CHAPTER 12

BACKGROUND FOR GÖDEL'S PROOF

> …when the drapery, which reveals its form more or less accurately, is torn away, there remains mathematics itself, which lives in human thought and is only symbolized by the signs of formalism.[1]

HERR WARUM

Kurt Gödel [1906–1978] and both his parents were born in what was then Brünn, Moravia, later Brno, Czechoslovakia. Unlike Jung's solid Swiss background, Gödel's was mixed. His parents were part of a large German-speaking community, and raised their children as part of that community, separated from the majority Czechs. His first sixteen years were spent in Brünn, the next sixteen in Vienna (where he became an Austrian citizen), and the remaining nearly 42 years in the United states (where he eventually became an American citizen).

Gödel's father was a prominent director of a textile factory, and his family lived in relative affluence. Even World War I—which was to change the region drastically—had little impact on the Gödel family. Kurt was a precocious boy known affectionately within his family by the pet name—*Herr Warum*[2]—for his insatiable curiosity. That quality was not to abate throughout his life.

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[1] Lucienne Félix, *The Modern Aspect of Mathematics*, p. 61.

[2] I.e., Mr. Why; a name that surely many another parent of bright children can sympathize with. *Kurt Gödel Collected Works, Vol. I, Publications 1929-1936,* Solomon Feferman, editor-in-chief (New York: Oxford University Press, 1986), p. 3.

During his university years in Vienna, Gödel came to spend a good deal of time with the *Vienna Circle*, a group of philosophers, logicians and linguists who in the 1920's and 1930's developed a school of philosophy known as *logical positivism*. The goal of logical positivism was "to purify philosophy—sifting out its metaphysical elements, and reconsituting the discipline with logic as its [guiding principle.][3] Logical positivism in part combined physicist Ernst Mach's empiricist philosophy with Bertrand Russell's logicist position. Logical positivism was most directly influenced , however, by the early ideas of a pupil of Russell's—Ludwig Wittgenstein—as published in Wittgenstein's *Tractatus, Logico-philosophicus* in 1919.[4]

Though Gödel was excited by the ideas being considered by the Vienna Circle, he had already formed his philosophical views before he came to Vienna, and those views were directly opposed to those of logical positivism. Gödel had little faith in any attempt to reduce philosophy and mathematics to logic; he was convinced that the *mathematical objects* with which mathematicians concern themselves, were every bit as real as the physical objects we encounter in everyday reality. These views were already central to Gödel's personal philosophy by the time he came to Vienna, and may have been formed as early as his teenage years, when he first read Kant.[5] Interesting for us and our discussion, Gödel evidently responded more to the archetypal

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[3] W. L. Reese, *Dictionary of Philosophy and Religion*, p. 314.

[4] By the time the Vienna Circle was developing logical positivism, Wittgenstein had already repudiated this ideas he had expressed in *Tractatus, Logico-philosophicus.* Wittgenstein is another figure of almost unequaled intellect, who like Jung and Gödel refused to accept early fame, and instead went his own direction. He figures in our history only peripherally, since his thought went a different direction that Jung and Gödel.

[5] Solomon Feferman, "Kurt Gödel: Conviction and Caution", in S. G. Shanker, editor, *Gödel's Theorem in Focus*, p. 100-1.

hypothesis which underlay Kant's philosophy, than to the specific philosophical details which were dated by Gödel's time.

GÖDEL ENCOUNTERS HILBERT'S PROGRAM

> Hilbert saw to the heart of the matter…he sought to develop a method that would yield demonstrations of consistency as much beyond genuine logical doubt as the use of finite models for establishing the consistency of certain sets of postulates—by an analysis of a finite number of structural features of expressions in completely formalized calculi.[6]

Of more immediate interest for Gödel than the ideas of the Vienna Circle, was a slim book by David Hilbert and Wilhelm Ackermann, called *Foundations of the Theory of Logic.*[7] When Hilbert defined his program in 1900, Cantor's set theory of transfinite numbers was both firmly entrenched in mathematics, yet under attack on all sides. Hilbert feared that not only Cantor's theory—but all mathematics—was in danger. Hilbert, however, was an optimist who hoped to preserve transfinite set theory by using a formal axiomatic method to develop the number system—including transfinite numbers—thus dealing with infinity through a finite number of definitions, rules, and operations. In contrast with Russell, Hilbert realized that infinity could still stick its head in through the back door, since those finite definitions, rules and operations could be used in an infinite number of ways to generate mathematical truths.

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[6] Ernest Nagel and James R. Newman, *Gödel's Proof* (New York: New York University Press, 1958), p. 32-3. Nagel's and Newman's volume is the best general introduction to Gödel's Proof, and will be drawn on extensively in the pages to follow.

[7] David Hilbert and Wilhelm Ackermann, *Grundzüge der theoretischen Logik* (Berlin: Springer, 1928).

Therefore, it was also necessary to find some way outside the system itself to demonstrate that the system was both *complete* and *consistent*.[8]

For example, by 1899 Hilbert had managed to develop a formal axiomatic system for geometry, which reduced geometry to arithmetic, thus completing the task Rene Descartes had begun two hundred fifty years earlier with his discovery of analytic geometry. Though Descartes had the great realization that marks all such revolutions in thought, it was much too early for him to realize that this equivalence needed to be formally demonstrated. Hilbert was able to do just that, demonstrating that if arithmetic was complete and consistent, so was geometry. Hence his program of 1900 to prove arithmetic also complete and consistent. The whole system was like a house of cards: if you pulled out one key card at the base—arithmetic—the house of cards would come tumbling down.

Almost three decades had passed since Hilbert first conceived his program, and identified ten significant problems mathematics needed to solve pursuant to that goal. Though his program had excited the interests of mathematicians world-wide, none of them were Gödel, and the major problems remained unresolved.

> …Cantor's doctrine, too, was attacked on all sides. So violent was this reaction that even the most ordinary and fruitful concepts and the simplest and most important deductive methods of mathematics were threatened and their employment was on the verge of being declared illicit.[9]

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[8] By complete, mathematicians mean that no true statements can be made within a system that cannot be derived from the axioms of the system. By consistent, mathematicians mean that if a statement can be derived from the axioms of a system, then its opposite cannot also be derived; i.e., a statement cannot both be true and false at the same time.

[9] David Hilbert, "On the Infinite", in Paul Benacerraf & Hilary Putnam, *Philosophy of Mathematics: Selected Readings*, p. 190.

Russell and Whitehead had taken a different direction, with their goal of subsuming all mathematics and philosophy within logic. Their *Principia Mathematica* purported to provide a system in which all mathematics was included within logic. Hilbert, like Gödel was too much the mathematician to believe this either possible or desirable.

> …We find ourselves in agreement with the philosophers, notably with Kant. Kant taught—and it is an integral part of his doctrine—that mathematics treats a subject matter which is given independently of logic. Mathematics, therefore, can never be grounded solely on logic.[10]

Hilbert regarded *Principia Mathematica* as only one possible formal axiomatic system for mathematical logic, and assumed that any such system would have to be demonstrated to be sound using mathematics. In their *Foundations of the Theory of Logic*, Hilbert and Ackermann presented still another formal axiomatic system for mathematical logic. Since Hilbert saw logic as a sub-set of mathematics, rather than the other way around, he was able to address epistemological issues which Russell and Whitehead never thought to question. It wasn't possible for them to both argue for the universality of logic, and to question whether logically derived systems, like their own, were complete and consistent. As Bertrand Russell wrote in a letter much later:

> You note that we were indifferent to attempts to prove that our axioms could not lead to contradictions. In this Gödel showed that we had been mistaken. But I thought that it

[10] David Hilbert, "On the Infinite", p. 192.

> must be impossible to prove than any given set of axioms does *not* lead to a contradiction, and, for that reason, I had paid little attention to Hilbert's work.[11]

Hilbert himself had deep problems with his own position, though he didn't yet realize it. At one and the same time, he believed every bit as much as Gödel that mathematics was concerned with archetypal forms, yet he still hoped to reduce such potentially infinite forms to empty symbols, devoid of meaning. This combination of beliefs was, of course, impossible, as Gödel was to prove.

> As a further precondition for using logical deduction and carrying out logical operations, something must be given in conception, viz., *certain extralogical concrete objects which are intuited as directly experienced prior to all thinking*.[12]

> Still it is consistent with our finitary viewpoint to deny any meaning to logical symbols, just as we denied meaning to mathematical symbols, and to declare that the formulas of the logical calculus are ideal statements which mean nothing in themselves.[13]

As the major problems from 1900 remained unresolved, they found their way into this new volume, in some cases reformed within the axiomatic system presented there. It was here that Gödel first encountered Hilbert's program and the problems whose resolutions he hoped would put mathematics on a sound footing. It was perhaps also here that Gödel first came in contact with Russell's and Whitehead's *Principia Mathematica*, though this is less clear. Of

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[11] From Bertrand Russell's letter to Leon Henkin of April 1, 1963, in John W. Dawson, Jr., "The Reception of Gödel's Incompleteness Theorem", in S. G. Shanker, *Gödel's Theorem in Focus*, p. 90.

[12] My emphasis—note that Hilbert is explicitly accepting *archetypes*, though he doesn't seem to realize it. David Hilbert, "On the Infinite", p. 192.

[13] Here Hilbert is trying to take back the archetypal meaning he just granted to symbols. David Hilbert, "On the Infinite", p. 197.

Hilbert’s ten problems, Gödel was to prove one[14], disprove another[15], and figured heavily in proving the undecidability of a third.[16] More importantly, Gödel’s contributions would effectively bring an end to Hilbert’s program.[17]

In the remainder of this chapter, we will discuss Gödel’s essential insight into what was wrong with both Hilbert’s and Russell’s separate approaches. We will deal with his famous incompleteness proof in chapter 16, then reserve further discussion of Cantor’s continuum hypothesis to the final chapter of this book.

GÖDEL'S INSIGHT

As you will recall, Bertrand Russell was never able to fully resolve the problem presented by the paradox he himself had discovered: the paradox of the set of all sets that do not include themselves. He was only able to achieve a partial resolution in *Principia Mathematica* by the trick of defining a hierarchy of classes of elements, classes of classes, classes of classes of classes, ad infinitum. Even Russell regarded his solution as a sorry state of affairs.

David Hilbert realized that any formal logical system, such as that presented in *Principia Mathematica* was itself subject to epistemological considerations of completeness and

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[14] Which is commonly termed the “completeness problem”. We will not deal with it in this book as its interest is largely mathematical except as it led Gödel to his incompleteness proof.

[15] The “consistency problem”, in the course of proving this to be false, Gödel produced his famous incompleteness proof.

[16] Cantor’s continuum hypothesis. Hilbert considered this important enough to identify as the first problem on his list. We dealt with this briefly in chapter 7 and will deal with its implications in depth in chapter 19.

[17] Though to this day there are still diehards to hope to get around Gödel’s results and still achieve Hilbert’s objectives.

consistency. Such issues were essentially *meta-mathematical*[18]; i.e., they transcended mathematics, and could only be resolved outside the system in question. Unless this was to lead to an infinite regress, the resolution itself had to be finite. For example, the statement that "arithmetic is consistent and complete" is a meta-mathematical statement. Hilbert recognized that this statement could not be resolved within arithmetic because it transcended arithmetic. Perhaps, however, it is overstating Hilbert's understanding of the issue to say he saw the meta-mathematical issues this clearly. If so, he would have realized the impossibility of his goal. Gödel was quite clear on all these issues. As he said at a later time:

> …A complete epistemological description of a language A cannot be given in the same language A, because the concept of truth of sentences of A cannot be defined in A.[19]

Gödel's succinct summary above implicitly acknowledges that the paradoxes which caused Cantor, Frege and Russell so much grief can never be defined away. Cantor's theory of transfinite sets had been attacked because of the paradox of the "set of all sets".[20] The logicians like Frege hoped to avoid the issue by reducing mathematics to mathematical logic. Russell then destroyed Frege's hopes when he discovered the paradox of the "set of all sets which do not contain themselves" within mathematical logic.[21] Russell in turn attempted to create a new logical system which resolved the paradox by creating a hierarchical chain of classes, classes of

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[18] *Meta* means above, beyond, transcending, as in metaphysics, which is the field of philosophy which deals with ultimate questions, questions which transcend physical explanation. Similarly, meta-mathematical statements are statements about mathematics which transcend mathematics.

[19] Quotation by Gödel in Solomon Feferman, "Kurt Gödel: Conviction and Caution", p. 105.

[20] See chapter 7.

[21] See chapter 9.

classes, etc., trailing off into infinity. Gödel saw that any such system was itself subject to the same problem. Russell could never succeed at pushing the issue away because it would always be possible to questions about his system which could not be resolved within his system. In particular, you could ask, as Hilbert and the formalists did, whether it was complete and consistent.

However, Gödel saw deeper than Hilbert. The problem wasn't infinity, nor was there a need to reduce mathematics to finite processes. Speaking of this problem, Gödel complained in 1967, of the "blindness (or prejudice, or whatever you may call it) of logicians at the time, according to which non-finitary reasoning was not accepted as a meaningful part of metamathematics."[22] Nor did Gödel accept that proving arithmetic (or any other system) consistent, automatically proved that every true statement could be determined within the system. The idea that there could be true statements that could neither be proved true nor false within an axiomatic system never even occurred to either Russell or Hilbert.

> How indeed could one think of *expressing* metamathematics *in* the mathematical systems themselves, if the latter are considered to consist of meaningless symbols which acquire some substitute of meaning only *through* metamathematics.[23]

Gödel was able to come to this realization because it was embedded within his belief system. Much like Pythagoras and Plato, much like Kant, Gödel believed that mathematics dealt with eternal archetypal entities such as number and shape. Gödel's difference from his predecessors lay largely in the degree of sophistication with which he approached this

[22] Solomon Feferman, "Kurt Gödel: Conviction and Caution", p. 102.

[23] Quotation by Kurt Gödel, in Solomon Feferman, "Kurt Gödel: Conviction and Caution", p. 107.

understanding. After all, twenty-three centuries separated Gödel from Plato. Even Kant's realization was a century and a half in the past. The concept of archetypes of a collective unconscious had slowly emerged over that period of time within the human psyche. It was finally beginning to appear in all fields, though except for Jung and, to a lesser extent Gödel, no one was yet able to step outside their own field and see that each field was saying similar things within their own language. In this book, we've emphasized Gödel and Jung because Gödel's mathematical expression of the archetypal hypothesis is the purest possible, and Jung's the most completely articulated, since he is describing the psyche within which the archetypal hypothesis found expression.

The fact that we have restricted ourselves to these two areas should not blind the reader to the underlying fact that, since the archetypal hypothesis was itself emerging as a symbolic expression within the human psyche, it inevitably had to emerge in parallel in all fields. For example, in art during the Middle ages, art was concerned less with outer reality than inner, spiritual truth. In the Renaissance artists shifted their gaze outwards to the world, and tried to accurately capture that reality. This inevitably led to increased abstraction, such as the brilliant attempt to use perspective to fool the eye into believing it was seeing a three-dimensional reality on a two-dimensional canvas. Step-by-step painting evolved in parallel with the evolution we've described in psychology and mathematics.

Late in the 19th century, at much the same time that the theory of the unconscious was beginning to emerge in psychology, Cantor's transfinite set theory in mathematics, abstract art began to emerge. Artists were struggling with the paradoxical realization that the physical world as it actually was, and the world as they conceived it in their minds, were separate and distinct, yet somehow came together in the art they produced on the canvas. That led briefly to a fully abstract art in which art was valued in and of itself. Still later, artists tried to advance past that position to an essentially archetypal recognition that there were artistic values that transcended the separation of world, mind and picture. The fact that the issue still remains unresolved in art can be readily seen in the multiplicity of artistic styles that jostle for position in the current art world.

A similar history could be followed in music, in architecture, in literature, etc. Of course, in order for any such history to be, it would need the same level of explicit parallels to be drawn as have been drawn in this book between the growth of psychology, and the growth of mathematics. But rest assured that those parallels would be there.

With this background for Gödel’s proof in place, it is now time to return to Jung and discuss his primary model of the individuation process.