{3} + 4[\cos x]_0^{\frac{\pi}{6}}$ $= \frac{\pi}{3} + 4(\frac{\sqrt{3}}{2} - 1) = \frac{\pi}{3} + 2\sqrt{3} - 4$.

(d) **(i)** Area $(ABCD) = AD.CD = 2\sqrt{9 - x^2}.x\tan 60^\circ = 2x\sqrt{27 - 3x^2}$

(ii) $V = \int_0^3 2x\sqrt{27 - 3x^2}\,dx = -\frac{1}{3}\int_0^3(-6x)\sqrt{27 - 3x^2}\,dx$ $= -\frac{1}{3}\int_0^3(\frac{d}{dx}(27 - 3x^2))(27 - 3x^2)^{1/2}\,dx = -\frac{2}{9}[(27 - 3x^2)^{3/2}]_0^3$ $= -\frac{2}{9}(0 - 81\sqrt{3}) = 18\sqrt{3}$

6. **(a)** **(i)** $I_0(x) = \int_0^x t^0e^{-t}\,dt = [-e^{-t}]_0^x = -e^{-x} - -1 = 0![1 - e^{-x}(\frac{x^0}{0!})] \therefore$ it is true for $n = 0$.
If it is true for $n = k, I_k(x) = \int_0^x t^ke^{-t}\,dt = k![1 - e^{-x}\sum_{j=0}^k \frac{x^j}{j!}]$
$\therefore I_{k+1}(x) = \int_0^x t^{k+1}e^{-t}\,dt = \int_0^x t^{k+1}(\frac{d}{dt}(-e^{-t}))\,dt$
$= [-t^{k+1}e^{-t}]_0^x - \int_0^x -e^{-t}(\frac{d}{dt}(t^{k+1}))\,dt = -x^{k+1}e^{-x} + (k+1)\int_0^x t^ke^{-k}\,dt$

$= -x^{k+1}e^{-x} + (k+1)I_k(x) = -x^{k+1}e^{-x} + (k+1)k![1 - e^{-x}\sum_{j=0}^{k}\frac{x^j}{j!}]$
$= (k+1)![1 - e^{-x}\sum_{j=0}^{k+1}\frac{x^j}{j!}]$
i.e., if it is true for $n = k$ then it is true for $n = k+1$.

Now I'll be like Darth Vader and be the chosen one to bring balance to "the force", I'll placate mathematicians by ending the proof here and say the statement is therefore true for all integers $n \geq 0$ by induction.$\square$

and to placate teachers I'll put the dreaded mantra here but say also that although I have for funny reasons decided to include it in this set of solutions it isn't part of the above proof:

"It is true for $n = 0 \therefore$ it is true for $n = 1 \therefore$ it is true for $n = 2$, etc., i.e., by induction it is true for all integers $n \geq 0$."

(ii) $0 \leq t \leq 1 \Rightarrow 0 \leq t^n e^{-t} \leq t^n \Rightarrow$
$0 \leq \int_0^1 t^n e^{-t}\,dt \leq \int_0^1 t^n\,dt = [\frac{t^{n+1}}{n+1}]_0^1 = \frac{1}{n+1}$.

(iii) From **(ii)**, $0 \leq I_n(1) = n![1 - e^{-1}\sum_{j=0}^{n}\frac{1}{j!}] \leq \frac{1}{n+1} \Rightarrow$
$0 \leq 1 - e^{-1}\sum_{j=0}^{n}\frac{1}{j!} \leq \frac{1}{(n+1)!}$

(iv) $\therefore 0 \leq \lim_{n\to\infty}(1 - e^{-1}\sum_{j=0}^{n}\frac{1}{j!}) = 1 - e^{-1}\sum_{j=0}^{\infty}\frac{1}{j!} \leq \lim_{n\to\infty}\frac{1}{(n+1)!} = 0$
$\therefore \sum_{j=0}^{\infty}\frac{1}{j!} = e$

(b) **(i)** $\omega^n = 1 \therefore \omega^n - 1 = (\omega - 1)\sum_{j=0}^{n-1}\omega^j = 0$ and $\omega \neq 1 \Rightarrow \sum_{j=0}^{n-1}\omega^j = 0$.
Hence $(\omega-1)\sum_{j=0}^{n-1}(j+1)\omega^j = \sum_{j=0}^{n-1}(j+1)(\omega-1)\omega^j$
$= n\omega^n - 1 + \sum_{j=1}^{n-1}(j-(j+1))\omega^j = n(1) - \sum_{j=0}^{n-1}\omega^j = n - 0 = n$

(ii) If $z = \text{cis}\,\theta$, $\frac{1}{\text{cis}\,2\theta - 1} = \frac{1}{z^2-1} = \frac{z^{-1}}{z - z^{-1}} = \frac{\text{cis}\,(-\theta)}{\text{cis}\,\theta - \text{cis}\,(-\theta)} = \frac{\cos\theta - i\sin\theta}{2i\sin\theta}$

(iii) If $\theta = \frac{\pi}{n}, \omega = \text{cis}\,\frac{2\pi}{n} = z^2 \Rightarrow \Re(\frac{1}{\omega-1}) = \Re(-\frac{1}{2}i\cot\theta - \frac{1}{2}) = -\frac{1}{2}$

(iv) If $n = 5$, $1 + 2\cos\frac{2\pi}{5} + 3\cos\frac{4\pi}{5} + 4\cos\frac{6\pi}{5} + 5\cos\frac{8\pi}{5}$
$= \Re(1 + 2\omega + 3\omega^2 + 4\omega^3 + 5\omega^4) = \Re(\frac{5}{\omega-1}) = 5(-\frac{1}{2}) = -\frac{5}{2}$

(v) $1 + 2\cos\frac{2\pi}{5} + 3\cos\frac{4\pi}{5} + 4\cos\frac{6\pi}{5} + 5\cos\frac{8\pi}{5}$
$= 1 + 2\cos\frac{2\pi}{5} - 3\cos\frac{\pi}{5} - 4\cos\frac{\pi}{5} + 5\cos\frac{2\pi}{5} = 1 + 7\cos\frac{2\pi}{5} - 7\cos\frac{\pi}{5} = -\frac{5}{2} \Rightarrow \cos\frac{2\pi}{5} - \cos\frac{\pi}{5} + \frac{1}{2} = 2\cos^2\frac{\pi}{5} - \cos\frac{\pi}{5} - \frac{1}{2} = 0 \therefore \cos\frac{\pi}{5} = \frac{1+\sqrt{1^2 - 4(2)(-\frac{1}{2})}}{2(2)} = \frac{1+\sqrt{5}}{4}$ ($\frac{\pi}{5}$ acute $\therefore \cos\frac{\pi}{5} > 0$).

7. **(a)** **(i)** $\angle BMP = \angle BNP = 90^\circ \therefore BNPM$ is cyclic (opp. $\angle$'s supp.)

(ii) $\angle MNP = \angle MBP$ ($\angle$'s standing on same interval MP in cyclic quad. $BNPM$)

$= 90° - \angle BPM$ ($BM \perp TP$)
$= 90° - \angle NAP$ (alt. seg. thm.)
$= \angle NPA$ ($TA \perp NP$)
$\therefore MN || PA$ (alt. $\angle$'s equal)

(iii) $\angle TNM = \angle TAP$ (corresp. $\angle$'s, $MN||PA$) and $\angle T$ common $\Rightarrow$ $\triangle TMN ||| \triangle TPA$ (AAA)
$\Rightarrow \frac{r}{r+s} = \frac{p+q}{p+q+u}$ (corresp. sides prop.)
$\therefore rp + rq + ru = rp + rq + sp + sq \therefore ru = s(p+q) \therefore \frac{s}{u} = \frac{r}{p+q} < \frac{r}{p}$

(iv) $\therefore s < \frac{ur}{p} = \frac{u\sqrt{p^2-t^2}}{p} < \frac{u\sqrt{p^2}}{p} = u$

(b) **(i)** $\ddot{x} = -\frac{k}{R^2} = -R\omega^2 = -R(\frac{2\pi}{T})^2 = -\frac{4\pi^2 R}{T^2} \Rightarrow k = \frac{4\pi^2 R^3}{T^2}$

(ii) $\frac{d}{dx}(\frac{1}{2}v^2) = -\frac{4\pi^2 R^3}{T^2 x^2} \Rightarrow \frac{1}{2}v^2 = \int -(\frac{4\pi^2 R^3}{T^2})x^{-2}\, dx = \frac{4\pi^2 R^3}{T^2 x} + C_1$ and when $x = R, v = 0 \Rightarrow C_1 = -\frac{4\pi^2 R^2}{T^2} \Rightarrow \frac{1}{2}v^2 = \frac{4\pi^2 R^3}{T^2 x} - \frac{4\pi^2 R^2}{T^2} = \frac{4\pi^2 R^2}{T^2}(\frac{R-x}{x}) \therefore v^2 = \frac{8\pi^2 R^2}{T^2}(\frac{R-x}{x})$

(iii) Satellite moves towards star $\Rightarrow \frac{dx}{dt} = -\frac{2\sqrt{2}\pi R}{T}\sqrt{\frac{R-x}{x}} \Rightarrow$
$dt = -\frac{T}{2\sqrt{2}\pi R}\sqrt{\frac{x}{R-x}}\, dx$
$\therefore t = -\frac{T}{2\sqrt{2}\pi R}\int \sqrt{\frac{x}{R-x}}\, dx = -\frac{T}{2\sqrt{2}\pi R}(R\sin^{-1}(\sqrt{\frac{x}{R}}) - \sqrt{x(R-x)}\,) + C_2$
and $t = 0 \Rightarrow x = R$, so $C_2 = \frac{TR\sin^{-1}1}{2\sqrt{2}\pi R} = \frac{T}{4\sqrt{2}}$ and hence
$t = \frac{T}{4\sqrt{2}} - \frac{T}{2\sqrt{2}\pi R}(R\sin^{-1}(\sqrt{\frac{x}{R}}) - \sqrt{x(R-x)}\,)$ so if the star were concentrated at a single point, if the satellite could reach that point, $x = 0$ and the time would be $\frac{T}{4\sqrt{2}}$.

8. **(a)** $a, b \in \mathbb{R}^+, f(x) = \frac{a+b+x}{3(abx)^{1/3}}$ for $x > 0$.

(i) $f'(x) = \frac{3(abx)^{1/3}(1) - (a+b+x)(abx)^{-2/3}(ab)}{9(abx)^{2/3}} = \frac{3abx - ab(a+b+x)}{9(abx)^{4/3}} = \frac{ab(2x-a-b)}{9(abx)^{4/3}}$ $= 0 \Rightarrow x = \frac{a+b}{2}$ and $f'((\frac{a+b}{2})^-) = 0^-,\ f'((\frac{a+b}{2})^+) = 0^+ \Rightarrow x = \frac{a+b}{2}$ minimises $f(x)$.

(ii) $c \in \mathbb{R}^+$ and $\frac{a+b}{2} \geq \sqrt{ab}$ and

$\left(\frac{a+b+c}{3\sqrt[3]{abc}}\right)^3 \geq \left(\frac{a+b+\frac{a+b}{2}}{3\sqrt[3]{ab(\frac{a+b}{2})}}\right)^3$
$= \left(\frac{3(a+b)/2}{3\sqrt[3]{ab(a+b)/2}}\right)^3$
$= \frac{(a+b)^3}{4ab(a+b)}$
$= \left(\frac{a+b}{2\sqrt{ab}}\right)^2$
$= \left(\frac{a+b}{2}\right)^2/ab$
$\geq \left(\sqrt{ab}\ \right)^2/ab$
$= ab/ab$
$= 1$

So $\frac{a+b+c}{3\sqrt[3]{abc}} \geq 1 \therefore \frac{a+b+c}{3} \geq \sqrt[3]{abc}$

(iii) $x^3 - px^2 + qx - r = 0$ has 3 positive real roots α, β, γ.
Sum of roots $= -\frac{-p}{1} = p \therefore \alpha + \beta + \gamma = p$ and $\frac{\alpha+\beta+\gamma}{3} = \frac{p}{3}$. Also, product of roots $= \alpha\beta\gamma = -\frac{-r}{1} = r \therefore$ from **(ii)**, $\frac{\alpha+\beta+\gamma}{3} = \frac{p}{3} \geq \sqrt[3]{\alpha\beta\gamma} = \sqrt[3]{r} \therefore p \geq 3\sqrt[3]{r} \therefore p^3 \geq 27r$.

(iv) Let $p = 2, q = 1, r = 1$ for $x^3 - px^2 + qx - r = 0 \therefore x^3 - 2x^2 + x - 1 = 0$. Now for $x \leq 0, x^3 - 2x^2 + x - 1 < 0$. Hence all the real roots (if they exist) of $x^3 - 2x^2 + x - 1 = 0$ are positive. From **(iii)**, if it has 3 real roots, $p^3 = 8 > 27(1) = 27$. But $8 < 27$ - contradiction. $\therefore$ it has at most 2 real roots (one of which is a double root). If it has a double root, it must also be a root of $3x^2 - 4x + 1 = (3x-1)(x-1) = 0 \therefore x = 1$ or $\frac{1}{3}$. But $1^3 - 2(1)^2 + 1 - 1 = -1 \neq 0$ & $(\frac{1}{3})^3 - 2(\frac{1}{3})^2 + \frac{1}{3} - 1 = -\frac{23}{27} \neq 0$ either. $\therefore$ it does not have a double root. Or perhaps one might consider it having 2 real roots and a complex root, but then the polynomial would have complex coefficients, and we know already that the coefficients are real. Also, it won't have more than 3 roots by the fundamental theorem of algebra. It has real coefficients and odd degree and therefore it has exactly 1 real root and by the cubic formula that root is $\left(\sqrt[3]{100 + 12\sqrt{69}} + \sqrt[3]{100 - 12\sqrt{69}} + 4\right)\Big/6 \approx 1.7548776662....$

(b) **(i)** $AP.PB = (b\sec\theta - b\tan\theta)(b\tan\theta + b\sec\theta) = b^2(\sec^2\theta - \tan^2\theta) = b^2$

(ii) $\angle ACP + \beta = \alpha$ (ext. $\angle$ = sum of int. opp. $\angle$'s) $\therefore \angle ACP = \alpha - \beta$
& $\frac{AP}{\sin(\alpha-\beta)} = \frac{CP}{\sin\angle CAP} = \frac{CP}{\sin(\pi-\angle OAB)} = \frac{CP}{\sin\angle OAB} = \frac{CP}{\cos\beta}$
$\therefore CP = \frac{AP\cos\beta}{\sin(\alpha-\beta)}$
$\angle xOB = \beta$ (by symmetry) and if CD intersects the x-axis at W, $\angle OWD = \angle PWx = \alpha$ (vert. opp.).
$\therefore \angle PDB = \alpha + \beta$ (ext. $\angle$ = sum of int. opp. $\angle$'s).
$\angle OBA = \angle OAB = \frac{\pi}{2} - \beta \therefore \frac{PD}{\sin(\frac{\pi}{2}-\beta)} = \frac{PD}{\cos\beta} = \frac{PB}{\sin(\alpha+\beta)}$
$\therefore PD = \frac{PB\cos\beta}{\sin(\alpha+\beta)}$

(iii) $CP.PD = \frac{AP\cos\beta}{\sin(\alpha-\beta)} \cdot \frac{PB\cos\beta}{\sin(\alpha+\beta)} = \frac{b^2\cos^2\beta}{\sin(\alpha-\beta)\sin(\alpha+\beta)}$ and therefore depends only on the value of α and not the position of P.

(iv) $CP = p, QD = q, PQ = r$, so $CP.PD = p(q+r) = pq + pr = c$, a constant. Likewise, $DQ.QC = q(p+r) = pq + qr = c$. Hence $pq + pr = qp + qr$. $\therefore p = q$.

(v) $UT = \lim_{r\to 0} p = \lim_{r\to 0} q = VT$. $\therefore T$ is the midpoint of UV.

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