You can also view this case study in the following formats:

Mathematica Maple MATLAB

Conic Sections and Quadric Surfaces

Text Reference: Section 7.2, p. 462

The purpose of this set of exercises is to show how quadratic forms and the Principal Axes Theorem may be used to classify conic sections and quadric surfaces.

Consider the **general quadratic equation in two variables** x_1 **and** x_2:

$$ax_1^2 + 2bx_1x_2 + cx_2^2 + dx_1 + ex_2 + f = 0 \tag{1}$$

The graph of this equation is called a **conic section**: the most interesting examples of conic sections are circles, ellipses, hyperbolas, and parabolas. These are called the **nondegenerate conic sections**. Each conic section has two axes; if these axes are identical to the coordinate axes, then the conic section is said to be in **standard position**. A conic section in standard position has a particularly simple equation; examples of the equations of nondegenerate conic sections in standard position are given in Table 1.

Name	Equation
Circle	$(x_1 - h)^2 + (x_2 - k)^2 = r^2$
Ellipse	$\frac{(x_1 - h)^2}{a^2} + \frac{(x_2 - k)^2}{b^2} = 1, a, b > 0$
Parabola	$(x_1 - h)^2 = 4p(x_2 - k)$ or $(x_2 - k)^2 = 4p(x_1 - h)$
Hyperbola	$\frac{(x_1 - h)^2}{a^2} - \frac{(x_2 - k)^2}{b^2} = 1, a, b > 0$ or $\frac{(x_2 - k)^2}{a^2} - \frac{(x_1 - h)^2}{b^2} = 1, a, b > 0$

Table 1: The Conic Sections in Standard Position

Other **degenerate conic sections** include single points and pairs of lines; this exercise set is only concerned with nondegenerate conic sections. The first goal of this exercise set is to classify and graph a general quadratic equation in two variables.

Begin by noting that Equation 1 may be written as

$$\mathbf{x}^T A\mathbf{x} + B\mathbf{x} + f = 0$$

where

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}, B = \begin{bmatrix} d & e \end{bmatrix}, \text{and } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The quadratic form $\mathbf{x}^T A\mathbf{x}$ is called the quadratic form associated with the quadratic equation.

Example 1: Consider the quadratic equation

$$8x_1^2 - 4x_1x_2 + 5x_2^2 - 24x_1 + 24x_2 = 0$$

This may be written as

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 8 & -2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -24 & 24 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

The quadratic form associated with this equation is thus $\mathbf{x}^T A\mathbf{x}$ with

$$A = \begin{bmatrix} 8 & -2 \\ -2 & 5 \end{bmatrix}$$

The Principal Axes Theorem says that there is an orthogonal change of variable $\mathbf{x} = P\mathbf{y}$ that transforms the quadratic form $\mathbf{x}^T A\mathbf{x}$ into a quadratic form $\mathbf{y}^T D\mathbf{y}$, where D is a diagonal matrix with the eigenvalues of A (with multiplicities) as its diagonal entries. Geometrically, since P is an orthogonal matrix, the change of variable $\mathbf{x} = P\mathbf{y}$ transforms the x_1 and x_2 axes into a new set of orthogonal axes (determined by the columns of P), which are denoted by y_1 and y_2. Performing this substitution gives the equation

$$(P\mathbf{y})^T A(P\mathbf{y}) + B(P\mathbf{y}) + f = 0$$

which may be rewritten as

$$\mathbf{y}^T(P^T AP)\mathbf{y} + (BP)\mathbf{y} + f = 0$$

or

$$\mathbf{y}^T D\mathbf{y} + (BP)\mathbf{y} + f = 0$$

Example 1(cont.): The eigenvalues of A are 4 and 9; one may show that the associated unit eigenvectors are

$$\lambda_1 = 4 : \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix} \qquad \lambda_2 = 9 : \begin{bmatrix} \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{bmatrix}$$

Thus

$$P = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{bmatrix}$$

Thus

$$D = P^T AP = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix} \text{ and } BP = \begin{bmatrix} -24 & 24 \end{bmatrix} P = \begin{bmatrix} \frac{24}{\sqrt{5}} & -\frac{72}{\sqrt{5}} \end{bmatrix}$$

so the equation may be written in terms of the new variables y_1 and y_2 as

$$\mathbf{y}^T D\mathbf{y} + (BP)\mathbf{y} = 4y_1^2 + 9y_2^2 + \frac{24}{\sqrt{5}}y_1 - \frac{72}{\sqrt{5}}y_2 = 0$$

Notice that the change of coordinates has eliminated the term containing x_1x_2; it is now much easier to see which type of conic section the graph of this equation is. Complete the square to find that the equation may be written

$$4\left(y_1 + \frac{3}{\sqrt{5}}\right)^2 + 9\left(y_2 - \frac{4}{\sqrt{5}}\right)^2 = 36$$

or

$$\frac{\left(y_1 + \frac{3}{\sqrt{5}}\right)^2}{9} + \frac{\left(y_2 - \frac{4}{\sqrt{5}}\right)^2}{4} = 1$$

so the graph of this equation is an ellipse. A graph of this ellipse showing both x_1 and x_2 axes and y_1 and y_2 axes is given in Figure 1.

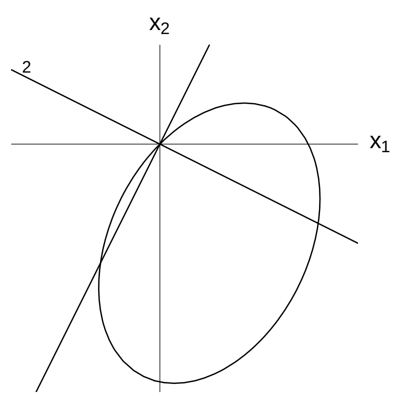

Figure 1: The ellipse of Example 1

The same analysis which has been done for the quadratic equation in two variables may be done to the **general quadratic equation in three variables** x_1**,** x_2**, and** x_3:

$$ax_1^2 + bx_2^2 + cx_3^2 + 2dx_1x_2 + 2ex_1x_3 + 2fx_2x_3 + gx_1 + hx_2 + ix_3 + j = 0 \quad (2)$$

The graph of this equation is called a **quadric surface**. There are quite a few types of nondegenerate quadric surfaces: ellipsoids, hyperboloids of one and two sheets, elliptic cones, elliptic paraboloids, and hyperbolic paraboloids. A quadric surface has three axes; when these axes coincide with the coordinate axes, the surface is in **standard position**, and has a straightforward equation. Examples

Name	Equation
Ellipsoid	$\frac{(x_1-h)^2}{a^2}+\frac{(x_2-k)^2}{b^2}+\frac{(x_3-l)^2}{c^2}=1, a, b, c>0$
Hyperboloid of One Sheet	$\frac{(x_1-h)^2}{a^2}+\frac{(x_2-k)^2}{b^2}-\frac{(x_3-l)^2}{c^2}=1, a, b, c>0$
Hyperboloid of Two Sheets	$\frac{(x_3-l)^2}{c^2}-\frac{(x_1-h)^2}{a^2}-\frac{(x_2-k)^2}{b^2}=1, a, b, c>0$
Elliptic Cone	$\frac{(x_1-h)^2}{a^2}+\frac{(x_2-k)^2}{b^2}-\frac{(x_3-l)^2}{c^2}=0, a, b, c>0$
Elliptic Paraboloid	$\frac{(x_1-h)^2}{a^2}+\frac{(x_2-k)^2}{b^2}=x_3-l, a, b>0$
Hyperbolic Paraboloid	$\frac{(x_2-k)^2}{b^2}-\frac{(x_1-h)^2}{a^2}=x_3-l, a, b>0$

Table 2: The Quadric Surfaces in Standard Position

of the equations of quadric surfaces in standard position are provided in Table 2. Other equations are possible by interchanging the positions of x_1, x_2, and x_3. The goal as before is to classify and graph a general quadratic equation in three variables.

Most of the analysis is identical to that done for conic sections; to begin, Equation 2 may be written as

$$\mathbf{x}^T A\mathbf{x} + B\mathbf{x} + j = 0$$

where

$$A = \begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix}, B = \begin{bmatrix} g & h & i \end{bmatrix}, \text{and } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The quadratic form $\mathbf{x}^T A\mathbf{x}$ is again called the quadratic form associated with the quadratic equation.

Example 2: Consider the quadratic equation

$$4x_1^2 + 4x_2^2 + 4x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3 - x_1 - x_2 - x_3 = 0$$

This may be written as

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

The quadratic form associated with this equation is thus $\mathbf{x}^T A\mathbf{x}$ with

$$A = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix}$$

As before the Principal Axes Theorem guarantees an orthogonal change of variable $\mathbf{x} = P\mathbf{y}$ that transforms the quadratic form $\mathbf{x}^T A\mathbf{x}$ into a quadratic form $\mathbf{y}^T D\mathbf{y}$, where D is a diagonal matrix with the eigenvalues of A (with multiplicities) as its diagonal entries. The matrix P exchanges the old x_1, x_2, and x_3 axes for a new set of y_1, y_2, and y_3 axes. The result of the change of variables is

$$\mathbf{y}^T D\mathbf{y} + (BP)\mathbf{y} + j = 0$$

Example 2(cont.): The eigenvalues of A are 3 and 6; one may show that an orthonormal set of associated unit eigenvectors is

$$\lambda_1 = 3 : \begin{bmatrix} \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \qquad \lambda_2 = 6 : \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

So

$$P = \begin{bmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$D = P^T AP = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} \text{ and } BP = \begin{bmatrix} -1 & -1 & -1 \end{bmatrix} P = \begin{bmatrix} \frac{4}{\sqrt{6}} & 0 & -\sqrt{3} \end{bmatrix}$$

The equation may be written in terms of the new variables y_1, y_2 and y_3:

$$\mathbf{y}^T D\mathbf{y} + (BP)\mathbf{y} = 3y_1^2 + 3y_2^2 + 6y_3^2 + \frac{4}{\sqrt{6}}y_1 - \sqrt{3}y_3 = 0$$

The change of coordinates has eliminated the cross-product terms; the graph using these new axes is in standard position. This graph is an ellipsoid, and completing the square produces an equation in graphable form. A graph of this ellipsoid using both x_1, x_2 and x_3 axes and the y_1, y_2 and y_3 axes is given in Figure 2.

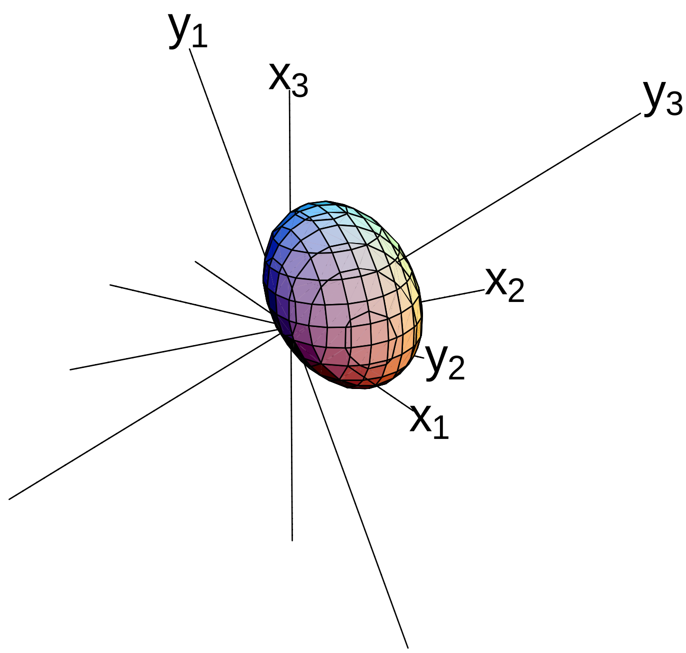

Figure 2: The ellipsoid of Example 2

Questions:

1. Perform an appropriate orthogonal change of variables $\mathbf{y} = P\mathbf{x}$ to remove the x_1x_2 term. Classify the conic section and give its equation in the new coordinate system.

   a) $x_1^2 + 4x_1x_2 - 2x_2^2 - 8 = 0$
   b) $x_1^2 + 2x_1x_2 + x_2^2 + x_1 + 8x_2 = 0$
   c) $5x_1^2 + 4x_1x_2 + 5x_2^2 - 9 = 0$

2. Perform an appropriate orthogonal change of variables $\mathbf{y} = P\mathbf{x}$ to remove the cross- product terms. Classify the quadric surface and give its equation in the new coordinate system.

   a) $4x_1x_2 + 4x_1x_3 + 4x_2x_3 - 12x_1 - 8x_2 - 12x_3 + 18 = 0$
   b) $2x_2x_3 + x_1 + 10x_2 - 6x_3 - 31 = 0$
   c) $5x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 + 2x_1x_3 - 4x_2x_3 - 2x_1 + 10x_2 - 26x_3 = 0$