SOME METHODS FOR CALCULATING STIFFNESS PROPERTIES OF PERIODIC STRUCTURES

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Abstract. We present a general numerical method for calculating effective elastic properties of periodic structures based on the homogenization method. Some concrete numerical examples are presented.

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1. Introduction

Numerical computations of the effective elastic moduli of heterogeneous structures have been considered in several papers (see e.g. [1], [3], [4], [5], [7], [11] and the references therein). Particularly in [11] some engineering and mathematical aspects of the homogenization method for such computations were discussed. In this paper we continue this discussion. The presentation of the theory concerning effective properties is made as simple as possible without involving complicated convergence processes. We focus on the fact that the formulations of the effective properties are quite natural from the physical point of view, something that is often hidden in modern mathematical literature of composite materials.

A general numerical method for the computation of effective elastic moduli of periodic composites is presented. This method is a variant of the "displacement method" given in [11]. We point out conditions under which the numerical treatment can be simplified. In particular, we study unidirectional fiber composites and discuss methods for deriving all elements of the corresponding effective stiffness matrix only by considering two-dimensional problems. Moreover, we consider conditions for which the periodic boundary conditions are reduced to piecewise constant Dirichlet boundary conditions and Neumann conditions, at least in a rotated coordinate system. In addition we present some concrete numerical examples where all the elastic moduli are computed.

2. Effective elastic moduli

For an isotropic material the shear modulus G and bulk modulus K in plane elasticity (plane strain) are related to the well known *Young's modulus* E and *Poisson's ratio* ν as follows:

$$K = \frac{E}{2(1+\nu)(1-2\nu)}, \quad G = \frac{E}{2(1+\nu)}.$$

The bulk modulus k of the three-dimensional theory is given by

$$k = \frac{E}{3}\frac{1}{1-2\nu}.$$

Thus the plane strain bulk modulus K can be expressed as

$$K = k + \frac{G}{3}.$$

If the material is a thin plate, we consider the plane-stress problem. In this case the plane stress bulk modulus is expressed as

$$K = \frac{E}{2}\frac{1}{1-\nu}.$$

The shear modulus G is independent of the dimension and also independent of whether we are dealing with the plane-strain or plane-stress problem.

Let us first consider the general case when the local stiffness matrix is symmetric and periodic relative to a cell $Y \subset \mathbb{R}^3$ and assumes the form

$$C = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1112} & C_{1123} & C_{1113} \\ C_{2211} & C_{2222} & C_{2233} & C_{2212} & C_{2223} & C_{2213} \\ C_{3311} & C_{3322} & C_{3333} & C_{3312} & C_{3323} & C_{3313} \\ C_{1211} & C_{1222} & C_{1233} & C_{1212} & C_{1223} & C_{1213} \\ C_{2311} & C_{2322} & C_{2333} & C_{2312} & C_{2323} & C_{2313} \\ C_{1311} & C_{1322} & C_{1333} & C_{1312} & C_{1323} & C_{1313} \end{bmatrix}.$$

Recall the cell problem corresponding to an average strain $\xi = \{\xi_{kl}\}$: Find the displacement

$$\mathbf{u}^\xi = \begin{bmatrix} u_1^\xi \\ u_2^\xi \\ u_3^\xi \end{bmatrix},$$

where $u_i^\xi \in W^{1,2}(Y)$ (the usual Sobolev space) with Y-periodic partial derivatives such that $\langle e_{kl}(\mathbf{u}^\xi)\rangle = \xi_{kl}$ and that for $i = 1, 2, 3$

$$\sum_j \frac{\partial}{\partial y_j}(\sigma_{ij}(\mathbf{u}^\xi)) = 0 \quad \text{(equilibrium of forces)},$$

where the stress $\sigma_{ij}(\mathbf{u}^\xi)$ is given by

$$\sigma_{ij}(\mathbf{u}^\xi) = \sum_{kl} C_{ijkl}(e_{kl}(\mathbf{u}^\xi)) \quad \text{(Hooke's Law)},$$

where the strain $e_{kl}(\mathbf{u})$ is given by

$$e_{kl}(\mathbf{u}) = \frac{1}{2}\Big(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\Big)$$

and $\langle\cdot\rangle$ denotes the average over the Y-cell. The effective stiffness parameters $\{C^*_{ijkl}\}$ are then found from the expression

$$\langle\sigma_{ij}(\mathbf{u}^\xi)\rangle = \sum_{kl} C^*_{ijkl}\langle e_{kl}(\mathbf{u}^\xi)\rangle. \tag{1}$$

It also appears that the following "energy" identity holds (see e.g. [8]):

$$\underbrace{\frac{1}{2}\sum_{ijkl}\langle e_{ij}(\mathbf{u}^\xi) C_{ijkl} e_{kl}(\mathbf{u}^\xi)\rangle}_{\text{average strain energy}} = \frac{1}{2}\sum_{ijkl}\langle e_{ij}(\mathbf{u}^\xi)\rangle C^*_{ijkl}\langle e_{kl}(\mathbf{u}^\xi)\rangle. \tag{2}$$

The solution $\mathbf{u}^\xi$ is understood as the (weak) solution of the weak formulation of the cell problem and does always exist under suitable conditions (see e.g. [8]). The strain $\{e_{kl}\}$ and the stress $\{\sigma_{ij}\}$ can be represented as symmetric matrices

$$e = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix}, \qquad \sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}.$$

This is used when C is represented as a 4th order tensor (as in (1) and (2)). The strain and stress can alternatively be represented as vectors

$$e = [e_{11}, e_{22}, e_{33}, \gamma_{12}, \gamma_{23}, \gamma_{13}]^T, \quad \gamma_{ij} = 2e_{ij}$$
$$\sigma = [\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{13}]^T.$$

This is used when C is represented as the above matrix, for which (1) and (2) takes the form

$$\langle \sigma(\mathbf{u}^\xi) \rangle = C^* \langle e(\mathbf{u}^\xi) \rangle \tag{3}$$

$$\underbrace{\frac{1}{2} \langle e(\mathbf{u}^\xi) \cdot Ce(\mathbf{u}^\xi) \rangle}_{\text{average strain energy}} = \frac{1}{2} \langle e(\mathbf{u}^\xi) \rangle \cdot C^* \langle e(\mathbf{u}^\xi) \rangle. \tag{4}$$

2.1. Physical interpretation.

In the mathematical theory of composites called the homogenization theory it is often usual to describe the above cell problem in a slightly different (but equivalent) way. Moreover, in the homogenization theory the cell problem is obtained as a consequence of a limiting process where the period of the structure is assumed to approach 0. However, it is important to observe that the cell problem formulated in the above way in itself serves as a natural definition of the effective parameters $\{C^*_{ijkl}\}$. For example, if we want to find C^*_{ij11} it would be natural to stretch the periodic structure in such a way that the average strain $\langle e \rangle = [1, 0, 0, 0, 0, 0]^T$, then measure the average stress $\langle \sigma \rangle = [\langle \sigma_{11} \rangle, \langle \sigma_{22} \rangle, \langle \sigma_{33} \rangle, \langle \sigma_{12} \rangle, \langle \sigma_{23} \rangle, \langle \sigma_{13} \rangle]^T$ and finally compute the coefficient C^*_{ij11} from the (effective) strain/stress relation

$$\begin{bmatrix} \langle \sigma_{11} \rangle \\ \langle \sigma_{22} \rangle \\ \langle \sigma_{33} \rangle \\ \langle \sigma_{12} \rangle \\ \langle \sigma_{23} \rangle \\ \langle \sigma_{13} \rangle \end{bmatrix} = \begin{bmatrix} C^*_{1111} & C^*_{1122} & C^*_{1133} & C^*_{1112} & C^*_{1123} & C^*_{1113} \\ C^*_{2211} & C^*_{2222} & C^*_{2233} & C^*_{2212} & C^*_{2223} & C^*_{2213} \\ C^*_{3311} & C^*_{3322} & C^*_{3333} & C^*_{3312} & C^*_{3323} & C^*_{3313} \\ C^*_{1211} & C^*_{1222} & C^*_{1233} & C^*_{1212} & C^*_{1223} & C^*_{1213} \\ C^*_{2311} & C^*_{2322} & C^*_{2333} & C^*_{2312} & C^*_{2323} & C^*_{2313} \\ C^*_{1311} & C^*_{1322} & C^*_{1333} & C^*_{1312} & C^*_{1323} & C^*_{1313} \end{bmatrix} \begin{bmatrix} \langle e_{11} \rangle \\ \langle e_{22} \rangle \\ \langle e_{33} \rangle \\ \langle \gamma_{12} \rangle \\ \langle \gamma_{23} \rangle \\ \langle \gamma_{13} \rangle \end{bmatrix}, \tag{5}$$

which gives $C^*_{ij11} = \langle \sigma_{ij} \rangle$.

2.2. Orthotropic composites.

Consider a linear elastic composite material in $\mathbb{R}^3$ with an effective stiffness matrix like that of an orthotropic material of the form

$$C^* = \begin{bmatrix} C^*_{1111} & C^*_{1122} & C^*_{1133} & 0 & 0 & 0 \\ C^*_{2211} & C^*_{2222} & C^*_{2233} & 0 & 0 & 0 \\ C^*_{3311} & C^*_{3322} & C^*_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & C^*_{1212} & 0 & 0 \\ 0 & 0 & 0 & 0 & C^*_{2323} & 0 \\ 0 & 0 & 0 & 0 & 0 & C^*_{1313} \end{bmatrix}.$$

The stiffness matrix is symmetric and the elements may be expressed as follows (see e.g. [15]):

$$\begin{aligned} &(6) \quad C^*_{1111} = \frac{1-\nu^*_{23}\nu^*_{32}}{\Delta E^*_2 E^*_3}, \quad C^*_{1122} = \frac{\nu^*_{21}+\nu^*_{31}\nu^*_{23}}{\Delta E^*_2 E^*_3}, \quad C^*_{1133} = \frac{\nu^*_{31}+\nu^*_{21}\nu^*_{32}}{\Delta E^*_2 E^*_3} \\ &\qquad C^*_{2222} = \frac{1-\nu^*_{13}\nu^*_{31}}{\Delta E^*_1 E^*_3}, \quad C^*_{2233} = \frac{\nu^*_{32}+\nu^*_{12}\nu^*_{31}}{\Delta E^*_1 E^*_3}, \quad C^*_{3333} = \frac{1-\nu^*_{12}\nu^*_{21}}{\Delta E^*_1 E^*_2} \\ &(7) \quad C^*_{1212} = G^*_{12}, \qquad C^*_{2323} = G^*_{23}, \qquad C^*_{1313} = G^*_{13}, \end{aligned}$$

where

$$\Delta = \frac{1-\nu^*_{12}\nu^*_{21}-\nu^*_{23}\nu^*_{32}-\nu^*_{31}\nu^*_{13}-2\nu^*_{21}\nu^*_{32}\nu^*_{13}}{E^*_1 E^*_2 E^*_3}.$$

Here, 'E^*_i' are the effective Youngs moduli, 'G^*_{ij}' are the effective shear moduli and 'ν^*_{ij}' are the effective Poisson's ratios. The inverse of the effective stiffness matrix (the complience matrix) which (certainly) also is symmetric is given by

$$\begin{bmatrix} 1/E^*_1 & -\nu^*_{12}/E^*_2 & -\nu^*_{13}/E^*_3 & 0 & 0 & 0 \\ -\nu^*_{21}/E^*_1 & 1/E^*_2 & -\nu^*_{23}/E^*_3 & 0 & 0 & 0 \\ -\nu^*_{31}/E^*_1 & -\nu^*_{32}/E^*_2 & 1/E^*_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G^*_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G^*_{23} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G^*_{13} \end{bmatrix}.$$

In the case of square honeycombs with locally isotropic material properties (local shear moduli G_o and G_I and local plane strain bulk moduli K_o and K_I with the corresponding volume fractions p_o and p_I, respectively, where the subscript o and I denote outer and inner material, respectively, see Fig. 1) the stiffness matrix reduces to a matrix of the form

$$
\begin{bmatrix}
K^* + G_T^* & K^* - G_T^* & l^* & 0 & 0 & 0 \\
K^* - G_T^* & K^* + G_T^* & l^* & 0 & 0 & 0 \\
l^* & l^* & n^* & 0 & 0 & 0 \\
0 & 0 & 0 & G_{T,45}^* & 0 & 0 \\
0 & 0 & 0 & 0 & G_L^* & 0 \\
0 & 0 & 0 & 0 & 0 & G_L^*
\end{bmatrix}. \tag{8}
$$

Here K^* is the effective transverse (also called “in-plane”) bulk modulus, G_T^*, $G_{T,45}^*$ are the effective transverse shear moduli and G_L^* is the longitudinal (also called “out-of plane”) shear modulus, see Fig. 2. In this case the complience matrix reduces to

$$
\begin{bmatrix}
1/E_T^* & -\nu_T^*/E_T^* & -\nu_L^*/E_L^* & 0 & 0 & 0 \\
-\nu_T^*/E_T^* & 1/E_T^* & -\nu_L^*/E_L^* & 0 & 0 & 0 \\
-\nu_L^*/E_L^* & -\nu_L^*/E_L^* & 1/E_L^* & 0 & 0 & 0 \\
0 & 0 & 0 & 1/G_{T,45}^* & 0 & 0 \\
0 & 0 & 0 & 0 & 1/G_L^* & 0 \\
0 & 0 & 0 & 0 & 0 & 1/G_L^*
\end{bmatrix}. \tag{9}
$$

Using (6) and the symmetry we obtain that

$$G_T^* = \frac{E_T^*}{2(1+\nu_T^*)}, \tag{10}$$

$$\frac{4}{E_T^*} = \frac{1}{G_T^*} + \frac{1}{K^*} + \frac{4(\nu_L^*)^2}{E_L^*} \tag{11}$$

and

$$l^* = \nu_L^* 2K^*, \quad n^* = E_L^* + 4(\nu_L^*)^2 K^*. \tag{12}$$

Moreover, it has been proved by Hill [6] that

$$E_L^* = p_o E_o + p_I E_I + \frac{4(\nu_o - \nu_I)^2}{\left(\frac{1}{K_o} - \frac{1}{K_I}\right)^2} \left(p_o \frac{1}{K_o} + p_I \frac{1}{K_I} - \frac{1}{K^*} \right), \tag{13}$$

$$\nu_L^* = p_o \nu_o + p_I \nu_I - \frac{\nu_o - \nu_I}{\frac{1}{K_o} - \frac{1}{K_I}} \left(p_o \frac{1}{K_o} + p_I \frac{1}{K_I} - \frac{1}{K^*} \right). \tag{14}$$

These two formulae were proved in [6] for the case of transverse isotropy. However, by following the proof in [6] it is easy to check that the same facts hold in our case.

We remark that (10), (11), (12), (13) and (14) hold for all two-component unidirectional fiber composites (oriented in the direction x_3) satisfying the property of

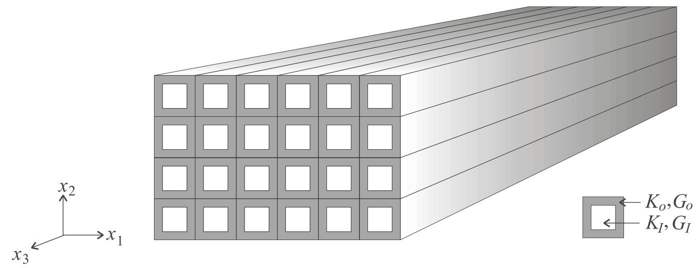

Figure 1. The structure of square honeycombs with locally isotropic material properties.

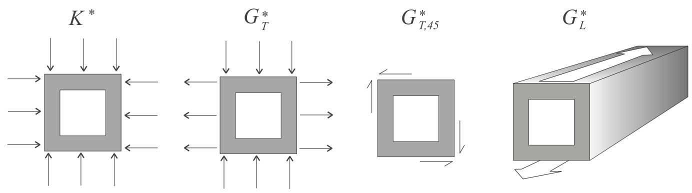

Figure 2. The 4 moduli measure resistence against the indicated average strains.

square symmetry, i.e. the case when the stiffness matrix is of the form (8). In order to compute all components of the stiffness matrix (9) we only have to compute 4 components (say G_L^*, G_T^*, $G_{T,45}^*$ and K^*). In the case of transverse isotropy (i.e. when $G_T^* = G_{T,45}^*$) this reduces to 3 components. The other two moduli l^* and n^* are then found by inserting the values of E_L^* (13) and ν_L^* (14) into (12). E_T^* and ν_T^* can be found by first evaluating E_T^* from (11) and finally ν_T^* from (10).

Note that G_L^* is found exactly as the effective conductivity in the similar 2-dimensional problem by letting G_o and G_I play the same role as the local conductivity for that problem (see e.g. [10]).

3. Numerical methods

In order to use conventional software to solve the cell problem numerically, e.g. by the finite element method, it is often necessary to "translate" the information of the strain into equivalent boundary conditions on the cell

$$Y = \prod_{j=1}^{n} \langle 0, y_j \rangle$$

for the displacement $\mathbf{u}^\xi$. This is easily obtained by letting each pair of points (x_-, x_+) (the latter point with the largest coordinates) on opposite faces with normal vector $\mathbf{n}_l$ be coupled to each other in such a way that

$$\mathbf{u}^\xi(x_+) = \mathbf{u}^\xi(x_-) + \mathbf{k}_l^\xi \qquad (15)$$

(together with putting $\mathbf{u}(0) = 0$ in order to obtain a unique solution) where

$$\mathbf{k}_l^\xi = \begin{bmatrix} k_{1l}^\xi \\ k_{2l}^\xi \\ k_{3l}^\xi \end{bmatrix}$$

is a constant vector. By integrating along the normal vector $\mathbf{n}_l$ we obtain that

$$\left\langle \frac{\partial u_k^\xi}{\partial x_l} \right\rangle = \frac{u_k^\xi(x_+) - u_k^\xi(x_-)}{y_l} = \frac{k_{kl}^\xi}{y_l},$$

thus, since $\langle e_{kl}(\mathbf{u}^\xi)\rangle = \xi_{kl}$, we obtain the useful relation

$$\frac{1}{2}\left(\frac{k_{kl}^\xi}{y_l} + \frac{k_{lk}^\xi}{y_k}\right) = \xi_{kl}. \tag{16}$$

Observe that k_{kl}^ξ and k_{lk}^ξ are not uniquely determined by this relation when $k \neq l$. Thus these constants can be chosen independently (as long as (16) is satisfied).

If the material properties are locally isotropic and symmetric in each coordinate with respect to the midpoint $((1/2)y_1, \ldots, (1/2)y_n)$ of the Y-cell problem (see Fig. 3) then we can often use simpler boundary conditions than (15).

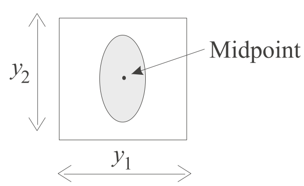

Figure 3. The Y-cell in the symmetric case.

For example, in the case when ξ is a normal strain, i.e. $\xi_{ij} = 0$ for $i \neq j$, we can in (15) put

$$u_l^\xi(x_-) = 0, \quad u_l^\xi(x_+) = k_{ll}^\xi, \quad l = 1, 2, 3,$$

and drop all the other boundary conditions. The latter is equivalent with setting a Neumann condition, $\partial u_i^\xi/\partial x_l = 0$, $i \neq l$ on the same faces. It is easy to see physically that these simplified boundary conditions hold by considering the deformation of the whole periodic structure, since it is obvious that the solution (which indeed represents the deformed body) must inherit the same symmetry as the material itself (see Fig. 4).

3.1. Coordinate transformation.

For the computation of effective moduli it is sometimes convenient to rotate the original coordinate system. Consider an orthonormal coordinate system with basis

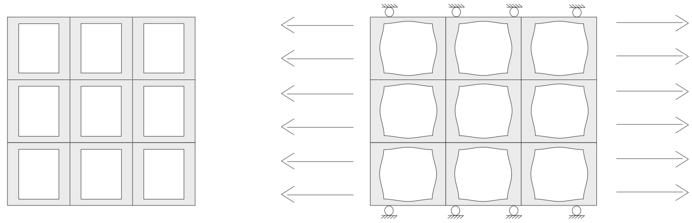

Figure 4. Deformation of the periodic structure in the symmetric case.

vectors

$$\mathbf{n}_1 = [n_{11}, n_{12}, n_{13}],$$
$$\mathbf{n}_2 = [n_{21}, n_{22}, n_{23}],$$
$$\mathbf{n}_3 = [n_{31}, n_{32}, n_{33}].$$

A vector with coordinates $x = (x_1, x_2, x_3)$ (relative to the usual coordinate system) will have coordinates $x' = (x_1', x_2', x_3')$ (relative to the new coordinate system) given by the relation

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \\ n_{31} & n_{32} & n_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(see Fig. 5). It is possible to show that the following relation between the strain $e = [e_{11}, e_{22}, e_{33,}\gamma_{12}, \gamma_{23,}\gamma_{13}]^T$, $\gamma_{ij} = 2e_{ij}$ (in the usual coordinate system) and $e' = [e_{11}', e_{22}', e_{33,}', \gamma_{12}', \gamma_{23}', \gamma_{13}']^T$, $\gamma_{ij}' = 2e_{ij}'$ (in the new coordinate system) holds

$$e' = \mathbf{T}e,$$

where

$$\mathbf{T} = \begin{bmatrix} n_{11}^2 & n_{12}^2 & n_{13}^2 & n_{11}n_{12} & n_{12}n_{13} & n_{11}n_{13} \\ n_{21}^2 & n_{22}^2 & n_{23}^2 & n_{21}n_{22} & n_{22}n_{23} & n_{21}n_{23} \\ n_{31}^2 & n_{32}^2 & n_{33}^2 & n_{31}n_{32} & n_{32}n_{33} & n_{31}n_{33} \\ 2n_{11}n_{21} & 2n_{12}n_{22} & 2n_{13}n_{23} & n_{11}n_{22}+n_{21}n_{12} & n_{12}n_{23}+n_{22}n_{13} & n_{11}n_{23}+n_{21}n_{13} \\ 2n_{21}n_{31} & 2n_{22}n_{32} & 2n_{23}n_{33} & n_{21}n_{32}+n_{31}n_{22} & n_{22}n_{33}+n_{32}n_{23} & n_{21}n_{33}+n_{31}n_{23} \\ 2n_{11}n_{31} & 2n_{12}n_{32} & 2n_{13}n_{33} & n_{11}n_{32}+n_{31}n_{12} & n_{12}n_{33}+n_{32}n_{13} & n_{11}n_{33}+n_{31}n_{13} \end{bmatrix}.$$

Moreover, we can obtain a similar relation between the corresponding stresses σ and σ':

$$\sigma' = \mathbf{T}^{-T}\sigma,$$

where $\mathbf{T}^{-T}$ is obtained from $\mathbf{T}$ by changing the factors of 2 in $\mathbf{T}$ symmetrically about the diagonal, i.e.

$$\mathbf{T}^{-T} = \begin{bmatrix} n_{11}^2 & n_{12}^2 & n_{13}^2 & 2n_{11}n_{12} & 2n_{12}n_{13} & 2n_{11}n_{13} \\ n_{21}^2 & n_{22}^2 & n_{23}^2 & 2n_{21}n_{22} & 2n_{22}n_{23} & 2n_{21}n_{23} \\ n_{31}^2 & n_{32}^2 & n_{33}^2 & 2n_{31}n_{32} & 2n_{32}n_{33} & 2n_{31}n_{33} \\ n_{11}n_{21} & n_{12}n_{22} & n_{13}n_{23} & n_{11}n_{22}+n_{21}n_{12} & n_{12}n_{23}+n_{22}n_{13} & n_{11}n_{23}+n_{21}n_{13} \\ n_{21}n_{31} & n_{22}n_{32} & n_{23}n_{33} & n_{21}n_{32}+n_{31}n_{22} & n_{22}n_{33}+n_{32}n_{23} & n_{21}n_{33}+n_{31}n_{23} \\ n_{11}n_{31} & n_{12}n_{32} & n_{13}n_{33} & n_{11}n_{32}+n_{31}n_{12} & n_{12}n_{33}+n_{32}n_{13} & n_{11}n_{33}+n_{31}n_{13} \end{bmatrix}.$$

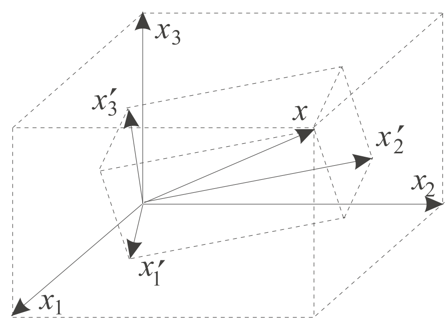

Figure 5. The two coordinate systems.

The stress-strain relation in the new coordinate system can therefore be written as

$$\sigma' = C'e',$$

where

$$C = \mathbf{T}^T C' \mathbf{T}.$$

Concerning these facts we refer e.g. to [2, p. 212].

In the plane-strain case we put all strains related to the x_3-variable equal to 0. If we also assume that the new coordinate system is obtained from the standard one by a rotation in the x_1-x_2-plane then we obtain the following simplified relations:

$$C = \begin{bmatrix} C_{1111} & C_{1122} & C_{1112} \\ C_{2211} & C_{2222} & C_{2212} \\ C_{1211} & C_{1222} & C_{1212} \end{bmatrix},$$

$$\mathbf{T} = \begin{bmatrix} n_{11}^2 & n_{12}^2 & n_{11}n_{12} \\ n_{21}^2 & n_{22}^2 & n_{21}n_{22} \\ 2n_{11}n_{21} & 2n_{12}n_{22} & n_{11}n_{22} + n_{21}n_{12} \end{bmatrix},$$

$$\mathbf{T}^{-T} = \begin{bmatrix} n_{11}^2 & n_{12}^2 & 2n_{11}n_{12} \\ n_{21}^2 & n_{22}^2 & 2n_{21}n_{22} \\ n_{11}n_{21} & n_{12}n_{22} & n_{11}n_{22} + n_{21}n_{12} \end{bmatrix}.$$

As an example, let us consider the square symmetric case, i.e. let the stiffness matrix be of the form

$$C^* = \begin{bmatrix} K^* + G_T^* & K^* - G_T^* & 0 \\ K^* - G_T^* & K^* + G_T^* & 0 \\ 0 & 0 & G_{T,45}^* \end{bmatrix}$$

(cf. (8)). Performing a rotation by 45°, i.e. $[n_{11}, n_{12}] = [\sqrt{2}/2, \sqrt{2}/2]$ and $[n_{21}, n_{22}] = [-\sqrt{2}/2, \sqrt{2}/2]$, the corresponding stiffness matrix in the rotated coordinate system becomes

$$C^{*\prime} = \mathbf{T}^{-T} C^* \mathbf{T}^{-1} = \begin{bmatrix} K^* + G_{T,45}^* & K^* - G_{T,45}^* & 0 \\ K^* - G_{T,45}^* & K^* + G_{T,45}^* & 0 \\ 0 & 0 & G_T^* \end{bmatrix},$$

i.e. $C^{*\prime}$ is the same as C^* except that the shear moduli $G_{T,45}^*$ and G_T^* have changed place. This explains the use of the index "45" and shows that we can calculate $G_{T,45}^*$ exactly as we calculate G_T^* except for rotating the coordinate system by 45°.

4. A computational example

In the case of square honeycombs with locally isotropic material properties (see Fig. 1) the structure is symmetric with respect to the midpoint of the Y-cell $Y = [-1, 1]^3$ in all the coordinates x_1, x_2, x_3, and also in the coordinates x_1', x_2', x_3 in the coordinate system obtained by a rotation of 45° in the x_1-x_2-plane. The effective stress/strain-relation (5) reduces in this case to

$$(17) \qquad \begin{bmatrix} \langle\sigma_{11}\rangle \\ \langle\sigma_{22}\rangle \\ \langle\sigma_{33}\rangle \\ \langle\sigma_{12}\rangle \\ \langle\sigma_{23}\rangle \\ \langle\sigma_{13}\rangle \end{bmatrix} = \begin{bmatrix} K^* + G_T^* & K^* - G_T^* & l^* & 0 & 0 & 0 \\ K^* - G_T^* & K^* + G_T^* & l^* & 0 & 0 & 0 \\ l^* & l^* & n^* & 0 & 0 & 0 \\ 0 & 0 & 0 & G_{T,45}^* & 0 & 0 \\ 0 & 0 & 0 & 0 & G_L^* & 0 \\ 0 & 0 & 0 & 0 & 0 & G_L^* \end{bmatrix} \begin{bmatrix} \langle e_{11}\rangle \\ \langle e_{22}\rangle \\ \langle e_{33}\rangle \\ \langle\gamma_{12}\rangle \\ \langle\gamma_{23}\rangle \\ \langle\gamma_{13}\rangle \end{bmatrix}.$$

In order to compute the effective in-plane moduli K^* and G_T^* we solve the cell problem for $\langle e\rangle = [1, 1, 0, 0, 0, 0]^T$ and $\langle e\rangle = [1, -1, 0, 0, 0, 0]^T$ and compute the corresponding values $K^* = \langle\sigma_{11}\rangle/2$ and $G_T^* = \langle\sigma_{11}\rangle/2$, respectively. Alternatively we can compute the average strain energy W and compute K^* and G_T^* from the identity (see (4))

$$W = \frac{1}{2}\langle e\rangle \cdot C^*\langle e\rangle,$$

which gives $K^* = W/2$ and $G_T^* = W/2$, respectively. For the computation it is enough to solve the two-dimensional cell problem using the plane strain. This is due to the fact that the solution of the three dimensional cell problem in both cases will be independent of the x_3-variable and we do not use any information of the strain or stresses in the x_3-direction for the computation of K^* and G_T^*. Because of the symmetries it suffices to solve the problem on 1/4 of the Y-cell, e.g. on the square $[0,1]^2$, and we can use uniform boundary conditions on each face of this square (see Section 3), i.e., for $\langle e \rangle = [1,1,0,0,0,0]^T$ the boundary conditions are

$$\begin{aligned} u_1(0,x_2) &= u_2(x_1,0) = 0, \\ u_1(1,x_2) &= u_2(x_1,1) = 1 \end{aligned}$$

and for $\langle e \rangle = [1,-1,0,0,0,0]^T$

$$\begin{aligned} u_1(0,x_2) &= u_2(x_1,0) = 0, \\ u_1(1,x_2) &= u_2(x_1,1) = -1. \end{aligned}$$

Due to symmetries the modulus $G_{T,45}^*$ can be found exactly as G_T^* except that we must rotate the coordinate system by 45° (see the last part of Section 3.1). The modulus G_L^* can be found by using $\langle e \rangle = [0,0,0,0,0,1]^T$ and computing the corresponding value $G_L^* = \langle \sigma_{13} \rangle$. The problem with doing this is that it often requires a full 3D FEM computation regardless of the fact that the solution of the cell problem is independent of the x_3-variable. Therefore it is often easier to solve the corresponding 2D heat-conductivity problem (see Remark 2.2). For more detailed information on computations of G_L^* in the case of honeycomb structures we refer to [12] and [13]. Once we have computed the 4 moduli K^*, G_T^*, $G_{T,45}^*$ and G_L^* we can easily obtain all the other moduli (see Remark 2.2).

As an example, let $p_o = p_I = 1/2$, $\nu_I = \nu_o = 0.3$, $E_I = 0.5$, $E_o = 1$ or equivalently $K_I = 0.48076923$, $K_o = 0.96153846$, $G_I = 0.19230769$, $G_o = 0.38461538$. By performing a FEM-calculation according to the above method we obtain the following effective moduli:

$$\begin{aligned} K^* &= 0.658455, \\ G_T^* &= 0.273775, \\ G_{T,45}^* &= 0.259418, \\ G_L^* &= 0.274426. \end{aligned}$$

Hence, by using (10), (11), (13) and (14) we obtain that

$$\nu_L^* = 0.3,$$
$$E_L^* = 0.75,$$
$$E_T^* = 0.70779659,$$
$$\nu_T^* = 0.29266111$$

and by (12)

$$l^* = 0.39507,$$
$$n^* = 0.98704.$$

Thus the effective stiffness matrix is written as

$$\begin{bmatrix} 0.932226 & 0.384677 & 0.39507 & 0 & 0 & 0 \\ 0.384677 & 0.932226 & 0.39507 & 0 & 0 & 0 \\ 0.395072 & 0.395072 & 0.98704 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.259418 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.274426 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.274426 \end{bmatrix}$$

and the effective complience matrix (9) takes the form

$$\begin{bmatrix} 1.41284 & -0.413482 & -0.4 & 0 & 0 & 0 \\ -0.413482 & 1.41284 & -0.4 & 0 & 0 & 0 \\ -0.4 & -0.4 & 1.33333 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3.8548 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3.64397 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3.64397 \end{bmatrix}.$$

In this example we have used the same Poisson's ratios in both phases. Let us consider the case when $\nu_I = 0.4$, $\nu_o = 0$, $E_I = 0.5$, $E_o = 1$ or equivalently $K_I = 0.89285714$, $K_o = 0.5$, $G_I = 0.17857143$, $G_o = 0.5$. In this case we obtain the following effective moduli:

$K^* = 0.66335$	$\nu_L^* = 0.22386372$
$G_T^* = 0.3038805$	$E_L^* = 0.79338859$
$G_{T,45}^* = 0.269322$	$E_T^* = 0.79193337$
$G_L^* = 0.307502$	$\nu_T^* = 0.3030342$

R e m a r k 4.1. In the first example (where Poisson's ratios are the same in both phases) ν_L^* and E_L^* equal the arithmetic mean of the corresponding phase properties (usually referred to as "the law of mixtures"). From the last example we observe that this need not be true when Poisson's ratios of the phases are different.

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