Alternative Solutions to Cambridge Extension 1 Year 12 Exercise 15E

by Derek Buchanan

March 8, 2020

I will use something outside the syllabus, namely the Penrose inverse for matrix equations for non-square coefficient matrices. Generally the line of best fit $y = mx + c$ for data $(x_1, y_1), \ldots, (x_n, y_n)$ can be found from the Penrose inverse $(A^T A)^{-1} A^T$ where $A = \begin{pmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{pmatrix}$ and A^T is the transpose of A, i.e., the matrix resulting from swapping rows and columns. Then $A \binom{m}{c} = b$ where $b = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$ and hence $\binom{m}{c} = (A^T A)^{-1} A^T b$.

1. $\binom{m}{c} = \left(\begin{pmatrix} -7 & 1 & 3 & 4 & 5 & 6 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -7 & 1 \\ 1 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \\ 6 & 1 \end{pmatrix} \right)^{-1} \begin{pmatrix} -7 & 1 & 3 & 4 & 5 & 6 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -3 \\ 0 \\ 2 \\ 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 53/112 \\ 3/56 \end{pmatrix}$ and hence $y = \frac{53}{112}x + \frac{3}{56} \approx 0.5x + 0.1$ (Note: The textbook (Pender, et al., 2019) has an incorrect answer $y = \frac{1}{2}x + 0$.)

2a. $\binom{m}{c} = \left(\begin{pmatrix} -2 & 0 & 1 & 3 & 4 & 6 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 0 & 1 \\ 1 & 1 \\ 3 & 1 \\ 4 & 1 \\ 6 & 1 \end{pmatrix} \right)^{-1} \begin{pmatrix} -2 & 0 & 1 & 3 & 4 & 6 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2/7 \\ 3/7 \end{pmatrix}$ and hence $y = \frac{2}{7}x + \frac{3}{7} \approx 0.3x + 0.4$

2b. $\binom{m}{c} = \left(\begin{pmatrix} -3 & -2 & 0 & 2 & 3 & 6 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -3 & 1 \\ -2 & 1 \\ 0 & 1 \\ 2 & 1 \\ 3 & 1 \\ 6 & 1 \end{pmatrix} \right)^{-1} \begin{pmatrix} -3 & -2 & 0 & 2 & 3 & 6 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -4 \\ -3 \\ 1 \\ 3 \\ 4 \\ 8 \end{pmatrix} = \begin{pmatrix} 75/56 \\ 9/56 \end{pmatrix}$ and hence $y = \frac{75}{56}x + \frac{9}{56} \approx 1.3x + 0.2$

2c. $\binom{m}{c} = \left(\begin{pmatrix} -4 & -2 & -1 & 0 & 1 & 2 & 4 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -4 & 1 \\ -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 4 & 1 \end{pmatrix} \right)^{-1} \begin{pmatrix} -4 & -2 & -1 & 0 & 1 & 2 & 4 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 7 \\ 6 \\ 1 \\ -1 \\ -3 \\ 1 \\ -4 \end{pmatrix} = \begin{pmatrix} -29/21 \\ 1 \end{pmatrix}$ and hence $y = -\frac{29}{21}x + 1 \approx -1.4x + 1.0$

2d. $\binom{m}{c} = \left(\begin{pmatrix} -2 & -1 & 0 & 1 & 2 & 6 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 6 & 1 \end{pmatrix} \right)^{-1} \begin{pmatrix} -2 & -1 & 0 & 1 & 2 & 6 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 6 \\ 3 \\ 4 \\ 2 \\ 0 \\ -3 \end{pmatrix} = \begin{pmatrix} -43/40 \\ 123/40 \end{pmatrix}$ and hence $y = -\frac{43}{40}x + \frac{123}{40} \approx -1.1x + 3.1$

Reference

Pender, B. et al., CambridgeMATHS Mathematics Extension 1 Year 12, Cambridge University Press, 2019.