On the solution of a four phase high contrast conductivity problem

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In this article we study the problem of finding the temperature distribution and effective conductivity for a domain divided in four subdomains, see Figure 1. The local conductivity depends on a variable k that is large. We see that the local conductivity function λ is constant equal to kb_1 in Ω_1, b_2/k in Ω_2, kb_3 in Ω_3 and b_4/k in Ω_4. We are searching for an estimate of the problem

$$\min_{w \in W_0^{1,2}(\Omega)} \int_\Omega \lambda(x) | \nabla (u+w)|^2 \mathrm{d}x, \tag{1}$$

where u is a function in $W^{1,2}(\Omega)$ satisfying the boundary conditions seen in Figure 1. This is the variational formulation of the problem

$$-\mathrm{div} \lambda \nabla u = 0.$$

If we try to find an approximation of (1) in some finite dimensional subspace of $W_0^{1,2}(\Omega)$, we see that (1) tends to infinity as k tends to infinity. This is due to the fact that finite dimensional spaces are closed and that our exact solutions tends to something that is larger than zero. See [1] for a numerical example.

The solution of our problem tends to the same limit as Mortola–Steffes [12] conjecture. Mortola and Steffe conjectured the homogenized effective conductivity of a four phase checkerboard. The proof was later found independently by Milton [11] and Craster & Obnosov [6]. In fact the effective conductivity (=energy) of our problem tends to the same limit as Mortola–Steffes conjecture. We are not surprised that our solution tends to this limit, since the energy concentrates around the point where the four phases meets, as k becomes large, hence our solution became increasingly less dependent on other parts of the domain. Theorem 3 is an example of this property. It is also interesting that we can give a very explicit estimate how fast the approximated energy will tend to the exact energy as $k \to \infty$. The ideas we use are relatively uncomplicated and it is not difficult to propose further generalizations. As an example, we suggest a very reasonable way to our result to three dimensions, but was not able to find a satisfactory proof that the approximated energy tends to the same limit as the exact solution as $k \to \infty$.

It is interesting to compare with an article of Keller [9] often referred to by later articles treating high contrast problems in conductivity. Here some less general cases (and much simpler cases, see also [1]) with only two phases were treated; a rectangular shaped and a paralellogram shaped checkerboard. Keller approximated the effective conductivity for these structures, when the conductivity ratio was large between the phases. This in some sense extended an earlier result of Dykhne [4], showing the exact effective conductivity of a square checkerboard with two phases.

Figure 1: Ω

1 The problem in two dimensions (See Figure 1)

We will use polar coordinates (r, θ) unless otherwise said. The considered domain Ω is a disc with radius 1, centered in the origin. The local conductivity function λ_k has the value $b_1 k$ in Ω_1, b_2/k in Ω_2, $b_3 k$ in Ω_3 and b_4/k in Ω_4. From elementary calculus we have the following relation

$$\nabla u = \Big[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\Big] = \Big[\frac{\partial u}{\partial r} \;\; \frac{\partial u}{r\partial \theta}\Big],$$

where $x = r\cos(\theta)$ and $y = r\sin(\theta)$. For u in $W^{1,2}(\Omega)$, we define the energy functional

$$E_k(u) = \int_\Omega \lambda_k(r,\theta)|\nabla u(r,\theta)|^2 \mathrm{d}\Omega. \tag{2}$$

For large k's, we will look for an approximation to the minimizer u_k of E_k in $W^{1,2}(\Omega)$, satisfying the boundary conditions in Figure 1. Suppose that q is in $W^{1,2}(\Omega)$. We will say that u is in $H_q(\Omega)$ if there exists a v in $W_0^{1,2}(\Omega)$, and, a constant c, such that u is the sum of q, v and c.

2 Results in the two dimensional case

Theorem 1. *Let Λ be a subset of $W^{1,2}(\Omega)$ such that for every $v \in W^{1,2}(\Omega)$, we have $v \in \Lambda$ if and only if the following conditions are satisfied:*

- *The function v satisfies the boundary conditions in Figure 1 up to a constant translation.*
- $\frac{\partial v}{\partial \theta} = 0$ *in Ω_1 and Ω_3.*
- $\frac{\partial v(r,\theta)}{\partial \theta}$ *is constant for each fixed r when $(r,\theta) \in (\Omega_2 \cup \Omega_4)$.*

Let u_k be a minimizer of E_k. If v_k is a minimizer of E_k in Λ, then

$$\tau(k)E_k(v_k) - \frac{3E_k(u_k)}{2k^2}\Big(\frac{b_2}{b_1} + \frac{b_2}{b_3} + \frac{b_4}{b_1} + \frac{b_4}{b_3}\Big) \le E_k(u_k),$$

where

$$\tau(k) = \frac{A_1^2 A_3^2}{(A_1 + A_2)(A_1 + A_4)(A_3 + A_2)(A_3 + A_4)}, \tag{3}$$

and

$$A_1 = \frac{b_1 k}{\theta_1},\ A_2 = \frac{b_2}{\theta_2 k},\ A_3 = \frac{b_3 k}{\theta_3} \ \ \textit{and} \ \ A_4 = \frac{b_4}{\theta_4 k}.$$

Observe that $\tau(k) \to 1$ as $k \to \infty$. What we want to know is if $|E(v_k) - E(u_k)| \to 0$ as k tends to infinity. The next theorem is useful to answer this question. First we should observe that from the definition of Λ, if v is in Λ, we know v for all of Ω if we know v only for two fixed angles θ_1' and θ_2', where $0 < \theta_1' < \theta_1$ and $\theta_1 + \theta_2 < \theta_2' < \theta_1 + \theta_2 + \theta_3$. This property of Λ makes it easier for us to derive some explicit results:

Theorem 2. *Let $S_k : W^{1,2}(\Omega) \to \mathbf{R}$ be defined as*

$$S_k(v) = E_k(v) - \int_{\Omega_2 \cup \Omega_4} \lambda_k(r,\theta)\Big(\frac{\partial v}{\partial r}\Big)^2 r \mathrm{d}r \mathrm{d}\theta.$$

If v is in Λ, then it follows that $S_k(v)$ is the radial energy of v. Define

$$v_k(r,\theta) = \frac{\phi_2}{\phi_1+\phi_2} r^{\frac{\phi_3(\phi 1+\phi_2)}{\phi_1\phi_2}} \quad in \quad \Omega_1,$$

and

$$v_k(r,\theta) = -\frac{\phi_1}{\phi_1+\phi_2} r^{\frac{\phi_3(\phi 1+\phi_2)}{\phi_1\phi_2}} \quad in \quad \Omega_3,$$

where

$$\phi_1 = kb_1\theta_1\,,\ \phi_3 = kb_3\theta_3 \quad and \quad \phi_2 = \frac{1}{k}\Big(\frac{b_2}{\theta_2}+\frac{b_4}{\theta_4}\Big).$$

Then v_k is a minimizer of S_k in Λ.

Remark 1. *Let $a = \frac{\phi_3(\phi 1+\phi_2)}{\phi_1\phi_2}$. A quick computation gives*

$$\int_{\Omega_2} \frac{b_2}{k}\Big(\frac{\partial v_k}{r\partial\theta}\Big)^2 r\mathrm{d}r\mathrm{d}\theta < \frac{\theta_2 b_2}{k}\int_0^1 (2ar^{a-1})^2 r\mathrm{d}r\mathrm{d}\theta < \frac{C}{k^2}.$$

Thus

$$|inf_{v\in\Lambda}S_k(v) - inf_{v\in\Lambda}E_k(v)| \to 0 \quad as \quad k\to\infty.$$

We conclude that $|E(v_k) - E(u_k)| \to 0$ as k tends to infinity.

Remark 2. *Let v_k be defined as in Theorem 2. Computations then gives*

$$S_k(v_k) = \sqrt{\frac{b_1b_3(b_2+b_4)}{b_1+b_3}}. \tag{4}$$

This formula has the the same form as the limit case of Mortola–Steffes conjecture. Observe that the right hand side of (4) does not depend on k.

3 A result for other shapes of the domain

There are many ways to generalize Theorem 2. Here we give only one example in the two dimensional case.

Theorem 3. *Let us use a Cartesian coordinate system. Let λ_k be the local conductivity function with $\lambda_k(x,y) = b_1k$ for $x,y > 0$, $\lambda_k(x,y) = b_2/k$ for $x,-y < 0$, $\lambda_k(x,y) = b_3k$ for $x,y < 0$ and $\lambda_k(x,y) = b_4/k$ for $x,-y > 0$. Let Ω be an open, bounded and connected domain in $\mathbf{R}^2$, with Lipschitz boundary and containing the origin. Let E_k be given by*

$$E_k(u) = \int_\Omega \lambda_k(x,y)|\bigtriangledown u|^2\mathrm{d}x.$$

Let q be a function in $W^{1,2}(\Omega)$ with $q = 1$ on the part of $\partial\Omega$ with $x, y > 0$,and, $q = 0$ on the part of $\partial\Omega$ with $x, y < 0$. Let E_k be given by

$$E_k(u) = \int_\Omega \lambda_k(x, y)| \nabla u|^2 \mathrm{d}x,$$

for $u \in W^{1,2}(\Omega)$. Then for u_k in $H^1_q(\Omega)$, we have

$$\min_{u_k \in H^1_q(\Omega)} E_k(u_k) \to \sqrt{\frac{b_1 b_3 b_2 + b_1 b_3 b_4}{b_1 + b_3}} \quad as \quad k \to \infty. \tag{5}$$

Proof. Let R_1 denote the region where $x, y > 0$, R_2 denote the region where $x, -y < 0$, R_3 denote the region where $x, y < 0$ and R_4 denote the region where $x, -y > 0$. We want to find a lower bound for E_k in $H^1_q(\Omega)$. Let D_1 and D_2 be two discs centred in the origin such that D_1 is a subset of Ω and Ω is a subset of D_2. Let us define

$$\Psi = \Big\{(D_2 \cap R_1) \cup (D_2 \cap R_3) \cup (D_1 \cap R_2) \cup (D_1 \cap R_4)\Big\}.$$

Let s be a function in $W^{1,2}(\Psi)$ such that $s = 1$ on the part of $\partial\Psi$ contained in R_1 and $s = 0$ on the part of $\partial\Psi$ contained in R_3. Let u_k be any function in $H^1_q(\Omega)$. Now let w_k be a function in $W^{1,2}(\Psi)$ such that $w_k = u_k$ on the part where Ω and Ψ overlap, and, such that w_k is constant on those parts of Ψ that do not overlap Ω. Clearly w_k is in $H^1_s(\Psi)$. Moreover, $E^\Psi_k(w_k) \leq E_k(u_k)$. By using similar methods as in the proof of Theorem 1 and Theorem 2, it is easy to show that

$$\lim_{k\to\infty} \inf_{w \in H^1_s(\Omega)} E_k(w) = \sqrt{\frac{b_1 b_3 b_2 + b_1 b_3 b_4}{b_1 + b_3}}.$$

Next, we want to find an upper bound for E_k in $H^1_q(\Omega)$. We will use polar coordinates. Let r' be the radius of D_1. In Theorem 2 we defined a function v_k. Define $v'_k(r, \theta)$ in D_1 as equal to $v_k(r, \theta)$ divided by $v_k(r', \theta)$. Then v'_k equals 1 on the part of ∂D_1 whics is in R_1 and v'_k equals 0 on the part of ∂D_1 which is in R_3. Then we can extend v'_k to a function in $H^1_q(\Omega)$, still labeled v'_k, such that

$$E_k(v'_k) = E^{D_1}_k(v'_k) + C/k,$$

where C does not depend on k and is equal to

$$C = b_2 \int_{R_2 \cap (\Omega/D_1)} | \nabla v'_k|^2 r \mathrm{d}r \mathrm{d}\theta + b_4 \int_{R_4 \cap (\Omega/D_1)} | \nabla v'_k|^2 r \mathrm{d}r \mathrm{d}\theta.$$

Obviously $E_k(v'_k)$ is an upper bound for $E_k(u_k)$. Moreover, we can easily verify that

$$\lim_{k\to\infty} E_k(v'_k) = \sqrt{\frac{b_1b_3b_2 + b_1b_3b_4}{b_1 + b_3}}.$$

□

4 A result in the 3 dimensional case

We suggest how the results in the 2–dimensional case can be extended to 3 dimensions. First we need to redefine the domain for our problem. Let Ω be a ball in $\mathbf{R}^3$ with radius $r = 1$ and with center in $r = 0$. Let Ω_1^s and Ω_2^s be sets on the boundary $\partial\Omega$ of Ω with $\mathrm{dist}(\Omega_1^s, \Omega_2^s) = d > 0$. Also suppose that Ω_1^s and Ω_2^s are open and connected sets with Lipscitz boundary, relative to the surface of Ω (not relative to $\mathbf{R}^3$). Let $\Omega_3^s = \partial\Omega \cap (\Omega_1^s \cup \Omega_2^s)^c$. Put $\Omega_i = \{(r, \theta, \phi) : (1, \theta, \phi) \in \Omega_i^s \text{ and } 0 < r < 1\}$ for $i = 1, 2, 3$. Define

$$|\Omega_i^s| = \int_{\Omega_i^s} \cos(\phi)\mathrm{d}\phi\mathrm{d}\theta,$$

for $i = 1, 2, 3$. Let λ_k be the local conductivity function with $\lambda_k(x) = b_1k$ in Ω_1, $\lambda_k(x) = b_2k$ in Ω_2 and $\lambda_k(x) = b_3/k$ in Ω_3. For $u \in W^{1,2}(\Omega)$ define the energy functional E_k as

$$E_k(u) = \int_\Omega \lambda_k(r, \theta, \phi)|\nabla u|^2 r^2 \cos(\phi)\mathrm{d}\theta\mathrm{d}\phi. \tag{6}$$

Let $w = u + q$, where u is the solution to the problem

$$E_1 = \min_{u\in H_q(\Omega_3^s)} \Big\{ \int_{\Omega_3^s} \Big(\Big(\frac{\partial u}{\partial\theta}\Big)^2 + \Big(\frac{\partial u}{\partial\phi}\Big)^2\Big) \cos(\phi)\mathrm{d}\theta\mathrm{d}\phi\Big\},$$

where q is in $W^{1,2}(\Omega)$, and, $q = 0$ on Ω_1^s and $q = 1$ on Ω_2^s (in the sense of trace). We do not know the analytic solution of the above problem in general. Now we define the approximation space:

Definition 1. *Let $u \in \Lambda^{3D}$ if and only if*

- $u \in H_q(\Omega)$.
- *$u(r, \theta, \phi)$ is constant with respect to θ and ϕ, for fixed r in Ω_1 and Ω_2.*
- *For $0 < r < 1$, let $(r, \theta_1, \phi_1) \in \Omega_1$ and $(r, \theta_2, \phi_2) \in \Omega_2$. Then for every $(r, \theta, \phi) \in \Omega_2$, we define $u(r, \theta, \phi) = (u(r, \theta_2, \phi_2) - u(r, \theta_1, \phi_1))w(\theta, \phi)$.*

Let $(1,\theta_1,\phi_1) \in \Omega_1^s$ and $(1,\theta_2,\phi_2) \in \Omega_2^s$. We see that any $u \in \Lambda^{3D}$ is uniquely defined by $u(x_1(r))$ and $u(x_2(r))$, where $x_1(r) = (r,\theta_1,\phi_1)$ and $x_2(r) = (r,\theta_2,\phi_2)$. For $u_k \in \Lambda^{3D}$, we define the functional T_k as

$$T_k(u) = \int_{\Omega_3} \lambda_k(r,\theta,\phi)\Big((\frac{\partial u}{r\partial\theta})^2 + (\frac{\partial u}{r\partial\phi})^2\Big)r^2\cos(\phi)\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi +$$

$$\int_{\Omega_1\cup\Omega_2} \lambda_k(r,\theta,\phi)\Big(\frac{\partial u_k}{\partial r}\Big)^2 r^2\cos(\phi)\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi$$

$$= \int_0^1 \Bigg(b_1k|\Omega_1^s|\Big(\frac{\partial u}{\partial r}\Big)^2 + b_2k|\Omega_2^s|\Big(\frac{\partial u}{\partial r}\Big)^2 + \frac{b_2E_1}{|\Omega_3^s|k}\Big(\frac{u(x_2(r)) - u(x_1(r))}{r}\Big)^2\Bigg)r\mathrm{d}r$$

Then by the steps from the proof of Theorem 2, we have

Theorem 4. *Define*

$$S_k(u_k) = E_k(u_k) - \int_{\Omega_3} \lambda_k(r,\theta,\phi)\Big(\Big(\frac{\partial u_k}{r\partial\theta}\Big)^2 + \Big(\frac{\partial u_k}{r\partial\phi}\Big)^2\Big)r^2\cos(\phi)\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi.$$

If u_k is a minimizer of S_k in Λ^{3D}, then

$$u_k(x_1(r)) + c = \frac{\rho_2}{\rho_1+\rho_2}r^{\frac{\rho_3(\rho_1+\rho_2)}{\rho_1\rho_2}} \quad \text{and} \quad u_k(x_2(r)) + c = -\frac{\rho_1}{\rho_1+\rho_2}r^{\frac{\rho_3(\rho_1+\rho_2)}{\rho_1\rho_2}},$$

where $\rho_1 = kb_1|\Omega_1^s|$, $\rho_2 = kb_2|\Omega_2^s|$ and $\rho_3 = \frac{b_3E_1}{k|\Omega_3^s|}$ and c is just some constant to assure that $u_k \in \Lambda^{3D}$.

Observe that by the *Maximum Modulus* principle, we have $0 \le w(\theta,\phi) \le 1$ for $(r,\theta,\phi) \in \Omega_3$. It follows that

$$u(r,\theta,\phi) = w(\theta,\phi)u(x_1(r)) + (1 - w(\theta,\phi))u(x_2(r)),$$

for $(r,\theta,\phi) \in \Omega_2$. Thus we can repeat the calculations we did for the 2-dimensional example to see that

$$|\min_{u\in\Lambda^{3D}} T_k(u) - \min_{u\in\Lambda^{3D}} E_k(u)| \to 0 \quad \text{as} \quad k \to \infty.$$

Is it true then that

$$|\min_{u\in\Lambda^{3D}} E_k(u_k) - \min_{u\in W_0^{1,2}(\Omega)} E_k(u+q)| \to 0 \quad \text{as} \quad k \to \infty$$

holds? It would be a very reasonable assumption, but to find a rigid proof is more complicated.

5 Proof of Theorem 1 and Theorem 2

5.1 Proof of Theorem 1

Proof. Let u_k be a minimizer of E_k. Define ω_1 and ω_2 such that

$$\int_0^1 \Big(\frac{\partial u_k(r,\omega_1)}{\partial r}\Big)^2 r\mathrm{d}r = \min_{\omega\in[0,\theta_1]} \int_0^1 \Big(\frac{\partial u_k(r,\omega)}{\partial r}\Big)^2 r\mathrm{d}r$$

$$\int_0^1 \Big(\frac{\partial u_k(r,\omega_2+\theta_1+\theta_2)}{\partial r}\Big)^2 r\mathrm{d}r = \min_{\omega\in[\theta_1+\theta_2,\theta_1+\theta_2+\theta_3]} \int_0^1 \Big(\frac{\partial u_k(r,\omega)}{\partial r}\Big)^2 r\mathrm{d}r.$$

Fix r' such that $0 < r' < 1$. We have

$$\int_{\omega_1}^{\theta_1} \Big(\frac{\partial u_k}{r'\partial\theta}\Big)^2 \mathrm{d}\theta = \big(\theta_1-\omega_1\big)\int_{\omega_1}^{\theta_1} \Big(\frac{\partial u}{r'\partial\theta}\Big)^2 \frac{\mathrm{d}\theta}{\theta_1-\omega_1}.$$

Using $\mathrm{d}\theta/(\theta_1-\omega_1)$ as a probability measure, we obtain from Jensens inequality that

$$\big(\theta_1-\omega_1\big)\Big(\int_{\omega_1}^{\theta_1} \frac{\partial u}{r'\partial\theta}\frac{\mathrm{d}\theta}{\theta_1-\omega_1}\Big)^2 \le \big(\theta_1-\omega_1\big)\int_{\omega_1}^{\theta_1} \Big(\frac{\partial u}{r'\partial\theta}\Big)^2 \frac{r'\mathrm{d}\theta}{\theta_1-\omega_1}.$$

Thus

$$\frac{(u_k(r',\theta_1)-u_k(r',\omega_1))^2}{r'(\theta_1-\omega_1)} \le \int_{\omega_1}^{\theta_1} \Big(\frac{\partial u_k(r',\theta)}{\partial\theta}\Big)^2 \mathrm{d}\theta.$$

By repeating the above argument a number of times we obtain

$$\begin{aligned} r'\int_0^{2\pi} \lambda_k(r',\theta)\Big(\frac{\partial u_k(r,\theta)}{r'\partial\theta}\Big)^2 r'\mathrm{d}\theta \ge{}& \frac{a_1(u_0-u_{11})^2}{\omega_1}+\frac{a_1(u_1-u_{11})^2}{\theta_1-\omega_1}+\frac{a_2(u_2-u_1)^2}{\theta_2}+\\ & \frac{a_3(u33-u_2)^2}{\omega_2}+\frac{a_3(u_{33}-u_3)^2}{\theta_3-\omega_2}+\frac{a_4(u_0-u_3)^2)}{\theta_4}\ge\\ & \frac{a_1(u_0-u_{11})^2}{\theta_1}+\frac{a_1(u_1-u_{11})^2}{\theta_1}+\frac{a_2(u_2-u_1)^2}{\theta_2}+\\ & \frac{a_3(u_{33}-u_2)^2}{\theta_2}+\frac{a_3(u_{33}-u_3)^2}{\theta_3}+\frac{a_4(u_0-u_3)^2}{\theta_4}\end{aligned}$$

, with $a_1 = b_1k$, $a_2 = b_2/k$, $a_3 = b_3k$, $a_4 = b_4/k$, $u_{11} = u_k(r',\omega_1)$, $u_{33} = u_k(r',\theta_1+\theta_2+\omega_2)$, $u_0 = u_k(r',0)$, $u_1 = u_k(r',\theta_1)$, $u_2 = u_k(r',\theta_2)$ and $u_3 = u_k(r',\theta_3)$. We can now get a lower bound for the last expression, call it W, of (7), expressed in terms of u_{11} and u_{22}. Just replace u_0 with a free variable s_0, minimize W with respect to s_0, insert the minimized variable into W and repeat the same procedure with u_1, u_2 and u_3. We will then obtain a very long expression W' such that $W \geq W'$. Now we show how to use this estimate. Define $v \in \Lambda$ such that

$$v_k(r',\omega_1) = u_k(r',\omega_1) \quad \text{and} \quad v_k(r',\omega_2+\theta_1+\theta_2) = u_k(r',\omega_2+\theta_1+\theta_2). \tag{7}$$

Then v is defined everywhere in Ω by our definition of Λ, and,

$$\int_0^{2\pi} \lambda_k(r',\theta)(\frac{\partial v_k}{r'\partial\theta})^2 r'\mathrm{d}\theta = \frac{(u_{11}-u_{33})^2}{r'}(\frac{a_2}{\theta_2}+\frac{a_4}{\theta_4}).$$

Let $A_1 = b_1k/\theta_1$, $A_2 = b_2/(\theta_1 k)$, $A_3 = b_3k/\theta_3$ and $A_4 = b_4/\theta_4 k$. Now

$$r'\frac{\int_0^{2\pi}\lambda_k(r',\theta)(\frac{\partial u_k(r',\theta)}{r'\partial\theta})^2 r'\mathrm{d}\theta}{(u_{33}-u_{11})^2(\frac{a_2}{\theta_2}+\frac{a_4}{\theta_4})} \geq \frac{W'}{(u_{33}-u_{11})^2(\frac{a_2}{\theta_2}+\frac{a_4}{\theta_4})} = \frac{Q_1}{Q_2}, \tag{8}$$

where

$$Q_1 = \left(2\,A_1\,A_2\,A_3^2 + 2\,A_1^2\,A_2\,A_3\right)\,A_4 + A_1^2\,A_3^2\,(A_4+A_2)$$

and

$$Q_2 = (A_4+A_2)\,(A_2+A_1)\,(A_3+A_2)\,(A_4+A_1)\,(A_4+A_3) - Q_3,$$

where

$$\begin{aligned} Q_3 = {} & \left((A_2+A_1)\,A_3 + A_2^2 + A_1\,A_2\right)A_4^3 + \left(\left(2\,A_2^2 + A_1\,A_2\right)A_3 + A_2^3 + 2\,A_1\,A_2^2\right)A_4^2 \\ & + \left(\left(A_2^3 + A_1\,A_2^2\right)A_3 + A_1\,A_2^3\right)A_4 + A_1\,A_2^3\,A_3. \end{aligned} \tag{9}$$

Then it is easy to see that

$$\frac{Q_1}{Q_2} \geq \tau(k) = \frac{A_1^2 A_3^2}{(A_1+A_2)(A_1+A_4)(A_3+A_2)(A_3+A_4)}. \tag{10}$$

Note that this result holds for every $0 < r < 1$ since r' was cancelled out. Trivially, if v'_k is a minimizer of E_k in Λ, then $\tau(k)E_k(v'_k) \leq \tau(k)E_k(v_k)$.

By our defintion of v_k in (7) we have that

$$\int_{\Omega_1\cup\Omega_3}\lambda_k(r,\theta)\Big(\frac{\partial v_k(r,\theta)}{\partial r}\Big)^2 r\mathrm{d}r\mathrm{d}\theta \leq \int_{\Omega_1\cup\Omega_3}\lambda_k(r,\theta)\Big(\frac{\partial u_k(r,\theta)}{\partial r}\Big)^2 r\mathrm{d}r\mathrm{d}\theta \tag{11}$$

For $0 \leq \alpha(\theta) = (\theta_1+\theta_2-\theta)/\theta_2 \leq 1$ we have in Ω_2 that

$$\begin{aligned} \Big(\frac{\partial v(r,\theta)}{\partial r}\Big)^2 &= \Big(\alpha(\theta)\frac{\partial u(r,\omega_2)}{\partial r} + (1-\alpha(\theta))\frac{\partial u(r,\omega_1)}{\partial r}\Big)^2 \leq \\ & \Big(\frac{\partial u(r,\omega_1)}{\partial r}\Big)^2 + \frac{\partial u(r,\omega_1)}{\partial r}\frac{\partial u(r,\omega_2)}{\partial r} + \Big(\frac{\partial u(r,\omega_2)}{\partial r}\Big)^2 \leq \\ & \frac{3}{2}\Big(\Big(\frac{\partial u(r,\omega_1)}{\partial r}\Big)^2 + \Big(\frac{\partial u(r,\omega_2)}{\partial r}\Big)^2\Big). \end{aligned} \tag{12}$$

By (11) and (12) we have that

$$\frac{b_2}{k}\int_{\Omega_2}\Big(\frac{\partial v_k}{\partial r}\Big)^2 r\mathrm{d}r\mathrm{d}\theta \leq \tag{13}$$

$$\frac{3b_2}{2k}\Big(\frac{1}{b_1 k}\int_{\Omega_1} b_1 k\Big(\frac{\partial u_k}{\partial r}\Big)^2 r\mathrm{d}r\mathrm{d}\theta + \frac{1}{b_3 k}\int_{\Omega_3} b_3 k\Big(\frac{\partial u_k}{\partial r}\Big)^2 r\mathrm{d}r\mathrm{d}\theta\Big) \leq$$

$$\frac{3b_2}{2k}\Big(\frac{1}{b_1 k}+\frac{1}{b_3 k}\Big)E_k(u_k).$$

The same kind of reasoning holds for Ω_4, of course. □

5.2 Proof of Theorem 2

Proof. Let $v \in \Lambda$. By the definition we have

$$\int_{\Omega_1\cup\Omega_3}\Big(\frac{\partial v}{r\partial\theta}\Big)^2 r\mathrm{d}r\mathrm{d}\theta = 0. \tag{14}$$

Define

$$T_k(v) = \int_{\Omega_1\cup\Omega_3}\lambda_k(r,\theta)\Big(\frac{\partial v}{\partial r}\Big)^2 r\mathrm{d}r\mathrm{d}\theta + \int_{\Omega_2\cup\Omega_4}\lambda_k(r,\theta)\Big(\frac{\partial v}{r\partial\theta}\Big)^2 r\mathrm{d}r\mathrm{d}\theta. \tag{15}$$

We will use the Euler–Lagrange equation to find a minimizer of T_k in Λ. Let L be defined by

$$L(x1,x2,x3,x4,x5) =$$

$$(1-c)^2\ p_2\, x_2^2\, x_5 + c^2\, p_1\, x_1^2\, x_5 + \frac{(1-c)^2\ p_3\, x_4^2}{x_5} + 2\frac{(1-c)\ cp_3\, x_3\, x_4}{x_5} + \frac{c^2\, p_3\, x_3^2}{x_5}. \tag{16}$$

Then the energy is given as

$$T_k(u,v) = \int_0^1 L\Big(\frac{\mathrm{d}u}{\mathrm{d}r},\frac{\mathrm{d}v}{\mathrm{d}r},u,v,r\Big). \tag{17}$$

This is associated with the following *Euler–Lagrange equation*

$$c\Big(-\phi_1 c\frac{\mathrm{d}^2u}{\mathrm{d}r^2}r - \phi_1 c\frac{\mathrm{d}u}{\mathrm{d}r} + \frac{u}{r}\phi_3(2c-1)\Big) + \tag{18}$$

$$(1-c)\Big(-\phi_2(1-c)\frac{\mathrm{d}^2v}{\mathrm{d}r^2}r - \phi_2(1-c)\frac{\mathrm{d}v}{\mathrm{d}r} + \frac{v}{r}\phi_3\Big) = 0.$$

After inserting $u = r^{a_u}$ and $v = r^{a_v}$ and splitting the equation in two parts we obtain

$$(-\phi_1 c a_u^2 + \phi_3) r^{a_u - 1} = 0 \qquad (19)$$
$$(-\phi_2(1-c) a_v^2 + \phi_3) r^{a_v - 1} = 0.$$

Thus

$$a_u = \pm\sqrt{\frac{\phi_3}{c\phi_1}} \quad \text{and} \quad a_v = \pm\sqrt{\frac{\phi_3}{(1-c)\phi_2}}. \qquad (20)$$

We are only interested in the positive parts of the root. Now we want to compute c. The most straightforward way to do this is to insert the new values of a_u and a_v into (17) and differentiate with respect c so that we can find minimum points. But it seems very difficult to find the roots of the resulting expression. Instead we can use the following approach: Fix a real value d such that $0 < d < 1$. Let $u = r^a$ and $v = r^b$, with

$$a = \sqrt{\frac{\phi_3}{d\phi_1}} \quad \text{and} \quad b = \sqrt{\frac{\phi_3}{(1-d)\phi_2}}. \qquad (21)$$

Insert u and v into (17). After computing the integral, we obtain

$$T_k(u,v) = \frac{c^2\,\phi_3}{2\,a} + \frac{2\,(1-c)\,c\,\phi_3}{b+a} + \frac{(1-c)^2\,\phi_3}{2\,b} + \frac{b\,(1-c)^2\,\phi_2}{2} + \frac{a\,c^2\phi_1}{2}. \qquad (22)$$

We want to find which c minimize $T(u,v)$. The parameter c will minimize (22) for the fixed functions u and v if

$$\frac{\mathrm{d}T_k(u,v)}{\mathrm{d}c} = -\frac{2\,c\phi_3}{b+a} + \frac{c\phi_3}{a} + \frac{2\,(1-c)\,\phi_3}{b+a} - \frac{(1-c)\,\phi_3}{b} - b\,(1-c)\,\phi_2 + a\,c\phi_1 = 0. \qquad (23)$$

By definition, the above equation holds for all $0 < d < 1$, in particular it will hold for $d = c$. Now we can insert the values from (21) into (23). Then replace d by c. Then use the facts that $a > 0$, $b > 0$, $0 < c$ and $0 < 1 - c$ to do a number of algebraic computations on (23). This will yield

$$\sqrt{1-c}\;\sqrt{c}\;(2\,c-1)\;\sqrt{\phi_1}\;\sqrt{\phi_2}\,\phi_3^2 =$$
$$\left((c-1)\,\phi_2 + c\phi_1\right)\phi_3^2 - \left(\left(c^2 - 2\,c + 1\right)\phi_2 - c^2\phi_1\right)\phi_3^2$$

Square both sides of the above expression. Then we obtain a fourth degree polynom. of which only one of the roots can be a solution to our problem, namely

$$c = \frac{\phi_2}{\phi_1 + \phi_2}.$$

□

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