Newton's Third Law of Motion:

The curved path of a projectile is called a trajectory. The equations for a vacuum trajectory where gravity is the only force acting on the projectile is quite simple. The trajectory of a projectile in a vacuum would inscribe nearly a parabolic path. The general shape of the trajectory is called a parabola because on a flat world and in a vacuum, an unresisting medium, absence of an atmosphere, the path of a projectile is actually a parabola, as was first proved by Galileo in 1638. Galileo noticed that, owing to the curvature of the earth, the force of gravity did not act in parallel lines, but acted instead in lines that converged to the center of the earth. As a consequence, the usual path of a projectile fired on the earth in the absence of air would be a portion of an ellipse. If the shot were fired horizontally from an eminence at any ordinary small arms velocity the elliptical path would intersect the earth’s surface and the projectile would fall to earth striking the surface. But if the velocity were to be 26,000 fps, the projectile would never return to earth, but would be itself a satellite with a circular orbit passing through its original point of firing once every seventeen revolutions. A circle is an ellipse that has both of its foci (plural of focus) at the same point in space and whose equation of the graph is \( r^2 = x^2 + y^2 \). If the velocity were greater than \( (>\) 26,000 fps and less than \( (<\) 36,000 fps, the orbit would be elliptical. An ellipse is the set of all points in the plane the sum of whose distances from two fixed points (foci), each called a focus, is a constant and whose equation of the graph is \( 1 = (x/a)^2 + (y/b)^2 \). If the velocity was exactly at 36,000 fps it would tolly-trop into space in a parabola. The geometric definition of a parabola is the set of points in the plane equidistant from a fixed point called the focus and a fixed line called the directrix and whose equation of the graph is \( y = ax^2 + bx + c \). And if the velocity were greater than \( (>\) 36,000 fps the trajectory would be a hyperbola. The geometric definition of a Hyperbola is the set of all points in the plane, the difference of whose distances from two fixed points, called the foci, is a constant and whose equation of the graph is \( 1 = (x/a)^2 - (y/b)^2 \). The hyperbola also has lines called asymptotes associated with it. Asymptotes are lines that the hyperbola approaches close to but never touches for ever larger values of \( x \) and \( y \). The only time the trajectory would be a straight line is if it was fired straight upward or downward.

For all practical purposes we will never shoot a bullet in a vacuum trajectory. So, why am I even going into showing a vacuum trajectory? Well, atmospheric trajectories display many of the same properties of a vacuum trajectory, but lack the convenience of a simple analytical solution. A vacuum trajectory is the simplest trajectory, only dealing with the force of gravity, which will show the basic fundamentals of trajectories without a bunch of clutter and the mathematical methods presented here will form the framework on which the higher approximations are based on.

Simply put, in a vacuum, the projectile leaves the bore, at the origin, with the barrel at a positive angle from a straight line to the target is called the inclination (slope) angle of departure. As the projectile climbs in height or gains altitude to the summit of its flight this is the ascending branch of its trajectory. The height between the origin and the summit of the projectile's flight path is it's maximum ordinate and the summit lays half way between the origin and the target. From the summit till the point of impact, the projectile's path is on the descending branch of the trajectory. The angle from the projectile's path on the descending branch of the trajectory to the straight line to the origin is the inclination angle of fall. The projectile's inclination angle of departure is the same angle as the inclination angle of fall. And the vertical velocity up ward and horizontal velocity at the point of origin will be the same as the vertical velocity down ward and horizontal velocity at the point of impact.

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• Gravitational Forces on a Projectile:

Let us first deal with the celestial bodies other than the earth itself. For our purposes we can consider all bodies of our solar system to be spherical and composed of concentric homogeneous spherical shells. Text on celestial mechanics shows that the gravitational field of such bodies, outside of their surfaces, is the same as though their masses were concentrated at their centers. We shall also suppose that all the bodies of our solar system moves in a circular orbit about their respective centers. For our purposes we can consider the orbit of the earth’s center about the sun’s center to have a radius of 93,000,000 miles and the radius of the earth to be about 4,000 miles. The angular velocity of the earth’s center about the sun is about:

Equation 1-1:
Angular velocity (in radians/sec.) = \(2\pi / \text{seconds in a year}\)
Angular velocity (in radians/sec.) = \(2 \times 3.1415926536 / 31,557,600\)
Angular velocity (in radians/sec.) = \(6.2831853072 / 31,557,600\)
Angular velocity =\(0.00000001991021277637\) or \(1.991 \times 10^{-7}\) rad. / sec.

Centrifugal acceleration of the earth’s center is:

Equation 1-2:
Centrifugal acceleration (\(ac\) in feet/sec²) = \((1.991 \times 10^{-7})^2 \times \text{radius of the earth’s orbit in feet}\)
Centrifugal acceleration (\(ac\) in feet/sec²) = \((1.991 \times 10^{-7})^2 \times (93,000,000 \text{ mi.} \times 5280 \text{ ft.})\)
Centrifugal acceleration (\(ac\) in feet/sec²) = \(3.9641657 \times 10^{-14} \times 491040000000 \text{ ft.}\)
\(ac \approx 0.0194656394\) or 0.019 ft. / sec.²

It is not difficult to see that for all points on the surface of the earth, this difference has its greatest value at the point “P” nearest to the sun. Centrifugal acceleration of the earth at point “P”:

Equation 1-3:
Centrifugal acceleration (\(ap\) in feet/sec²) = \((1.991 \times 10^{-7})^2 \times \text{radius of the point “P” orbit in feet}\)
Centrifugal acceleration (\(ap\) in feet/sec²) = \((1.991 \times 10^{-7})^2 \times (92,996,000 \text{ mi.} \times 5280 \text{ ft.})\)
Centrifugal acceleration (\(ap\) in feet/sec²) = \(3.9641657 \times 10^{-14} \times 491018880000 \text{ ft.}\)
\(ap \approx 0.0194648022\) or 0.019 ft. / sec.²

At this point the acceleration \(ap\) has the same direction as \(ac\), and its magnitude is greater in the ratio of 93,000,000² to 92,996,000², since the radius of the earth is about 4,000 miles. The ratio differs from unity by about 1 / 11,624, and \(| ap |\) is about 0.019 ft. per sec. per sec., so \(| ap - ac |\) cannot exceed 0.0000018 ft / sec², which is entirely negligible for ballistics purposes. Like wise for the moon’s attraction, though somewhat larger, is also negligible, while the effects of the other planets are far smaller.

• Vacuum Trajectory:

The Negative Force of Gravity on a Projectile:

Where the acceleration due to gravity is pulling the projectile in the downward direction towards the ground to end it's flight.
The maximum gun elevation that will give the maximum range is found by differentiating the equation:

Equation 1-4:
\[ X_w = \frac{(V_o^2 \times \sin(2 \times \theta_o))}{g} \]

with respect to \( \theta_o \), setting the derivative equal to zero, and solving for \( \theta_o \):

Equation 1-5:
\[ \frac{dX}{d\theta_o} = \frac{2 \times V_o^2}{\cos(2 \times \theta_o)} = 0 \]

The solution of this equation gives us:

Equation 1-6:
\[ 0 = \cos(2 \times \theta_o) \]

So that:

\[ 90^\circ = 2 \times \theta_o \]
\[ 45^\circ = \theta_o \]

Therefore, by setting \( \theta_o \) equal to 45° you can find out how far the projectile will go. And if X is made in small enough increments for a given \( \theta_o \) you can plot the trajectory of the projectile. Remember that the equations along with the initial conditions dictate the projectile's trajectory.

The first thing we need to find is the inclination angle of departure:

Equation 1-4:
\[ X_w = \frac{(V_o^2 \times \sin(2 \times \theta_o))}{g} \]

Where \( X_w \) is the range to the target in feet,
\( V_o \) is muzzle velocity in fps,
and \( g \) is the acceleration due to gravity (32.1734 ft per sec²).

By rearranging the formula to give \( \theta_o \) we have:

\[ \frac{(V_o^2 \times \sin(2 \times \theta_o))}{g} = X_w \]
\[ \sin(2 \times \theta_o) = \frac{(X_w \times g)}{V_o^2} \]
\[ 2 \times \theta_o = \csc\left(\frac{(X_w \times g)}{V_o^2}\right) \]
\[ \theta_o = \frac{\csc\left(\frac{(X_w \times g)}{V_o^2}\right)}{2}; \text{ (csc is the same as the inverse sin X or sin}^{-1}X) \]

OR

Equation 1-7:
\[ \theta_o = \frac{\sin^{-1}\left(\frac{(X_w \times g)}{V_o^2}\right)}{2} \]

Let's set our initial conditions, for we know our: muzzle velocity (\( V_o \)) = 2800 fps and range (\( X_w \)) = 300 yards or 900 ft.

\[ \theta_o = \frac{\sin^{-1}\left(\frac{(900 \text{ ft} \times 32.1734 \text{ ft/sec}^2)}{2800^2 \text{ ft}^2/\text{sec}^2}\right)}{2} \]
\[ \theta_o = \frac{\sin^{-1}\left(\frac{(900 \text{ ft} \times 32.1734 \text{ ft/sec}^2)}{2800^2 \text{ ft}^2/\text{sec}^2}\right)}{2} \]
\[ \theta_o = \frac{\sin^{-1}\left(\frac{28956.06}{2800^2}\right)}{2} \]
\[ \theta_o = \frac{\sin^{-1}(0.003693375)}{2} \]
\[ \theta_o = 0.2116152808 \div 2 \]
\[ \theta_o = 0.1058076404^\circ \]
There are actually two elevation angles which satisfy the equation:

Equation 1-4:
\[ X_w = \left( V_o^2 \cdot \sin(2 \cdot \Omega_o) \right) / g \]

Only the lower angle solution is given by the equation:

Equation 1-7:
\[ \Omega_o = \sin^{-1} \left( X_w \cdot g / V_o^2 \right) / 2 \]

The higher-angle solution (denoted by \( \Omega'_o \)) is given as:

\[ \Omega'_o = 90^\circ - \Omega_o = 90^\circ - \sin^{-1} \left( X_w \cdot g / V_o^2 \right) / 2 \]
\[ \Omega'_o = 90^\circ - 0.1058076404^\circ = 90^\circ - \sin^{-1} \left( 900 \text{ ft} \cdot 32.1734 \text{ ft/sec}^2 / 2800^2 \text{ ft}^2/\text{sec}^2 \right) / 2 \]
\[ \Omega'_o = 89.89419236^\circ = 90^\circ - \sin^{-1} \left( 900 \cdot 32.1734 / 2800^2 \right) / 2 \]
\[ \Omega'_o = 89.89419236^\circ = 90^\circ - 0.1058076404 \]
\[ \Omega'_o = 89.89419236^\circ = 89.89419236^\circ \]

The higher-angle is commonly encountered in the use of mortars. Mortars are often used to attack targets inaccessible to direct weapons fire, such as on the other side of a hill. In addition, mortars are relatively large, heavy, low pressure, low velocity weapons, and the vacuum trajectory is often a good approximation to the actual flight of heavy, low velocity projectiles.

Now we're on our way. Range \( X_w \) is 900 ft and at that range our rifle is zeroed. We would expect the height \( Y_w \) to be zero at that range. Let's test it by using the formula that does not use Time Of Flight \( T_w \) as a variable:

Equation 1-8:
\[ Y_w = X_w \cdot \tan \Omega_o - \left( \left( g \cdot X_w^2 \right) / \left( 2 \cdot V_o^2 \cdot \cos^2 \Omega_o \right) \right) \]
\[ Y_w = 900 \text{ ft} \cdot \tan 0.1058076404^\circ - \left( \left( 32.1734 \text{ ft/sec}^2 \cdot 900^2 \text{ ft}^2 \right) / \left( 2 \cdot 2800^2 \text{ ft}^2/\text{sec}^2 \cdot \cos^2 0.1058076404 \right) \right) \]
\[ Y_w = 900 \text{ ft} \cdot 0.018466938 - \left( (32.1734 \text{ ft/sec}^2 \cdot 15680000 \text{ ft}^2) / (2 \cdot 7840000 \cdot 0.9999982949^2) \right) \]
\[ Y_w = 1.662024417 \text{ ft} - (26060454 / (15680000 \cdot 0.9999965897)) \]
\[ Y_w = 1.662024417 \text{ ft} - 1.662024418 \]
\[ Y_w = 0.0000000005 \text{ ft or rounded to 0.00} \]

Well, that looks like it's a zero to me. Sence we know the range, \( X_w \), and we found the inclination angle of departure, \( \Omega_o \). We're ready to find the Time Of Flight, \( T_w \), with the formula:
Equation 1-9:
\[ X_w = T_w \cdot V_o \cdot \cos \theta_o \]
\[ T_w = X_w \div (\cos \theta_o \cdot V_o) \]
\[ T_w = 900 \text{ ft} \div (\cos 0.1058076404^\circ \cdot 2800 \text{ ft/sec}) \]
\[ T_w = 900 \div (0.9999982949 \cdot 2800 \text{ ft/sec}) \]
\[ T_w = 900 \div 2799.995226 \text{ ft/sec} \]
\[ T_w = 900 \div 2799.995226 \text{ sec} \]
\[ T_w = 0.3214291195 \text{ sec} \quad \text{(There, the Time Of Flight.)} \]

We have another formula for \( Y_w \) but this is dealing with \( T_w \):

Equation 1-10:
\[ Y_w = T_w \cdot V_o \cdot \sin \theta_o - (g \cdot T_w^2 \div 2) \]
\[ Y_w = 0.3214291195 \text{ sec} \cdot 2800 \text{ ft/sec} \cdot \sin 0.1058076404^\circ - (32.1734 \text{ ft/sec}^2 \cdot 0.3214291195^2 \text{ sec}^2 \div 2) \]
\[ Y_w = 0.3214291195 \cdot 2800 \text{ ft} \cdot 0.0018466906 - (32.1734 \text{ ft} \cdot 0.1033166789 \div 2) \]
\[ Y_w = 900.0015346 \text{ ft} \cdot 0.0018466906 - (3.324048836 \text{ ft} \div 2) \]
\[ Y_w = 1.662024417 \text{ ft} - 1.662024418 \text{ ft} \]
\[ Y_w = -0.00000001 \text{ ft or rounded to 0.00} \]

And here is the formula for \( X_w \) this also is dealing with \( T_w \):

Equation 1-10:
\[ X_w = T_w \cdot V_o \cdot \cos \theta_o - (g \cdot T_w^2 \div 2) \]
\[ X_w = 0.3214291195 \text{ sec} \cdot 2800 \text{ ft/sec} \cdot \cos 0.1058076404^\circ - (32.1734 \text{ ft/sec}^2 \cdot 0.3214291195^2 \text{ sec}^2 \div 2) \]
\[ X_w = 0.3214291195 \cdot 2800 \text{ ft} \cdot 0.9999982949 - (32.1734 \text{ ft} \cdot 0.10331667886254528025 \div 2) \]
\[ X_w = 900.0015346 \text{ ft} \cdot 0.9999982949 - (3.3240488357162143195935 \text{ ft} \div 2) \]
\[ X_w = 900.0015346 \text{ ft} - 1.662024417858107159797675 \text{ ft} \]
\[ X_w = 898.337975589525246 \text{ ft} \text{ or rounded to 898 ft} \quad \text{(This one is not as close as I would like it to be, but it is quite good, to less than 1%).} \]

The angle between the inclination angle of origin on the ascending branch and the inclination angle of fall on the descending branch of the trajectory at impact can be verified by differentiating the equation:

Equation 1-8:
\[ Y_w = X_w \cdot \tan \theta_o - \left[(g \cdot X_w^2) \div (2 \cdot V_o^2 \cdot \cos^2 \theta_o)\right] \]

with respect to \( X \):

Equation 1-11:
\[ dY \div dX = \tan \theta_o - \left((g \cdot X_w) \div (V_o^2 \cdot \cos^2 \theta_o)\right) \]

At impact:

Equation 1-4:
\[ X_w = \sin (2 \cdot \theta_o) \cdot (V_o^2 \div g) \]

and substituting into the above equation, we have:

Equation 1-12:
\[ (dY \div dX)_I = \tan \theta_I = \tan \theta_o - (sin (2 \cdot \theta_o) \div \cos^2 \theta_o) \]

With the help of the trigonometric identity we get \( [\tan \theta_I = -\tan \theta_o] \). Thus the angle of fall on level ground is always the negative of the angle of departure, for any vacuum trajectory. This result can be generalized to show that the ascending and descending branches of any vacuum trajectory are symmetric about a vertical line passing through the summit.

You already know enough to find the trajectory summit. But, here are some other formulas to finding the trajectory summit. The Time Of flight to the summit \( (T_s) \) is:
Equation 1-13:
\[ T_s = \frac{(V_o \cdot \sin \Theta_o)}{g} \]
\[ T_s = \frac{(2800 \text{ ft/sec} \cdot \sin 0.1058076404^\circ)}{32.1734 \text{ ft/sec}^2} \]
\[ T_s = \frac{(2800 \text{ sec} \cdot 0.0018466906)}{32.1734} \]
\[ T_s = 5.170733815 \div 32.1734 \]
\[ T_s = 0.1607145597 \] (To check it multiply by 2 and compare to \( T_w \), we're 0.0000000001 off. I think it checks out good.)

The Range to the summit \( (X_s) \) is:

Equation 1-14:
\[ X_s = \frac{[V_o^2 \cdot \sin (2 \cdot \Theta_o)]}{(2 \cdot g)} \]
\[ X_s = \frac{(2800^2 \text{ ft}^2/\text{sec}^2 \cdot \sin (2 \cdot 0.1058076404^\circ))}{(2 \cdot 32.1734 \text{ ft/sec}^2)} \]
\[ X_s = \frac{(7840000 \text{ ft}^2/\text{sec}^2 \cdot \sin 0.2116152808)}{64.3468 \text{ ft/sec}^2} \]
\[ X_s = \frac{(7840000 \text{ ft}^2/\text{sec}^2 \cdot 0.003693375)}{64.3468 \text{ ft/sec}^2} \]
\[ X_s = 28956.06 \text{ ft} \div 64.3468 \]
\[ X_s = 450 \text{ ft} \] (That's half of \( T_w \), 900 ft, checks out good.)

The height of the projectile at the summit \( (Y_s) \), maximum ordinates, is:

Equation 1-15:
\[ Y_s = \frac{(V_o^2 \cdot \sin^2 \Theta_o)}{2 \cdot g} = \frac{g \cdot T_w^2}{8} \]
\[ Y_s = \frac{(2800^2 \text{ ft}^2/\text{sec}^2 \cdot \sin^2 0.1058076404^\circ)}{2 \cdot 32.1734 \text{ ft/sec}^2} = \frac{32.1734 \text{ ft/sec}^2 \cdot 0.3214291195^2}{2} \]
\[ Y_s = \frac{(7840000 \text{ ft}^2/\text{sec}^2 \cdot 0.0018466906^2)}{2 \cdot 64.3468 \text{ ft/sec}^2} = \frac{32.1734 \text{ ft/sec}^2 \cdot 0.1033166789 \text{ sec}^2}{2} \]
\[ Y_s = \frac{26.73648819 \text{ ft}}{64.3468} = 3.324048836 \text{ ft} \div 8 \]
\[ Y_s = 0.4155061043 \text{ ft} \div 64.3468 = 0.0000000002 \text{ ft of each other, checks out good.)} \]

- Envelope of Vacuum Trajectories:

For a fixed muzzle velocity, every vacuum trajectory will, at some point, be tangent to a curve that is defined as the envelope of trajectories. This envelope of trajectories defines the danger space, to aircraft and ground personnel, associated with a firing range. Although actual projectiles will not travel as far nor as high as in the vacuum envelope, the curve of an actual envelope is strikingly similar in appearance to that of Figure below.

![Figure 1-4](https://example.com/figure1-4.png)
Let's define an envelope of vacuum trajectory for a .30-06, 180 grain bullet, with a muzzle velocity of 2800 fps. We'll work the formula out for a muzzle angle of 90º to the horizon, 65º to the horizon, and 45º to the horizon.

The equation for this envelope is:

**Equation 1-16:**
\[ Y_e = \left(\frac{V_o^2}{2 \times g}\right) - \left(\frac{g \times X_w^2}{2 \times V_o^2}\right) \]

First we need to find the range of a projectile with a velocity of 2800 fps at an angle of 65º and 45º. The range of an angle of 90º to the horizon is zero.

**Equation 1-4:**
\[ X_w = \left(\frac{V_o^2 \times \sin(2 \times \theta)}{g}\right) \]

First:
\[ X_w = \left(\frac{2800^2 \text{ ft}^2/\text{sec}^2 \times \sin(2 \times 65^\circ)}{32.1734 \text{ ft/sec}^2}\right) \]
\[ X_w = \left(\frac{7840000 \text{ ft} \times 0.7660444431}{32.1734}\right) \]
\[ X_w = 6005788.434 \text{ ft} \approx 35.35404809 \text{ miles} \]

**AND**

Second:
\[ X_w = \left(\frac{2800^2 \text{ ft}^2/\text{sec}^2 \times \sin(2 \times 45^\circ)}{32.1734 \text{ ft/sec}^2}\right) \]
\[ X_w = \left(\frac{7840000 \text{ ft} \times 0.707107}{32.1734}\right) \]
\[ X_w = 243679.5614 \text{ ft} \approx 46.15143208 \text{ miles} \]

We now plug all three of our \( X_w \) into the envelope of vacuum trajectory formula.

First:
\[ X_w \text{ of zero for 90º:} \]

**Equation 1-16:**
\[ Y_e = \left(\frac{V_o^2}{2 \times g}\right) - \left(\frac{g \times X_w^2}{2 \times V_o^2}\right) \]
\[ Y_e = \left(\frac{2800^2 \text{ ft}^2/\text{sec}^2}{2 \times 32.1734 \text{ ft/sec}^2}\right) - \left(\frac{32.1734 \text{ ft/sec}^2 \times 0^2 \text{ ft}^2}{2 \times 2800^2 \text{ ft}^2/\text{sec}^2}\right) \]
\[ Y_e = \left(\frac{7840000 \text{ ft}^2/\text{sec}^2}{2 \times 32.1734 \text{ ft/sec}^2}\right) - \left(32.1734 \text{ ft/sec}^2 \times 0 \text{ ft}^2 / (2 \times 7840000 \text{ ft}^2/\text{sec}^2)\right) \]
\[ Y_e = 121839.7807 \text{ ft} \]
\[ Y_e = 121839.7807 \text{ ft} = 23.07571604 \text{ miles}; \text{ (X}_w = 0 \text{ ft, Y}_e = 121839.7807 \text{ ft}) \]

Second:
\[ X_w \text{ of 186669.3739 ft for 65º:} \]

**Equation 1-16:**
\[ Y_e = \left(\frac{V_o^2}{2 \times g}\right) - \left(\frac{g \times X_w^2}{2 \times V_o^2}\right) \]
\[ Y_e = \left(\frac{2800^2 \text{ ft}^2/\text{sec}^2 \times \sin(2 \times 180^\circ)}{32.1734 \text{ ft/sec}^2}\right) - \left(\frac{32.1734 \text{ ft/sec}^2 \times 186669.3739^2 \text{ ft}^2}{2 \times 2800^2 \text{ ft}^2/\text{sec}^2}\right) \]
\[ Y_e = \left(\frac{7840000 \text{ ft}^2/\text{sec}^2 \times 64.3468 \text{ ft/sec}^2}{2 \times 32.1734 \text{ ft/sec}^2}\right) - \left(32.1734 \text{ ft/sec}^2 \times 0 \text{ ft}^2 / (2 \times 7840000 \text{ ft}^2/\text{sec}^2)\right) \]
\[ Y_e = \left(\frac{7840000 \text{ ft}^2/\text{sec}^2 \times 64.3468}{2 \times 32.1734 \text{ ft/sec}^2}\right) - (0 \text{ ft} / (2 \times 7840000 \text{ ft}^2/\text{sec}^2)) \]
\[ Y_e = 121839.7807 \text{ ft} \]
\[ Y_e = 121839.7807 \text{ ft} - 23.07571604 \text{ miles; (X}_w = 186669.3739 \text{ ft, Y}_e = 121839.7807 \text{ ft}) \]

Third:
$X_w$ of 243679.5614 ft for 45º:

Equation 1-16:

$$Y_e = \left( \frac{V_o^2}{2} - \frac{X_w^2}{2} \right) - (g * X_w^2 / (2 * V_o^2))$$

$$Y_e = \left( 2800^2 \text{ ft}/\text{sec}^2 / (2 * 32.1734 \text{ ft/sec}^2) \right) - (32.1734 \text{ ft/sec}^2 * 243679.5614^2 \text{ ft}^2 / (2 * 2800^2 \text{ ft}^2/\text{sec}^2))$$

$$Y_e = \left( 7840000 \text{ ft}^2/\text{sec}^2 / (2 * 32.1734 \text{ ft/sec}^2) \right) - (32.1734 \text{ ft/sec}^2 * 59379728644.09636996 \text{ ft}^2 / (2 * 7840000 \text{ ft}^2/\text{sec}^2))$$

$$Y_e = \left( 7840000 \text{ ft} / (2 * 32.1734) \right) - (32.1734 \text{ ft} * 59379728644.09636996 / (2 * 7840000))$$

$$Y_e = 121839.7807 \text{ ft} - 121839.7807 \text{ ft}$$

$$Y_e = 0 \text{ ft} = 0 \text{ miles}; (X_w = 243679.5614 \text{ ft}, Y_e = 0 \text{ ft})$$

If this were plotted, the plot would look something like this:

The reason why you only see two trajectories is that the third is overlaid on the 'Y' axes.

- **Flat-Fire Approximation to the Vacuum Trajectory:**

  The equation for motion in a vacuum is:

  Equation 1-8:

  $$Y_w = X_w * \tan \theta_o - (g * X_w^2 / (2 * V_o^2 * \cos^2 \theta_o))$$

  This equation can be rewritten as:

  Equation 1-17:

  $$Y_w = X_w * \tan \theta_o - ((g * X_w^2 / (2 * V_o^2)) * \sec^2 \theta_o)$$

  Back before the turn of the century up till the end of World War 2 (WWII) the numeric calculation were very time consuming and costly, therefore the approximations were the fastest and less costly way to get satisfactory results to the problems of motion. The flat-fire Approximation was fine assuming that the height of Y was everywhere vary close to the line from the firearm bore to the target, X, as long as one could live with a little less accuracy. The derivative of the trajectory height with respect to elevation angle, at a fixed range, is:

  Equation 1-18:

  $$dY / d\theta_o = X_w * (1 - ((g * X_w / V_o^2) * \tan \theta_o) * \sec^2 \theta_o)$$
If \( \tan^2 \theta_o \ll 1 \) then \( \sec^2 \theta_o \) may be replaced by unity with one more than a one % error. \( \tan^2 \theta_o \) being restricted to a vacuum trajectory for which \( \theta_o < 5 \) degrees, i.e., for which \( \tan \theta_o < 0.1 \), may be treated as a flat-fire trajectory, we have:

Equation 1-19:
\[
Y_w = X_w \cdot \tan \theta_o - \left( \frac{g \cdot X_w^2}{2 \cdot V_o^2} \right)
\]

Equation 1-20:
\[
\frac{dY}{d\theta_o} = X_w \left( 1 - \frac{g \cdot X_w}{V_o^2} \cdot \tan \theta_o \right)
\]

And for shorter ranges, where \( (g \cdot X_w / V_o^2) \cdot \tan \theta_o \ll 1 \), then the equation may be further approximated as:

Equation 1-21:
\[
\frac{dY}{d\theta_o} = X_w
\]

The above equation is usually called the "rigid trajectory" approximation, because a change in elevation angle produces a change in trajectory height that increases in direct proportion to an increasing range, and the trajectory appears to rotate "rigidly" about the origin. The rigid trajectory approximation is valid for a vacuum trajectory if:

Equation 1-22:
\[
\left( \frac{g \cdot X_w}{V_o^2} \right) \cdot \tan \theta_o \ll 1
\]

The error in trajectory height from the flat-fire approximation is the difference between:

Equation 1-19:
\[
Y_w = X_w \cdot \tan \theta_o - \left( \frac{g \cdot X_w^2}{2 \cdot V_o^2} \right)
\]
AND

Equation 1-8:
\[
Y_w = X_w \cdot \tan \theta_o - \left( \frac{g \cdot X_w^2}{2 \cdot V_o^2 \cdot \cos^2 \theta_o} \right)
\]

That is, the approximation:

Equation 1-19:
\[
Y_w = X_w \cdot \tan \theta_o - \left( \frac{g \cdot X_w^2}{2 \cdot V_o^2} \right)
\]
is to high by the amount:

Equation 1-23:
\[
E_y = \left( \frac{g \cdot X_w^2}{2 \cdot V_o^2} \right) \cdot \tan^2 \theta_o
\]

and this vertical error in the flat-fire approximation to the vacuum trajectory increases rapidly with increased range and elevation angle.

For example, let us use the same example as we did before and that was our initial conditions is a muzzle velocity \( (V_o) = 2800 \) fps and range \( (X_w) = 300 \) yards or 900 ft. We will find the angle of departure by the formula:

Equation 1-4:
\[
X_w = \left( \frac{V_o^2 \cdot \sin (2 \cdot \theta_o)}{g} \right)
\]

and after solving for \( \theta_o \), our equation looks like:

Equation 1-7:
\[
\theta_o = \sin^{-1} \left( \frac{(X_w \cdot g) \div V_o^2}{2} \right)
\]
\[
\theta_o = \sin^{-1} \left( \frac{(900 \cdot 32.1734 \text{ ft/sec}^2) \div 2800^2 \text{ ft}^2/\text{sec}^2}{2} \right)
\]
\[
\theta_o = \sin^{-1} \left( \frac{(900 \cdot 32.1734 \text{ ft/sec}^2) \div 2800^2 \text{ ft}^2/\text{sec}^2}{2} \right)
\]
\[
\theta_o = \sin^{-1} \left( \frac{(900 \cdot 32.1734) \div 2800^2}{2} \right)
\]
\[
\theta_o = \sin^{-1} \left( \frac{28956.06 \div 2800^2}{2} \right)
\]
Now the same shot but zeroed at a range \((X_w)\) of 600 yards or 1800 ft, for a comparison.

Equation 1-7:
\[
\Theta_o = \sin^{-1} \left( \frac{(X_w \times g)}{V_o^2} \right) \div 2
\]
\[
\Theta_o = \sin^{-1} \left( \frac{(1800 \text{ ft} \times 32.1734 \text{ ft/sec}^2)}{2800^2 \text{ ft}^2/\text{sec}^2} \right) \div 2
\]
\[
\Theta_o = \sin^{-1} \left( \frac{(1800 \times 32.1734) \div 2800^2}{2} \right)
\]
\[
\Theta_o = \sin^{-1} \left( \frac{57912.12 \div 2800^2}{2} \right)
\]
\[
\Theta_o = \sin^{-1} \left( \frac{57912.12 \div 7840000}{2} \right)
\]
\[
\Theta_o = \sin^{-1} \left( 0.00738675 \right) \div 2
\]
\[
\Theta_o = 0.4232334483 \div 2
\]
\[
\Theta_o = 0.2116167241^\circ
\]

The error for each of these ranges resulting from using the flat-fire approximation is:

Equation 1-23:
\[
E_y = \left( \frac{(g \times X_w^2) \div (2 \times V_o^2)}{\tan^2 \Theta_o} \right)
\]
\[
E_y = \left( \frac{(32.1734 \text{ ft/sec}^2 \times 900^2 \text{ ft}^2 \div (2 \times 2800^2 \text{ ft}^2/\text{sec}^2)) \times \tan^2 0.1058076404^\circ}{2} \right)
\]
\[
E_y = \left( \frac{(32.1734 \text{ ft} \times 900^2) \div (2 \times 2800^2)}{2} \right) \times 0.0018466938^2
\]
\[
E_y = \left( \frac{26060454 \div 15680000}{0.0000034103} \right)
\]
\[
E_y = 166201875 \times 0.0000034103
\]
\[
E_y = .0000056679 \text{ ft} = .0000680154 \text{ inches}
\]

Equation 1-23:
\[
E_y = \left( \frac{(g \times X_w^2) \div (2 \times V_o^2)}{\tan^2 \Theta_o} \right)
\]
\[
E_y = \left( \frac{(32.1734 \text{ ft/sec}^2 \times 1800^2 \text{ ft}^2) \div (2 \times 2800^2 \text{ ft}^2/\text{sec}^2)}{2} \right) \times \tan^2 0.2116167241^\circ
\]
\[
E_y = \left( \frac{(32.1734 \text{ ft} \times 1800^2) \div (2 \times 2800^2)}{2} \right) \times 0.0036934254^2
\]
\[
E_y = \left( \frac{104241816 \div 15680000}{0.000003414} \right)
\]
\[
E_y = 6.648075 \times 0.0000034103
\]
\[
E_y = 0.000090689 \text{ ft} = 0.0010882679 \text{ inches}
\]

By doubling the range we nearly doubled the angle of departure. But the error jumped by a factor of 2^4 power or 16 times.

If we take the primary equation and add the error equation.

Equation 1-8:
\[
Y_w = X_w \times \tan \Theta_o - \left( \frac{(g \times X_w^2) \div (2 \times V_o^2 \times \cos^2 \Theta_o)}{} \right)
\]

PLUS

Equation 1-23:
\[
E_y = \left( \frac{(g \times X_w^2) \div (2 \times V_o^2)}{\tan^2 \Theta_o} \right)
\]

We will get the same result as if we used the Flat-Fire Approximation to the Vacuum Trajectory alone. Let's see, we will use one of the examples from above:

Equation 1-8:
\[
Y_w = X_w \times \tan \Theta_o - \left( \frac{(g \times X_w^2) \div (2 \times V_o^2 \times \cos^2 \Theta_o)}{} \right)
\]
\[
Y_w = 1800 \text{ ft} \times \tan 0.2116167241^\circ - (32.1734 \text{ ft/sec}^2 \times 1800^2 \text{ ft}^2 \div (2 \times 2800^2 \text{ ft}^2/\text{sec}^2 \times 0.0036934254^2)
\]
\[
Y_w = 1800 \text{ ft} \times 0.2116167241^\circ - (32.1734 \text{ ft} \times 1800^2 \div (2 \times 2800^2 \times 0.9999931794^2))
\]

\[
Y_w = 1800 \text{ ft} \times 0.0036934254 - (32.1734 \text{ ft} \times 1800^2 \div (2 \times 2800^2 \times 0.9999931794^2))
\]
Equation 1-23:
\[ Y_w = 6.64816572 \text{ ft.} - (32.1734 \text{ ft/sec}^2 \times 1800^2 \text{ ft}^2) ÷ (2\times 2800^2 \text{ ft}^2/\text{sec}^2) \times \tan^2 0.2116167241^\circ \]
\[ Y_w = (32.1734 \text{ ft} \times 3240000) ÷ (2 \times 7840000) \times 0.0036934254^2 \]
\[ Y_w = 6.648075 \text{ ft.} \times 0.0000136414 \]
\[ Y_w = 0.0000906891 \text{ ft.} \]

And we just add the two together and we get:

Equation 1-24:
\[ Y_w + E_y \]
\[ 0.000000031 \text{ ft.} + 0.0000906891 \text{ ft.} \]
\[ 0.0000907201 \text{ ft.} \]

- Uphill/downhill Firing in a Vacuum:

Special Note: The uphill/downhill section is a very difficult subject to understand and especially hard to simplify were it makes any sense. For this reason I have used almost verbatim form the text in "Modern Exterior Ballistics" by Robert L. McCoy, pages 47 - 51.

The vacuum trajectory of uphill and downhill Firing for all practical purposes is the same. But I will show the more interested reader some very important differences. First of all, we will only be adding one term, "A", to the above formulas of motion and that is an angle called the "superelevation." This is where the (X,Y) plain is inclined at an angle + or - A, relative to the horizontal. To form a new coordinate system that is offset by "A", (Xa,Ya). This angle is positive for uphill firing, and negative for downhill firing.

The gravitational acceleration vector now has an angle 'A' components to it, -g * sin A, -g * cos A. This changes our equations of motion slightly to the form of:

Equation 1-25:
\[ X_a = T_w \times V_o \times \cos \theta_o - (g \times T_w \times \sin A ÷ 2) \]

and
Equation 1-26:
\[ Y_a = T_w \cdot V_o \cdot \sin \Theta_o - (g \cdot T_w^2 \cdot \cos A \div 2) \]

Note: If angle 'A' is set to zero these two equations reduce to the original formula of:

Equation 1-27:
\[ X_w = T_w \cdot V_o \cdot \cos \Theta_o \]

and

Equation 1-28:
\[ Y_w = T_w \cdot V_o \cdot \sin \Theta_o - (g \cdot T_w^2 \div 2) \]

The elimination of time from the equations is more complex than it is for the level ground trajectory because both \( X_a \) and \( Y_a \) vary quadratically with time. There are two special cases of the uphill-downhill vacuum trajectory problem that reduces to relatively simple analytical forms, and these special cases readily illustrate the interesting nature of the problem.

The first case involves an expression for the dependence of slant range to impact, \( R_a \), on the two angles, \( A \) and \( \Theta_o \). Through some Mathematical wizardry the equation:

Equation 1-26:
\[ Y_a = T_w \cdot V_o \cdot \sin \Theta_o - (g \cdot T_w^2 \cdot \cos A \div 2) \]

is transformed into:

Equation 1-29:
\[ R_s \div R = [1 - \tan \Theta_o \cdot \tan A] \cdot \sec A. \]

\( R_s \) is the range at the uphill/downhill angle \( A \), and \( R \) is the same range along the level ground. If \( R_s \div R \) is > 1 than the slant range to impact along the incline will exceed the level ground range, and if \( R_s \div R \) is < 1 than the slant range will be shorter than the level ground range. The equation:

Equation 1-29:
\[ R_s \div R = [1 - \tan \Theta_o \cdot \tan A] \cdot \sec A \]

Illustrates some very interesting properties of vacuum trajectory for uphill and downhill firing.

Since:

Equation 1-30:
\[ \tan (-A) = -\tan A \]

and

Equation 1-31:
\[ \sec (-A) = \sec A, \]

than

Equation 1-32:
\[ R_s \div R > 1 \]

Equation 1-33:
for \(-90° < A < 0\).

This means that the slant range will always exceed the ground range for downhill firing. This makes sense, because gravity ends up aiding the projectile vice retarding it during uphill firing. For uphill firing where

Equation 1-34:
\[ 90° > A > 0 \]
with very small $\theta_o$

Equation 1-32:
$\frac{R_s}{R} > 1$.

However, for larger superelevation angles:

Equation 1-35:
$\frac{R_s}{R} < 1$,

and the slant range will then be less than the level ground range. There is one particular value of $\theta_o$ for which:

Equation 1-36:
$\frac{R_s}{R} = 1$

at any given positive angle '$A$', and this critical value is when:

$\frac{R_s}{R}$

is set to 1 in the equation:

Equation 1-29:
$\frac{R_s}{R} = [1 - \tan \theta_o \cdot \tan A] \cdot \sec A$

and solving for $\theta_{o-cr}$ of:

Equation 1-37:
$\theta_{o-cr} = \tan^{-1} \cdot [(1 - \cos A) \cdot \cot A]$

Where $\theta_{o-cr}$ = critical superelevation angle for $\frac{R_s}{R} = 1$.

For uphill vacuum trajectories if:

Equation 1-38:
$\theta_o < \theta_{o-cr}$

than

Equation 1-32:
$\frac{R_s}{R} > 1$,

if $\theta_o = \theta_{o-cr}$

than

Equation 1-36:
$\frac{R_s}{R} = 1$,

and

if $\theta_o > \theta_{o-cr}$

than

Equation 1-35:
$\frac{R_s}{R} < 1$.

The table along with the two figures below illustrate the interesting effect of various angles of 'A' on the vacuum trajectory.
<table>
<thead>
<tr>
<th>A (Degrees)</th>
<th>$\Theta_{cr}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>15</td>
<td>7.25</td>
</tr>
<tr>
<td>30</td>
<td>13.06</td>
</tr>
<tr>
<td>45</td>
<td>16.33</td>
</tr>
<tr>
<td>60</td>
<td>16.10</td>
</tr>
<tr>
<td>75</td>
<td>11.23</td>
</tr>
<tr>
<td>90</td>
<td>0</td>
</tr>
</tbody>
</table>

The equation $\Theta_{cr} = \tan^{-1} \left( \frac{(1 - \cos A) \cdot \cot A}{\cos A} \right)$ is indeterminate at $A = 0$, and L'Hopital's rule must be used to find $\Theta_{cr}$ at $A = 0$.

Figure 1-7

Figure 1-8
The flat-fire approximation to the equation:

Equation 1-29:
\[ R_s \div R = [1 - \tan \Theta_o \cdot \tan A] \cdot \sec A \]

provides another interesting and useful result. For

Equation 1-39:
|\[ \tan \Theta_o \cdot \tan A | << 1 \]

and thus reduces to the simple form of:

\[ R_s \cdot R \approx \sec A, \text{ or } (R_s \cdot \cos A) \div R \approx 1 \]

("\approx" means approximately equal to).

The second form of these two equations is referred to as the rifleman's rule for uphill/downhill firing; if the slant range to the target is \( R_s \), the rifle sights should be set for the equivalent horizontal range \( R_s \cdot \cos A \), in order to hit the target.

Figure 1-9

However, the flat-fire approximation for uphill/downhill firing is more restrictive than for the level ground case. For moderate 'A' angles, \( \tan A \) is of order unity, which requires that \( \tan \Theta_o << 1 \), which is more restrictive than the corresponding condition, \( \tan^2 \Theta_o << 1 \), for flat-fire across level ground.

The second special case of the uphill-downhill problem is an expression for the height of impact, \( Y_a \), of the vacuum trajectory, at a slant range equal to the corresponding level ground impact range. Solving the equation:

Equation 1-25:
\[ X_a = T_w \cdot V_o \cdot \cos \Theta_o - (g \cdot T_w^2 \cdot \sin A \div 2) \]

for time by means of the quadratic formula we get:

Equation 1-40:
\[ t = ((V_o \cdot \cos \Theta_o) \div (g \cdot \sin A)) \cdot [1 \pm \text{squar root of } (1 - ((2 \cdot g \cdot X_a \cdot \sin A) \div (V_o^2 \cdot \cos^2 \Theta_o))) \]
The root corresponding to the negative sign before the radical in the above equation is the correct solution. After some algebraic manipulation, the equation may be written in the alternative form:

Equation 1-41:
\[ t = (2 \times X_a) \div ((V_o \times \cos \theta_o) \times [1 + \text{the square root of} \,(1 - ((2 \times g \times X_a \times \sin A) \div (V_o^2 \times \cos^2 \theta_o)))]) \]

Now, for:

Equation 1-42:
\[ X = R = (V_o^2 \div g) \times \sin (2 \times \theta_o) \]

Equation 1-43:
\[ [t = (4 \times V_o \times \sin \theta_o) \div (g \times (1 + v))] \]

Where

Equation 1-44:
\[ [v = \text{the square root of}(1 - (4 \times \tan \theta_o \times \sin A))] \]

Substituting the equation:
Equation 1-43:
\[ [t = (4 \times V_o \times \sin \theta_o) \div (g \times (1 + v))] \]
into the equation:
Equation 1-28:
\[ [Y_w = T_w \times V_o \times \sin \theta_o - (g \times T_w^2 \div 2)] \]

We get the equation:

Equation 1-45:
\[ Y_{a1} = ((4 \times V_o^2 \times \sin \theta_o)^2) \div (g \times (1 + v))) \times [1 - ((2 \times \cos A) \div (1 + v))] \]

Before we explore the properties of equations:

Equation 1-44:
\[ [v = \text{the square root of}(1 - (4 \times \tan \theta_o \times \sin A))] \]

and

Equation 1-45:
\[ Y_{a1} = ((4 \times V_o^2 \times \sin \theta_o)^2) \div (g \times (1 + v))) \times [1 - ((2 \times \cos A) \div (1 + v))] \]

We must first address the restriction implied in the \( v \) equation. The parameter \( v \) can be real only if the quantity \( 4 \times \tan \theta_o \times \sin A < 1 \). For downhill firing, \( \sin (-A) = -\sin A \), thus \( v \) is always real for any \( A < 0 \). However, for firing uphill, the parameter \( v \) can be real only if \( \tan \theta_o < (\csc A) \div 4 \). This inequality defines another restriction on \( \theta_o \); the projectile cannot reach an uphill target if the superelevation angle exceeds a maximum value, given by the equation:

Equation 1-46:
\[ \theta_o \text{ max} = \tan^{-1}(\frac{\csc A}{4}) \]

The angle \( \theta_o \text{ max} \) is the superelevation angle above which the uphill vacuum trajectory can never reach the target. The figure below illustrates the variation of the two critical superelevation angles, \( \theta_o \text{ cr} \) and \( \theta_o \text{ max} \), with uphill angle of site, \( A \), and shows how the two curves bound the various solution regions for vacuum trajectories.
Several interesting properties of equations:

Equation 1-44:
\[ v = \sqrt{1 - (4 \cdot \tan \theta_0 \cdot \sin A)} \]

and

Equation 1-45:
\[ Y_{a1} = \left(\frac{4 \cdot V_0^2 \cdot (\sin \theta_0)^2}{g \cdot (1 + v)} \right) \cdot \left[1 - \frac{2 \cdot \cos A}{1 + v}\right] \]

are listed below:

(a) If \( A = 0 \), then \( v = 1 \), and \( Y_{a1} = 0 \). The impact height vanishes, as it should, for level ground firing.

(b) If, \( \theta_0 = \theta_{0\text{cr}} = \tan^{-1} \left[\left(1 - \cos A\right) \cdot \cot A\right], v = (2 \cdot \cos A) - 1 \), and \( Y_{a1} = 0 \). The impact height correctly vanishes for the critical superelevation angle, where \( R_s \div R = 1 \).

(c) For flat-fire (\( \tan \theta_0 \ll 1 \)), \( v = 1 \), and the approximate impact height is given by the equation:

Equation 1-47:
\[ Y_{a1} = \left(\frac{2 \cdot V_0^2 \cdot (\sin \theta_0)^2}{g} \right) \cdot \left[1 - \cos A\right] \]

Now, \( \cos(-a) = \cos A \), and we observe that for flat-fire, the trajectory will always intersect the target above center, and the projectile will strike equally high for either uphill or downhill firing.
"Example:"

An air gun fires a heavy projectile at a muzzle velocity of 250 feet per second. Sight settings have been obtained for ranges between 50 yards and 400 yards, on level ground. Determine the impact locations on target placed at the same range, but along a 30 degree uphill incline.

\[
\begin{array}{c|c|c}
\text{R (Yards)} & \text{R (Feet)} \\
50 & 150 \\
100 & 300 \\
200 & 600 \\
300 & 900 \\
400 & 1200 \\
\end{array}
\]

The first step in the solution is to find the gun elevation angles required to hit the level ground target, using equation:

**Equation 1-7:**

\[
\theta_o = \sin^{-1} \left( \frac{Xw \cdot g}{V_o^2} \right) / 2
\]

\[
\text{Table 1-3:}
\begin{array}{c|c|c}
\text{R (Yards)} & \text{R (Feet)} & \theta_o \text{(Degrees)} \\
50 & 150 & 2.214284152 \\
100 & 300 & 4.441937084 \\
200 & 600 & 8.995410874 \\
300 & 900 & 13.80002990 \\
400 & 1200 & 19.07525171 \\
\end{array}
\]

For \( A = 30 \) degrees use equation:

**Equation 1-29:**

\[
\frac{R_s}{R} = \left[ 1 - \tan \theta_o \cdot \tan A \right] \cdot \sec A
\]

to find the ratio of slant range to level ground range as illustrated in Table 1-4.

\[
\begin{array}{c|c|c|c|c}
\text{R (Yards)} & \text{R (Feet)} & \theta_o \text{(Degrees)} & R_s \div R & R_s \\
50 & 150 & 2.214284152 & 1.128923338 & 56.446 \\
100 & 300 & 4.441937084 & 1.102912457 & 110.291 \\
200 & 600 & 8.995410874 & 1.049165647 & 209.833 \\
300 & 900 & 13.80002990 & 0.990951130 & 297.285 \\
400 & 1200 & 19.07525171 & 0.924168947 & 369.668 \\
\end{array}
\]

The projectile will hit high on the 50, 100, and 200 yard uphill targets, and low on the 300 and 400 yard targets. To find how high or low the impacts will be, we will use the exact parametric equations:

**Equation 1-44:**

\[
v = \sqrt{1 - (4 \cdot \tan \theta_o \cdot \sin A)}
\]

and
**Equation 1-45:**
\[ Y_{a1} = \frac{(4 \cdot V_o^2 \cdot (\sin \theta_o)^2)}{(g \cdot (1 + v))} \cdot [1 - \frac{(2 \cdot \cos A)}{(1 + v)}] \]

The flat-fire approximation to the impact height use equation:

**Equation 1-47:**
\[ Y_{a1} = \frac{(2 \cdot V_o^2 \cdot (\sin \theta_o)^2)}{g} \cdot [1 - \cos A] \]

The flat-fire approximation equation is included for a comparison. Table 1-5 illustrates these three equations below.

<table>
<thead>
<tr>
<th>R (Yards)</th>
<th>50</th>
<th>100</th>
<th>200</th>
<th>300</th>
<th>400</th>
</tr>
</thead>
<tbody>
<tr>
<td>(v = ) the square root of ((1 - (4 \cdot \tan \theta_o \cdot \sin A)))</td>
<td>(0.9605562963)</td>
<td>(0.9190406717)</td>
<td>(0.8266772807)</td>
<td>(0.7132683761)</td>
<td>(0.5553424413)</td>
</tr>
<tr>
<td>(Y_{a1} = \frac{(4 \cdot V_o^2 \cdot (\sin \theta_o)^2)}{(g \cdot (1 + v))} \cdot [1 - \frac{(2 \cdot \cos A)}{(1 + v)}]) (Inches)</td>
<td>(8.2749906450)</td>
<td>(28.399015430)</td>
<td>(64.645577100)</td>
<td>(-33.94890446)</td>
<td>(-727.4769881)</td>
</tr>
<tr>
<td>(Y_{a1} = \frac{(2 \cdot V_o^2 \cdot (\sin \theta_o)^2)}{g} \cdot [1 - \cos A]) (Inches)</td>
<td>(9.324423086)</td>
<td>(37.46675126)</td>
<td>(152.7011399)</td>
<td>(355.3998250)</td>
<td>(667.1241391)</td>
</tr>
</tbody>
</table>

At 50 yard range, flat-fire is a valid assumption for the low velocity air gun, and the error in equation:

**Equation 1-47:**
\[ Y_{a1} = \frac{(2 \cdot V_o^2 \cdot (\sin \theta_o)^2)}{g} \cdot [1 - \cos A] \]

is just over one inch in impact height. For the longer ranges, all of which violate the flat-fire restriction, the accuracy of equation:

**Equation 1-47:**
\[ Y_{a1} = \frac{(2 \cdot V_o^2 \cdot (\sin \theta_o)^2)}{g} \cdot [1 - \cos A] \]

degraded rapidly, and at the two longest ranges, it predicts a high impact on the target, when in fact, the impact will be low.

Try repeating the calculations of the above example for firing downhill along a minus 30 degree incline.

Enough detail has been included in this section to demonstrate that firing uphill and downhill is not a trivial problem, even with the simplifying assumption of a vacuum trajectory. The fact that actual atmosphere trajectories behave in a remarkably similar fashion is sufficient reason to understand the behavior of uphill and downhill vacuum trajectories.
An atmosphere is a complex mixture of gases and vapors with varying atomic weights and characteristics. These gases combined with the gravitational forces of a planet that has a diverse topography and dynamic weather patterns make for a very difficult job of calculating trajectory through an atmosphere. The earth has an atmosphere that we call air. Air is a mixture of gases and vapors that is comprised mostly of about 78.084% Nitrogen (N₂), 20.946% Oxygen (O₂), 0.934% Argon (Ar), 0.033% Carbon dioxide (CO₂), and water vapor and other trace gases, gases like Helium, Hydrogen, Krypton, Neon, Xenon, and Carbon Monoxide. Because of the force of gravity pulling everything down ward to a point at the center of the earth, air has a weight and this weight changes with altitude. At sea level air exerts a force of 14.7 pounds per square inch (psi), 29.92 Inches of Mercury (in Hg), 750 Millimeters of Mercury (mm Hg), 1013 Millibars (mb), or 1.01325 X 10⁵ Newtons Per Square Meters (N/m²).

Because of this complexity with an atmosphere that is made up of various gases, the topography of the landscape, and gravity, drag turns out to be a very complicated function of the size, shape, velocity, and angular velocity of the bullet, and of the temperature, density, and altitude of the air through which it moves.

In classical literature of exterior ballistics the term "particle trajectory" will be found; it has the same meaning as the term "point-mass trajectory" that is found in modern literature. Either term implies a non-spinning, non-lifting projectile whose mass is concentrated at a mathematical point in space. Such a projectile could experience no aerodynamic force other than drag. In the nineteenth to twenty-first centuries exterior ballistics therefore referred to any trajectory affected solely by the forces of aerodynamic drag and gravity as a "particle" or "point-mass" trajectory.

It was Johann Bernoulli (1667 - 1748) of Switzerland, in 1711, first applied analytical solution of the differential equations in describing a point-mass trajectory. Bernoulli’s solution assumed constant air density and drag coefficient, thus it was only valid for low velocities and flat-fire trajectories.

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Measurement of drag became possible in 1740 when Benjamin Robins (1707-1751), invented his ballistic pendulum. An instrument designed to measure the velocity of a projectile directed at it. Knowing the weight of the pendulum, H, with target, K, the weight of the shot fired, and the distance the pendulum moved when struck (measured by strap L), it was possible to calculate the velocity of the shot. By performing experiments at various distances Robins was able to determine the loss of velocity as range increased, and therefore the effects of air density and gravity. The instrument had its faults, but at least its design was based upon Newton’s laws of motion, not on rules of thumb or guesswork. Robins published his findings in the New Principles of Gunnery 1742. Had more notice been taken of them at the time the technical development of ordnance might have progressed much faster. Charles Hutton (1737-1823), who succeeded Robins at Woolwich in London, England, obtained drag results for spheres between 1787 and 1791 that showed close agreement with Robins’ measurements.

Another Swiss mathematician, Leonhard Euler (1707 - 1783), in about 1753 developed the mean-value, short-arc method for solving  systems of ordinary differential equations, and his method of solving elementary point-mass trajectories allowed the use of variable drag coefficients, air densities, and temperatures thus represents the first general solution of the point-mass trajectory.

Both Bernoulli and Euler’s method required the use of quadratures, the approximation of definite integrals by the summation of small squares. Although Euler’s method was successfully used from about 1753 until the latter part of the nineteenth century, the extreme tediousness of the manual arithmetic computations eventually forced exterior ballistics to develop simpler approximate methods for practical trajectory calculations.

A more accurate means of the measurement of drag became possible in the late 1800's after the invention of the electric chronograph and the pressure gauge. In about 1860, Captain Commandant P. Le Boulenger of the Belgian Artillery, invented the electric chronograph which also bears his name. And the Rodman pressure gauge was invented in 1861.

One of the most useful approximate methods was devised around 1880 by Cornal Francesco Siacci of Italy. Siacci's method for flat-fire trajectories with angles of departure of less than 20 degrees was abandoned as impractical for artillery fire by the end of the First World War, its use in direct-fire weapons such as small arms and tank gunnery persisted in the U.S. Army Ordnance until the middle of the twentieth century. The Siacci method is still in almost universal use in the U.S. sporting arms and ammunition industry, and for short-range, flat-fire trajectories of sporting projectiles, its accuracy is sufficient for most practical purposes, see "Trajectory Part 3" on Siacci Method for Flat-Fire Trajectory "Ballistic Coefficient."

Shortly after the start of World War I trajectories with initial elevations up to 45° began to be commonly used, and the previous approximation methods became quite inadequate. It became necessary to find some other way to integrate the differential equations of ballistics. The birth of the computer age made possible the numerical integration method of the differential equation of motion that was adopted in this country in the middle of World War I which was modeled for ballistics from earlier uses in astronomy by Professor F. R. Moulton (1872 - 1952) and his associates. Like the previous methods the numerical integration method was still one of approximation, but it could be refined to give whatever accuracy was needed, and it was applicable to all trajectories, see see "Trajectory Part 4" on the Numerical Integration Method for Flat-Fire Trajectory.
Atmospheric Drag Forces:

Drag Force:

The drag force is in opposition to the forward velocity of the projectile, and thus is always negative. In classical exterior ballistics this was referred to as “air resistance.”

Bullet Diameter:

This refers to the straight cylindrical section of a bullet called the shank also called the reference diameter.

The equation of motion for a vacuum is fairly straightforward but when an atmosphere is introduced the complexity increases by an exponential amount making the equation of motion unsolvable. But there is a silver lining around this dark cloud and that is approximation of the equation of motion that can, in modern days, get the accuracy that is needed.

The equations for a vacuum trajectory where gravity is the only force acting on the projectile inscribes a parabolic path. In an atmosphere trajectory the projectile no longer inscribes a parabolic path but instead inscribes what is called a "ballistic curve". A ballistic curve is much the same as a vacuum trajectory but with some differences. These differences will be shown in red. A ballistic curve is were the projectile leaves the bore, at the origin, with the barrel at a slightly greater positive angle from a straight line to the target, than the vacuum trajectory. this is called the inclination (slope) angle of departure. As the projectile climbs in height or gains altitude to the summit of its flight this is the ascending branch of its trajectory. The height between the origin and the summit of the projectile's flight path is it's maximum ordinate and the summit lies a little farther than half way between the origin and the target. From the summit till the point of impact, the projectile's path is on the descending branch of the trajectory. The angle from the projectile's path on the descending branch of the trajectory to the straight line to the origin is the inclination angle of fall. The projectile's inclination angle of departure is no longer the same angle as the inclination angle of fall. The projectile’s inclination angle of fall is now greater than the inclination angle of departure and the distance from the origin to the summit is now greater than the distance from the summit to the target. And the vertical velocity upward and horizontal velocity at the point of origin will be greater than the vertical velocity downward and horizontal velocity at the point of impact.

The equation of motion for an atmosphere is:

\[ m \frac{dV}{dt} = -\sum F - mg \]

By dividing both sides by the mass (m) of the projectile we get:

\[ \frac{dV}{dt} = -(\sum \frac{F}{m}) - g \]
Where \( \frac{dV}{dt} \) is the vector acceleration, \( V \) is velocity, \( t \) is time, \( \sum F \) is the vector sum of all the atmospheric dynamic forces acting on the projectile, \( m \) is the projectile mass, \( g \) is the acceleration due to gravity. In a point-mass trajectory there is only gravity and atmospheric dynamic forces acting on a projectile so we will disregard the spin and lifting forces of the projectile and the Coriolis force or acceleration due to the earth’s rotation (Coialius Effect).

In the case of small arms the projectile is already accelerated to a velocity when it leaves the muzzle of the barrel. The \( \frac{dV}{dt} \) is the rate at which this muzzle velocity is reduced by. Therefore, the sum of all the dynamic atmospheric forces acting on the projectile must be a negative and the acceleration due to gravity is in the downward direction and it too must by a negative value making the answer a negative to reduce the muzzle velocity. Now it is this reducing factor of the atmosphere that is of prime concern to us. Lets look at these forces now.

The forces that affect a trajectory other than gravity are air density \( (p) \), projectile reference area \( (S) \), and a dimensionless drag coefficient \( (C_D) \).

The air density is made up of atmospheric pressure \( (P) \), the universal gas constant \( (R^*) \), absolute temperature \( (T) \), and the mean molecular weight of the air \( (M) \). The air density is dependent on altitude and humidity. With an increase in altitude this will cause a decrease in the absolute temperature and the acceleration due to gravity. While the fall of the absolute temperature will cause a drop in the humidity, the drop in the acceleration due to gravity will cause the decrease in the atmospheric pressure.

Equation 2-3:
\[
p = \frac{(M \times P)}{(R^* \times T)}
\]

Where \( p \) is the air density, \( P \) is the atmospheric pressure, \( R^* \) is the universal gas constant, and \( T \) is the absolute temperature. It is to be noted that \( M \), the mean molecular weight of air, is assumed to be constant up to an altitude of 90 km, while above this altitude \( M \) varies because of increasing dissociation and diffusive separation.

Since
\[
M_0 \div M = 1, \\
M_0 = M, \\
and \\
M = 28.9644
\]

\[
R^* = 8.31432 \text{joules/(°K) mol}
\]

\[
T_t = 273.15° \text{K}, \\
T_t = 459.67° \text{R},
\]

Equation 2-4:
\[
T = T_t + t
\]

Where \( t \) is the local temperature.

\[
T = T_t + t \\
T = 273.15° K + 15° C \\
T = 288.15° K
\]

The \( P \) at sea level standard atmosphere is 101325 newtons/m².

\[
p_0 = \frac{(M \times P)}{(R^* \times T)} \\
p_0 = \frac{(28.9644 \times 101325.0 \text{newtons/m²})}{(8.31432 \text{joules/(°K)mol} \times 288.15° \text{K})}
\]
\[ p_0 = 2934817.83 \text{ newtons/m}^2 \div 2395.771308 \text{ joules/(°K)/(°K)mol} \]
\[ p_0 = 2934817.83 \text{ newtons/m}^2 \div 2395.771308 \text{ joules/mol} \]
\[ p_0 = 1224.999156 \text{ (newtons/m}^2 \div \text{joules/mol}) \]
\[ p_0 = 1224.999156 \text{ (newtons mol/joules m}^2) \]
\[ p_0 = 1224.999156 \text{ (10}^{-3}\text{kg*kg*m*sec}^2/\text{sec}^2*\text{kg*m}^2/\text{m}^2) \]
\[ p_0 = 1.224999156 \text{ kg/m}^3 = .001224999156 \text{ g/cm}^2 \]

The ratio of the normal air density \( p(y) \) at the altitude \( y \) above sea level to the standard density \( p_0 \) at sea level has been determined from the average of many observations. The value of this ratio usually accepted for ballistics is:

Equation 2-5:
\[ H(y) = 10^{-0.000045y} = e^{-0.0001036y} \]

Equation 2-6:
\[ p(y) = p_0 \times H(y) \]

When the altitude \( y \) is given in meters. Normal air densities at all altitudes rarely or probably never occur simultaneously in nature. But a trajectory can be computed for normal densities and then corrected to account for the variations from normal densities at different altitudes at the time of fire. One should note that \( H(y) \) is also the ratio of normal air densities at any two altitudes \( y \) meters apart since two such densities have values of the form \( p_s H(y_1 + y) \) and \( p_s H(y_1) \)

Equation 2-7:
\[ H(y_1 + y) = H(y_1) \times H(y) \]

The acceleration due to gravity can be calculated using equation 7.

Equation 2-8:
\[ g = G \times m_e \div r_e^2 \]

Where \( g \) is the acceleration due to gravity, \( G \) is the universal constant \((6.673290052 \times 10^{-11} \text{ N * m}^2/\text{kg}^2 \) in the SI notation or \( 6.673290052 \times 10^{-8} \text{ dyne * cm}^2/\text{g}^2 \) in the cgs notation or \( 3.442654153 \times 10^{-8} \text{ lb. * ft}^2/\text{slug}^2 \) in the English notation), \( m_e \) is the mass of the earth \((5.9815436569 \times 10^{24} \text{ kg}) \), and \( r_e \) is the radius of the earth, from the center to the surface. If we wanted to calculate the acceleration due to gravity at the surface of the earth at zero altitude we would use the 6,380,000 meters or 20,926,400 feet for \( r_e \) and the acceleration due to gravity at any altitude we would add the altitude to this figure.

0.0 meters of altitude
\[ g = G \times m_e \div r_e^2 \]
\[ g = 6.673290052 \times 10^{-11} \times 5.9815436569 \times 10^{24} \div (6.38 \times 10^6)^2 \]
\[ g = 6.673290052 \times 10^{-11} \times 5.9815436569 \times 10^{24} \div 4.07044 \times 10^{13} \]
\[ g = 9.80645232 \times \text{N/kg.} \]

Convert
\[ g = 9.80645232 \times \text{kg/m/s}^2 \]
\[ g = 9.80645232 \times \text{m/s}^2 \]

5,000 ft or 1,524 m of altitude
\[ r_e = 6,380,000 \text{ m} + 1,524 \text{ m} \]
\[ r_e = 6,381,524 \text{ m} \]
\[ g = G \times m_e \div r_e^2 \]
\[ g = 6.673290052 \times 10^{-11} \times \text{N} \times \text{m}^2/\text{kg}^2 \times 5.9815436569 \times 10^{24} \text{kg} \div (6,381,524 \text{ m})^2 \]
\[ g = 6.673290052 \times 10^{-11} \times \text{N} \times \text{m}^2/\text{kg}^2 \times 5.9815436569 \times 10^{24} \text{kg} \div 4.0723848562576 \times 10^{13} \text{ m}^2 \]
\[ g = 3.99165757811945 \times 10^{14} \times \text{N}/\text{kg} \div 4.0723848562576 \times 10^{13} \text{ m}^2 \]
\[ g = 9.801769034638 \times \text{N/kg}. \]

Convert
\[ g = 9.801769034638 \times \text{kg} \times \text{m/kg} \times \text{s}^2 \]
\[ g = 9.801769034638 \times \text{m/s}^2 \]

It is common for many people to relate bullet velocity to a speed over land as in so many feet per second but this practice leads to errors in trajectories. While there is a place to know the velocity over land, calculating trajectories should be made with regard to mach numbers. Mach numbers can be calculated by taking the bullet’s velocity divided by the speed of sound. In order to calculate the Mach number the speed of sound must be known. The speed of sound, \( C_s \), can be calculated by:

\[ C_s = \left[ \frac{\gamma \times (R* \div M_o) \times T_M}{2} \right]^{1/2} \]
\[ C_s = [1.40 \times (8.31432 \text{ J/mol (°K)} \div 28.9644) \times 288.15° \text{ K}]^{1/2} \]
\[ C_s = [1.40 \times 0.287053072 \text{ J/mol (°K)} \times 288.15° \text{ K}]^{1/2} \]
\[ C_s = [0.4018743009 \text{ J/mol (°K)} \times 288.15° \text{ K}]^{1/2} \]
\[ C_s = [115.8000798 \text{ J/mol}]^{1/2} \]
\[ C_s = [115.8000798 \text{ kg} \times \text{m}^2/\text{sec}^2 \div 10^{-3} \times \text{kg}]^{1/2} \]
\[ C_s = [115.8000798 \text{ kg} \times \text{m}^2 \times 10^3 \div \text{sec}^2 \times \text{kg}]^{1/2} \]
\[ C_s = [115.8000798 \times \text{m}^2 \times 10^3 \div \text{sec}^2]^{1/2} \]
\[ C_s = 340.2941078 \times \text{m}^2 \div \text{sec}^2 \]
\[ C_s = 340.2941078 \text{ m/sec} \]

Equation 2-10:
\[ T_M = (M_o \div M) \times T \]
\[ M_o \div M = 1 \]
Therefore,
\[ T_M = T \]

Where \( \gamma \) is the ratio of specific heat of air at constant pressure to that at constant volume, and is taken to be 1.40 exact (dimensionless), (\( R^* \), \( M_o \), and \( T_M \) are as defined by the above definition).

Thus the sum of all the aerodynamic drag forces can be rewritten in the form:

Equation 2-11:
\[ \Sigma F = \left[ (p \times S \times C_D) \div (2 \times m) \right] \]
\[ = \left[ (p \times \pi \times C_D \times d^2) \div (8 \times m) \right] \]

Equation 2-12:
\[ dV \div dt = -\left[ (p \times S \times C_D) \div (2 \times m) \right] \times V \times V_{(x, y, z)} - g \]
\[ = -\left[ (p \times \pi \times C_D \times d') \div (8 \times m) \right] \times V \times V_{(x, y, z)} - g \]

Equation 2-13:
\[ V = \text{Square root of} \left[ V_x^2 + V_z^2 + V_y^2 \right] \]

Equation 2-14:
\[ dV_x \div dt = -\left[ (p \times S \times C_D) \div (2 \times m) \right] \times V \times V_{(x)} \]

Equation 2-15:
\[ dV_y \div dt = -\left[ (p \times S \times C_D) \div (2 \times m) \right] \times V \times V_{(y)} - g \]
Equation 2-16:
\[
dV_z \div dt = -\left(\frac{p \cdot S \cdot C_D}{2 \cdot m}\right) \cdot V \cdot V(z)
\]

- **Maximum Gun Elevation:**

  In actual firing in an atmosphere the maximum range of small arms is obtained with an inclination angle of between 29º and 35º. For mortars that have big heavy, low-velocity shells their trajectory is basically the same in an atmosphere as it is in a vacuum and their maximum range is still obtained by an inclination angle of 45º. The only difference is that the shell will fall about 29% short of its vacuum range.

  The first view on the retardation of a bullet was the “square of the velocity” and it was simply taking the velocity times itself and that was the air resistance. This is a fairly close figure as long as the projectile’s velocity is less than about 900 fps.

- **Flat-Fire Approximation:**

  A flat-fire trajectory is defined as a trajectory that is restricted to lie everywhere close to the X-axis, (with no crosswind). Therefore equation 2-13 can be reduced to:

  Equation 2-17:
  \[
  V = \text{Square root of } [V(x)^2 + V(y)^2] \\
  = V(x) \cdot \text{Square root of } [1 + (V(y) \div V(x))^2]
  \]

  It can be shown that \(V\) and \(V(x)\) differ by less than 1/2 of one percent if the following inequality is satisfied:

  \[
  |V(y) \div V(x)| < 10^{-1}
  \]

  \(V(y) \div V(x) = \tan \theta\) is a strict mathematical definition of flat-fire requires both the angle of departure and the angle of fall of the trajectory to be less that 5.7 degrees above the horizontal but in practice, angles as large as fifteen degrees can be tolerated without incurring serious trajectory errors.

  Substituting the flat-fire approximation \(V \sim V(x)\) into equations 2-14 and 2-15 yields the differential equations of motion for a flat-fire trajectory:

  Equation 2-18:
  \[
  \frac{dV_x}{dt} = -\left(\frac{p \cdot S \cdot C_D}{2 \cdot m}\right) \cdot V(x)^2
  \]

  Equation 2-19:
  \[
  \frac{dV_y}{dt} = -\left(\frac{p \cdot S \cdot C_D}{2 \cdot m}\right) \cdot V(x) \cdot V(y) - g
  \]

  Time is the independent variable in equation 2-18 and 2-19. But downrange distance, X, if a much more convenient independent variable than is time. Equations 2-18 and 2-19 are readily transformed into the new equations with distance X as the independent variable:

  Equation 2-20:
  \[
  V_x = -\left(\frac{p \cdot S \cdot C_D}{2 \cdot m}\right) \cdot V(x) \cdot X
  \]

  Equation 2-21:
  \[
  V_y = -\left(\frac{p \cdot S \cdot C_D}{2 \cdot m}\right) \cdot V(y) - (g \div V(x))
  \]
• Constant Drag Coefficient (Square law of Air Resistance)

\[ F = k \]

Equation 2-22:
\[ F = \left( \frac{p \times S \times C_D}{2 \times m} \right) = k, \text{ where } C_D \text{ is constant.} \]

Equation 2-23:
\[ V_x = V_{xo} \times e^{-kt_x} = V_{xo} \times \left[ 1 + (V_{xo} \times k \times t_x) \right] \]

Equation 2-24:
\[ t_x = \left( \frac{X}{V_{xo}} \right) \times \left[ (V_{xo} \div V_x) - 1 \right] \div \left[ \ln(V_{xo} \div V_x) \right] \]

Equation 2-25:
\[ \tan \varnothing = \tan \varnothing_o - \left\{ \left[ \frac{g \times t_x}{V_{xo}} \right] \times \left[ (1 \div 2) \times \left[ 1 - (V_{xo} \div V_x) \right] \right] \right\} \]

Equation 2-26:
\[ Y = Y_o + X \times \tan \varnothing_o - \left\{ (1 \div 2) \times g \times t_x^2 \times \left[ (1 \div 2) + [(V_{xo} \div V_x) - 1]^{-1} + [(V_{xo} \div V_x) - 1]^{-2} \times \ln(V_{xo} \div V_x) \right] \right\} \]

In modern exterior ballistics the approximation with constant drag coefficient is often a useful tool in flat-fire trajectory. Most projectiles at subsonic speeds have nearly constant drag coefficients as well as projectiles flying at hypersonic speeds. Over short distances, the variation in drag coefficient is usually small for any projectile at any flight speed, and a constant drag coefficient is often an adequate approximation in free-flight ballistic range work.

Now for what you all have been waiting for, let's do an example:

Let's take a .45 ACP with a 230-grain hollow point, with a muzzle velocity of 900 fps. The projectile reference diameter is 0.451 inch. The ICAO sea-level standard air density is 0.076474 lb/ft³, and the average measured value of the projectile's drag coefficient is 0.185 at subsonic speeds. We will use equation 2-22 through 2-24 to construct a ballistic table, table 2-1, of velocity and time of flight out to 200 yard range, in 25 yard intervals.

The reference area, S, the projectile's mass, m, and k are:

\[ S = \left( \frac{\pi}{4} \right) \times d^2 \\
S = \left( 3.1415926536 \div 4 \right) \times (0.451 \div 12)^2 \\
S = 0.7853981634 \times (0.0375833)^2 \\
S = 0.7853981634 \times 0.00141250443889 \\
S = 0.001109378392 \text{ ft}^2 \\
\]

\[ m = \text{bullet weight (in grains)} \div 7000 \\
m = 230 \div 7000 \\
m = 0.0328571 \text{ lb} \\
\]

\[ k = \left( \frac{p \times S \times C_D}{2 \times m} \right) \\
k = \left[ \left( 0.076474 \text{ lb/ft}^3 \times 0.001109378392 \text{ ft}^2 \right) \div (2 \times 0.0328571 \text{ lb}) \right] \times 0.185 \\
k = \left[ 8.438394225709889 \times 10^{-5} \text{ lb/ft} \div 0.0657142 \text{ lb} \right] \times 0.185 \\
k = 0.00129102390578912929 \times 0.185 \\
k = 2.388394225709889 \times 10^{-5} \text{ /ft} \\
\]

\[ e = 2.7182818 \]
The downrange striking velocity and time of flight are calculated by equations 2-22 and 2-24. In both equations X is in feet.

**Equation 2-23:**

\[ V_x = V_{xo} \cdot e^{-kX} \]

\[ V_x = 900 \cdot 2.7182818^{-0.0002388394225709889X} \]

**Equation 2-24:**

\[ t_x = \left( \frac{X}{V_{xo}} \right) \cdot \left[ \frac{V_{xo}}{V_x} - 1 \right] \cdot \frac{1}{\ln(V_{xo}/V_x)} \]

\[ t_x = \left( \frac{X}{900} \right) \cdot \left[ \frac{900}{V_x} - 1 \right] \cdot \frac{1}{\ln(900/V_x)} \]

<table>
<thead>
<tr>
<th>Table 2-1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Range (Yards)</td>
</tr>
<tr>
<td>---</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>25</td>
</tr>
<tr>
<td>50</td>
</tr>
<tr>
<td>75</td>
</tr>
<tr>
<td>100</td>
</tr>
<tr>
<td>125</td>
</tr>
<tr>
<td>150</td>
</tr>
<tr>
<td>175</td>
</tr>
<tr>
<td>200</td>
</tr>
</tbody>
</table>

The top of our front sight blade is 0.80225 inch above the centerline of the bore, \( Y_o = -0.80225 \) inch = -0.0668542 ft. We will use equation 26 and the results from table 2-1 to find the elevation angle of the gun in order to zero the pistol at 75 the yard range.

Since the pistol is to be zeroed at 75 yards, \( Y = 0 \) when \( X = 225 \) ft. the time of flight \( (t_x) \) to 75 yards is 0.257 seconds, and \( V_x = 853 \) fps.

**Equation 2-26:**

\[ Y = Y_o + X \cdot \tan \Theta_o - \{(1 \div 2) \cdot g \cdot t_x^2 \cdot [(1 \div 2) + [(V_{xo} \div V_x) - 1]^{-1} - [(V_{xo} \div V_x) - 1]^2 \cdot \ln(V_{xo} \div V_x)]\} \]

\[ 0 = -0.0668542 + 225 \cdot \tan \Theta_o - \{(1 \div 2) \cdot 32.1734 \cdot 0.257^2 \cdot [(1 \div 2) + [(900 \div 853) - 1]^{-1} - [(900 \div 853) - 1]^2 \cdot \ln(900 \div 853)]\} \]

\[ 0 = -0.0668542 + 225 \cdot \tan \Theta_o - \{0.5 \cdot 32.1734 \cdot 0.066049 \cdot [0.5 + 18.14893617 - 329.3838841 \cdot 0.0536352158]\} \]

\[ \tan \Theta_o = \left[0.0668542 + \{0.5 \cdot 32.1734 \cdot 0.066049 \cdot [0.5 + 18.14893617 - 329.3838841 \cdot 0.0536352158]\}\right] \div 225 \]

\[ \tan \Theta_o = \left[0.0668542 + \{0.5 \cdot 32.1734 \cdot 0.066049 \cdot [0.5 + 18.14893617 - 17.66657572]\}\right] \div 225 \]

\[ \tan \Theta_o = \left[0.0668542 + \{0.5 \cdot 32.1734 \cdot 0.066049 \cdot 0.9823604544\}\right] \div 225 \]
\[
\tan \ddot{\Theta}_o = \frac{[0.0668542 + 1.043768247]}{225}
\]
\[
\tan \ddot{\Theta}_o = 1.110622447 \div 225
\]
\[
\tan \ddot{\Theta}_o = 0.0049360998
\]
\[
\ddot{\Theta}_o = \tan^{-1}0.0049360998
\]
\[
\ddot{\Theta}_o = 0.2828153867^\circ = 16.9689 \text{ minutes}
\]

Equation 2-26:
\[
Y = Y_o + X \times \tan \ddot{\Theta}_o - \{(1 \div 2) \times g \times t_x^2 \times \left(1 \div 2 + \left(\frac{V_{xo}}{V_x} \div 1\right)^{-1} - \left(\frac{V_{xo}}{V_x} \div 1\right)^{-2} \times \ln\left(\frac{V_{xo}}{V_x}\right)\right)\}
\]
\[
Y = -0.0668542 + 0.0 \times \tan \ddot{\Theta}_o - \{0.5 \times 32.1734 \times 0.0^2 \times \left[0.5 + \left(\frac{900}{900} \div 1\right)^{-1} - \left(\frac{900}{900} \div 1\right)^{-2} \times \ln\left(\frac{900}{900}\right)\}\}
\]
\[
Y = -0.0668542 - \{0.5 \times 32.1734 \times 0 \times [0.5 + 0 - 0 \times 0]\}
\]
\[
Y = -0.0668542 - 0
\]
\[
Y = -0.0668542 \text{ ft.}
\]
\[
Y = -0.0668542 \text{ ft.} \times 12
\]
\[
Y = -0.8022504 \text{ in.}
\]

<table>
<thead>
<tr>
<th>Table 2-2</th>
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<tbody>
<tr>
<td>Range (Yards)</td>
</tr>
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<tr>
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</tr>
<tr>
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</tr>
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</tr>
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</tr>
</tbody>
</table>
• Drag Coefficient Inversely proportional to Mach Number
(Linear law of Air Resistance)

\[ \Sigma F = k \div M \]

Equation 2-27:
\[ \Sigma F = \left( \frac{p \times S \times C_D}{2 \times m \times M} \right) \]
\[ = k \div M \]

For flat-fire, \( M = \frac{V_x}{C_s} \), where \( M \) is the Mach number, \( C_s \) is the speed of sound.

Equation 2-28:
\[ k_1 = \left( \frac{p \times S \times C_D \times C_s}{2 \times m} \right) \]

Equation 2-29:
\[ \Sigma F = \left( \frac{p \times S \times C_D \times C_s}{2 \times m \times V_x} \right) \]
\[ = k_1 \div V_x \]

Equation 2-30:
\[ V_x = V_{xo} - (k_1 \times X) \]

Equation 2-31a:
\[ t_x = \left[ \frac{X}{V_{xo}} \right] \times \ln(V_{xo} \div V_x) \div \left[ 1 - \left( \frac{V_x}{V_{xo}} \right) \right] \]

Equation 2-31b:
\[ \tan \theta = \tan \theta_o - \left\{ \left[ (g \times t_x) \div V_{xo} \right] \times \left\{ \left[ \ln(V_{xo} \div V_x) - 1 \right] \div \left( V_x \div V_{xo} \right) \right\} \right\} \]

Equation 2-31c:
\[ Y = Y_o + X \times \tan \theta_o - \left\{ \left[ (1 \div 2) \times g \times t_x^2 \times \left[ \left. \ln(V_{xo} \div V_x) \right] \times \left[ 1 - \left( \frac{V_x}{V_{xo}} \right) \right] \right] \right\} \]

Let's take a 260 Remington with a 160-grain open tip Dragonfly bullet (by LattieStone Ballistics), with a muzzle velocity of 2530 fps. The projectile reference diameter is 0.264 inch. The ICAO sea-level standard air density is 0.076474 lb/ft³, and the average measured value of the projectile's drag coefficient is 0.164 at subsonic speeds. We will use equation 2-30 through 2-31c to construct a ballistic table, table 2-3, of velocity and time of flight out to 600 yard range, in 25 yard intervals. The reference area, \( S \), the projectile's mass, \( m \), and \( k \) are:

\[ S = \left( \frac{\pi}{4} \right) \times d^2 \]
\[ S = (3.1415926536 \div 4) \times (0.264 \text{ in} \div 12)^2 \]
\[ S = 0.7853981634 \times (0.022 \text{ ft})^2 \]
\[ S = 0.7853981634 \times 0.000484 \text{ ft}^2 \]
\[ S = 0.000380132711 \text{ ft}^2 \]

\[ m = \text{bullet weight (in grains)} \div 7000 \]
\[ m = 160 \div 7000 \]
\[ m = 0.0228571429 \text{ lb} \]

\[ \gamma = 1.40 \]
\[ R^* = 8.31432 \text{ J/mol (°K)} \]
\[ M_o = 28.9644 \]

Equation 2-10:
\[ T_M = T \]
\[ T = T_t + t \]
\[ T = 273.15 \degree K + 15 \degree C \]
\[ T = 288.15 \degree K \]

Equation 2-9:
\[ C_s = \left[ \gamma \left( R * \frac{M_o}{M} \right) * T \right]^{\frac{1}{2}} \]
\[ C_s = \left[ 1.40 * \frac{8.31432 \text{ J/mol (°K)} \div 28.9644} {288.15 \degree K} \right]^{\frac{1}{2}} \]
\[ C_s = \left[ 0.4018743009 \text{ J/mol (°K)} \div 288.15 \degree K \right]^{\frac{1}{2}} \]
\[ C_s = \left[ 115.8000798 \text{ J/mol} \right]^{\frac{1}{2}} \]
\[ C_s = \left[ 115.8000798 \text{ kg} \div \text{m}^2 \div \text{sec}^2 \right]^{\frac{1}{2}} \]
\[ C_s = 340.2941078 \text{ kg} \div \text{m}^2 \div \text{sec}^2 \]
\[ C_s = 340.2941078 \text{ m/sec} = 1116.450485 \text{ ft/sec} \]

Equation 2-28:
\[ k_1 = \left( \frac{p \times S \times C_{D} \times C_s}{2 \times m} \right) \]
\[ k_1 = \left( \frac{0.076474 \text{ lb/ft}^2 \times 0.000380132711 \text{ ft}^2 \times 0.164 \times 1116.450485 \text{ ft/sec}} {2 \times 0.0228571429 \text{ lb}} \right) \]
\[ k_1 = 0.1164341596 \text{ /sec} \]

The downrange striking velocity and time of flight are calculated by equations 2-30 and 2-31a. In both equations \( X \) is in feet.

Equation 2-30:
\[ V_x = V_{xo} - (k_1 \times X) \]
\[ V_x = 2530 \text{ ft/sec} - (0.5559021154 \text{ /sec} \times X \text{ [in ft.]}) \]

Equation 2-31a:
\[ t_x = \left[ \left( X \div V_{xo} \right) \times \ln(V_{xo} \div V_x) \right] \div \left[ 1 - (V_x \div V_{xo}) \right] \]
\[ t_x = \left[ \left( X \text{ [in ft.]} \div 2530 \right) \times \ln(2530 \div V_x) \right] \div \left[ 1 - (V_x \div 2530) \right] \]

The center of our scope is 2.03 inch above the centerline of the bore, \( Y_o = -2.03 \text{ inch} = -0.1691667 \text{ ft.} \)
We will use equation 2-31c and the results from table 2-3 to find the elevation angle of the gun in order to zero the rifle at 300 the yard range.

Since the rifle is to be zeroed at 300 yards, \( Y = 0 \) when \( X = 900 \text{ ft.} \), the time of flight \( (t_x) \) to 300 yards is 0.3633082502 seconds, and \( V_x = 2030 \text{ fps} \).

Equation 2-31c:
\[ Y = Y_o + X \times \tan \theta_o - \left( \left( \frac{1}{2} \right) \times g \times t_x^2 \times \left( 2 \div \ln(V_{xo} \div V_x) \right) \times \left[ 1 - \left( 1 - (V_x \div V_{xo}) \right) \div \ln(V_{xo} \div V_x) \right] \right) \div X \]
\[ \tan \theta_o = \left[ Y - Y_o + \left( \frac{1}{2} \right) \times g \times t_x^2 \times \left( 2 \div \ln(V_{xo} \div V_x) \right) \times \left[ 1 - \left( 1 - (V_x \div V_{xo}) \right) \div \ln(V_{xo} \div V_x) \right] \right] \div X \]
\[ \tan \theta_o = \left[ 0 - (-0.1691666667) + \left( 0.5 \times 32.1734 \times (0.3633082502)^2 \times \left( 2 \div \ln(2530 \div 2425.209256) \right) \times \left[ 1 - \left( 1 - (2425.209256 \div 2530) \right) \div \ln(2530 \div 2425.209256) \right] \right) \right] \div 900 \]
\[ \tan \theta_o = \left[ 0.1691666667 + \left( 0.5 \times 32.1734 \times 0.1319928847 \times \left( 2 \div \ln(1.043208949) \right) \times \left[ 1 - \left( 1 - 0.9585807337 \right) \div \ln(1.043208949) \right] \right) \right] \div 900 \]
\[\tan \theta_o = \left[0.1691666667 + \left\{0.5 \times 32.1734 \times 0.1319928847 \times \left[\frac{2}{0.0423014908} \times \left(1 - \frac{0.0414192663}{0.0423014908}\right)\right]\right\}\right] \div 900\]

\[\tan \theta_o = \left[0.1691666667 + \left\{2.123329938 \times 47.27965759 \times \left[1 - \frac{0.9791443632}{1}\right]\right\}\right] \div 900\]

\[\tan \theta_o = \left[0.1691666667 + \left\{2.123329938 \times 47.27965759 \times 0.020856368\right\}\right] \div 900\]

\[\tan \theta_o = 2.262870564 \div 900\]

\[\theta_o = \tan^{-1}0.0025143006\]

\[\theta_o = 0.1440585107^\circ\]

Equation 2-31c:

\[Y = Y_o + X \times \tan \theta_o - \{(1 + 2) \times g \times t_x^2 \times \left[2 \div \ln(V_{xo} \div V_x)\right] \times \left[1 - \left\{\left[1 - (V_x \div V_{xo})\right] \div \ln(V_{xo} \div V_x)\right\}\right]\}\]

\[Y = -0.1691666667 + X \times 0.0025143006 - \{0.5 \times 32.1734 \times t_x^2 \times \left[2 \div \ln(2530 \div V_x)\right] \times \left[1 - \left\{\left[1 - (V_x \div 2530)\right] \div \ln(2530 \div V_x)\right\}\right]\}\]

<table>
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Drag Coefficient Inversely proportional to The Square Root of Mach Number (3/2 Power Law of Air Resistance) -

Equation 2-32:
\[ CD = K_3 \div \text{Square root of } M \]

where \( M \) = Mach number, and \( K_3 \) is a constant.

Equation 2-33:
\[ \Sigma F = [(p \times S \times CD) \div (2 \times m)] = [(p \times S \times K_3) \div (2 \times m \times \text{Square root of } M)] \]

Equation 2-34:
\[ k = [(p \times S \times K_3 \times \text{Square root of } a) \div (2 \times m)] \]

Equation 2-35:
\[ \Sigma F = [(p \times S \times K_3 \times \text{Square root of } [a \div V_x]) \div (2 \times m)] = k \div \text{Square root of } V_x \]

Equation 2-36:
\[ V_x = \left[ \text{Square root of } (V_{xo}) - (.5 \times k \times X) \right]^2 \]

Equation 2-37:
\[ t_x = (X \div V_{xo}) \times \text{Square root of } [V_{xo} \div V_x] \]

Equation 2-38:
\[ \tan \theta = \tan \theta_0 - \left\{ [g \times t_x] \div V_{xo} \right\} \times [1 \div 3] \times \{1 \div \text{Square root of } [V_{xo} \div V_x] + (V_{xo} \div V_x)\}] \]

Equation 2-39:
\[ Y = Y_0 + X \times \tan \theta_0 - [1 \div 2] \times g \times t_x^2 \times [(1 \div 3) \times \{1 \div 2 \times \text{Square root of } [V_x \div V_{xo}]\}] \]

In their classic textbook "Exterior Ballistics" McShane, Kelley, and Reno were first to derive equation 2-39 and noted the usefulness of this approximation in computing the gravitational drop of flat-fire trajectories.

In the book "Modern Practical Ballistics" Mr. A. J. Pejsa takes advantage of the fact that the drag coefficients of many modern small arms projectiles are accurately described by equation 2-32 for most of their supersonic flight path.

Figure 2-2 demonstrates that the equation \( CD = (K_3 \div \text{Square root of } M) \) closely agrees with the spark range measured drag coefficient curve for the U.S. service rifle of World War Two the .30 caliber Ball M2 with a 150 grain flat-base spitzer bullet which was fired at a velocity of 2800 fps for a \( k_3 = 0.491 \) in the velocity range between Mach 1.2 to Mach 3.

Let's take a .30-06 with a 180-grain boattail spitzer bullet, with a muzzle velocity of 2800 fps that has a drag coefficient \( K_3 = 0.584 \). Using the International Civil Aviation Organization (ICAO) of 0.076474 lb/ft^3 air density and 1116.45 ft/sec for the speed of sound in air, to calculate tables of terminal velocity, time of flight, gun elevation angle, and terminal angle of fall out to 800 yards range in 50 yard intervals.
The reference area, $S$, the projectile's mass, $m$, and $k$ are:

$$S = \left(\frac{\pi}{4}\right) d^2$$
$$S = (3.14159 \times 26536 \div 4) \times (0.308 \text{ in} \div 12)^2$$
$$S = 0.7853981634 \times (0.02566667 \text{ ft})^2$$
$$S = 0.7853981634 \times 0.0006587778 \text{ ft}^2$$
$$S = 0.0005174029 \text{ ft}^2$$

$$m = \text{bullet weight (in grains)} \div 7000$$
$$m = 180 \div 7000$$
$$m = 0.0257142857 \text{ lb}$$

Equation 2-34:

$$k = \left[ p * S * K_3 * \sqrt{a} \right] \div \left(2 * m\right)$$
$$k = \left[ (0.076474 \text{ lb/ft}^3 \times 0.0005174029 \text{ ft}^2 \times 0.584 \times \sqrt{1116.45 \text{ ft/sec}} \right] \div \left(2 \times 0.0257142857 \text{ lb}\right)$$
$$k = \left[ 0.0007721028 \text{ [ft/sec]^{1/2} / ft} \div 0.0514285714 \text{ lb}\right]$$
$$k = 0.0150131108 \text{ (ft/ft}^2\text{-sec})^{1/2}$$
$$k = 0.0150131108 \text{ (/ft-sec)}^{1/2}$$
$$k = 0.0150131108 \text{ (ft-sec)}^{-1/2}$$

The downrange striking velocity and time of flight are calculated by equations 2-36 and 2-37. In both equations the rang, $X$, is in feet. Equation 2-39 is set to zero, $Y = 0$, to find the gun's angle, $\theta_o$, so that it will be zeroed at range, $X$ in feet, 500 yards. The center of our scope is 2.03 inch above the centerline of the bore, $Y_o = -2.03$ inch $= -0.1691667$ ft. We will use equations 38 and 39 to find the angle of fall and the hight of the bullet with regard to the line of sight.

Equation 2-36:

$$V_x = \left[ \sqrt{V_{xo} - \left(0.5 \times k \times X\right)} \right]^2$$
$$V_x = \left[ \sqrt{2800 \text{ ft/sec} - \left(0.5 \times 0.0150131108 \text{ (ft/ft}^2\text{-sec})^{1/2} \times 1500 \text{ ft}\right) \right]^2$$
$$V_x = \left[ 52.91502622 \text{ (ft/sec)}^{1/2} - \left(0.5 \times 0.0150131108 \text{ (ft/ft}^2\text{-sec})^{1/2} \times 1500 \text{ ft}\right) \right]^2$$
$$V_x = \left[ 52.91502622 \text{ (ft/sec)}^{1/2} - 11.2598331 \text{ (ft/ft}^2\text{-sec})^{1/2} \right]^2$$
$$V_x = \left[ 52.91502622 \text{ (ft/sec)}^{1/2} - 11.2598331 \text{ (ft/ft}^2\text{-sec})^{1/2} \right]^2$$
$$V_x = [41.65519312 \text{ (ft/ft}^2\text{-sec)})^{1/2}]$$
$$V_x = 1735.155114 \text{ ft/sec}$$

Equation 2-37:

$$t_x = \left(\frac{X}{V_{xo}}\right) \times \sqrt{\left[V_{xo} - V_x\right]}$$
$$t_x = \left(1500 \text{ ft} \div 2800 \text{ ft/sec}\right) \times \sqrt{\left[2800 \text{ ft/sec} - 1735.155114 \text{ ft/sec}\right]}$$
$$t_x = (1500 \text{ sec} \div 2800) \times \sqrt{[1.613688585]}$$
$$t_x = 0.5357142857 \text{ sec} \times 1.270310428$$
$$t_x = 0.6805234438 \text{ sec}$$

Since the rifle is to be zeroed at 500 yards, $Y = 0$ when $X = 1500$ ft., the time of flight ($t_x$) to 500 yards is 0.3633082502 seconds, and $V_x = 2800$ fps.

Equation 2-39:

$$Y = Y_o + X \times \tan \theta_o - (0.5 \times g \times t_x^2 \times [(1 \div 3) \times \{1 + (2 \times \text{Square root of}[V_x + V_{xo})]\}])$$
$$\tan \theta_o = \{Y - Y_o + (0.5 \times g \times t_x^2 \times [(1 \div 3) \times \{1 + (2 \times \text{Square root of}[V_x + V_{xo})]\}] \} \div X$$
$$\tan \theta_o = \{(0 - (-0.1691667) \text{ ft}) + (0.5 \times 32.1734 \text{ ft/sec}^2 \times (0.6805234438 \text{ sec})^2 \times [(1 \div 3) \times \{1 + (2 \times \text{Square root of}[V_x + V_{xo})]\}] \} \div X$$
\[
\sqrt{1735.155114 \text{ ft/sec} ÷ 2800 \text{ ft/sec}} \div 1500 \text{ ft}
\]

\[
\tan \theta_o = \left\{0.1691667 \text{ ft} + (0.5 \times 32.1734 \text{ ft/sec}^2 \times 0.4631121576 \text{ sec}^2 \times \left[\frac{1}{3} \times \{1 + (2 \times \sqrt{0.619698255})\}\right]\right\} \div 1500 \text{ ft}
\]

\[
\tan \theta_o = \left\{0.1691667 \text{ ft} + (0.5 \times 32.1734 \text{ ft/sec}^2 \times 0.4631121576 \text{ sec}^2 \times \left[\frac{1}{3} \times \{1 + 1.574418312\}\right]\right\} \div 1500 \text{ ft}
\]

\[
\tan \theta_o = \left\{0.1691667 \text{ ft} + (7.449946346 \text{ ft} \times 0.8581394371)\right\} \div 1500 \text{ ft}
\]

\[
\tan \theta_o = \{0.1691667 \text{ ft} + 6.393092764 \text{ ft}\} \div 1500 \text{ ft}
\]

\[
\tan \theta_o = 6.562259464 \div 1500 \text{ ft}
\]

\[
\tan \theta_o = 0.0043748396
\]

\[
\theta_o = \tan^{-1}0.0043748396
\]

\[
\theta_o = 0.2506582459^\circ
\]

Equation 2-36:
\[
V_x = \left[\sqrt{(V_{xo}) - (0.5 \times k \times X)}\right]^2
\]

\[
V_x = \left[\sqrt{(2800 \text{ ft/sec}) - (0.5 \times 0.0150131108 \text{ (ft-sec)}^{1/2} \times X \text{ ft})}\right]^2
\]

Equation 2-37:
\[
t_x = \left(\frac{X}{V_{xo}}\right) \times \text{Square root of } [V_{xo} / V_x]
\]

\[
t_x = \left(\frac{X \text{ ft}}{2800}\right) \times \text{Square root of } [2800 / V_x]
\]

Equation 2-38:
\[
\tan \theta = \tan \theta_o - \left\{\left[(g \times t_x) ÷ V_{xo}\right] \times \left(\frac{1}{3}\right) \times \{1 + \text{Square root of } [V_{xo} ÷ V_x] + (V_{xo} ÷ V_x)\}\right\}
\]

\[
\tan \theta = 0.0043748396 - \left\{[32.1734 \text{ ft/sec}^2 \times t_x] ÷ 2800 \text{ ft/sec}] \times \left(\frac{1}{3}\right) \times \{1 + \text{Square root of } [2800 \text{ ft/sec ÷ V_x}] + (2800 \text{ ft/sec ÷ V_x})\}\right\}
\]

\[
\theta = \tan^{-1}\theta_o
\]

Equation 2-39:
\[
Y = Y_o + X \times \tan \theta_o - \left[0.5 \times g \times t_x^2 \times \left(\frac{1}{3}\right) \times \{1 + (2 \times \text{Square root of } [V_x ÷ V_{xo}]\)\}\]
\]

\[
Y = -0.1691667 \text{ ft} + X \times 0.0043748396 - \left[0.5 \times 32.1734 \text{ ft/sec}^2 \times t_x^2 \times [0.3333333333 \times \{1 + \{2 \times \text{Square root of } [V_x ÷ 2800 \text{ ft/sec}]\}\}\]
\]

\[
Y (\text{in feet}) = Y \times 12 = Y (\text{in inches})
\]
Table 2-4

<table>
<thead>
<tr>
<th>Range (Yards)</th>
<th>X (Feet)</th>
<th>V_x (ft/sec)</th>
<th>t_x (sec)</th>
<th>θ° (Minutes)</th>
<th>Y (Inches)</th>
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</tr>
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<td>0.05474</td>
<td>12.83002</td>
<td>5.27</td>
</tr>
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<td>100</td>
<td>300</td>
<td>2566.7</td>
<td>0.11191</td>
<td>10.41974</td>
<td>11.37</td>
</tr>
<tr>
<td>150</td>
<td>450</td>
<td>2453.9</td>
<td>0.17167</td>
<td>7.78530</td>
<td>16.15</td>
</tr>
<tr>
<td>200</td>
<td>600</td>
<td>2343.6</td>
<td>0.23422</td>
<td>4.90003</td>
<td>19.47</td>
</tr>
<tr>
<td>250</td>
<td>750</td>
<td>2235.9</td>
<td>0.29975</td>
<td>1.73335</td>
<td>21.23</td>
</tr>
<tr>
<td>300</td>
<td>900</td>
<td>2130.7</td>
<td>0.36847</td>
<td>-1.74988</td>
<td>21.24</td>
</tr>
<tr>
<td>350</td>
<td>1050</td>
<td>2028.0</td>
<td>0.44063</td>
<td>-5.59022</td>
<td>19.33</td>
</tr>
<tr>
<td>400</td>
<td>1200</td>
<td>1927.8</td>
<td>0.51650</td>
<td>-9.83463</td>
<td>15.31</td>
</tr>
<tr>
<td>450</td>
<td>1350</td>
<td>1830.2</td>
<td>0.59635</td>
<td>-14.53767</td>
<td>8.96</td>
</tr>
<tr>
<td>500</td>
<td>1500</td>
<td>1735.2</td>
<td>0.68052</td>
<td>-19.76296</td>
<td>0.0</td>
</tr>
<tr>
<td>550</td>
<td>1650</td>
<td>1642.6</td>
<td>0.76937</td>
<td>-25.58502</td>
<td>-11.84</td>
</tr>
<tr>
<td>600</td>
<td>1800</td>
<td>1552.6</td>
<td>0.86330</td>
<td>-32.09151</td>
<td>-26.91</td>
</tr>
</tbody>
</table>

In table 2-4 we can see that, unlike the vacuum trajectory, the projectile's inclination angle of fall, at 500 yards, is no longer the same as the inclination angle of departure, at zero yards, but is now greater indicating that the trajectory is no longer symmetric. We can also see, by both the inclination angle of the bullet and the height above the line of sight, that the summit is no longer at the mid-point of our zero range but is in fact at a slightly greater range. The negative drag of air is what gives us the graphic shape of the trajectory that is known as the ballistic curve.

- Effects of Wind in The Calculations of The Flat-Fire Trajectory -

The wind completes the components aspect of the atmospheric equation of motion. Besides the complexity of an atmosphere the wind is the next most complex component. While an atmospheric drag is always in the negative direction to the bullets flight the wind component can be either in the negative or in the positive direction to the bullets flight. The wind components of these equations are the same as the component of windage found in the “Windage and Elevation” page on this site. Therefore I will only give the form of the equations and not go into the developing of the problem, why reinvent the wheel.

Equation 2-40:
\[
\vec{W} = \sqrt{V_x^2 + V_y^2 + V_z^2};
\]
Full equation for velocity.

Equation 2-41:
\[
\vec{W} = \sqrt{(V_x - W_x)^2 + (V_y - W_y)^2 + (V_z - W_z)^2};
\]
Full equation for velocity with wind.

Equation 2-41a:
\[
\vec{W} = (V_x - W_x) \times \sqrt{1 + \{(V_y - W_y) / (V_x - W_x)\}^2 + \{(V_z - W_z) / (V_x - W_x)\}^2};
\]

Equation 2-42:
\[
\vec{W} = -\sum \vec{F} \times \nabla V_x (V_x - W_x)
\]

Equation 2-43:
\[ \dot{V}_y = -\mathbf{F} \cdot \dot{V}_y (V_y - W_y) - g \]

Equation 2-44:
\[ \dot{V}_z = -\mathbf{F} \cdot \dot{V}_z (V_z - W_z) \]

With equation 2-41a expanded in series using the binomial theorem and some inequalities understood and satisfied, equations 2-42 through 2-44 can be linearly uncoupled with equation 2-41. \( \dot{V}_n \) and \( (V_n - W_n) \) differ by less than one percent for flat-fire trajectories and equations 2-42 through 2-44 may be rewritten to produce the following three equations.

Equation 2-45:
\[ \dot{V}_x = -\mathbf{F} \cdot (V_x - W_x)^2 \]

Equation 2-46:
\[ \dot{V}_y = -\mathbf{F} \cdot (V_x - W_x) \cdot (V_y - W_y) - g \]

Equation 2-47:
\[ \dot{V}_z = -\mathbf{F} \cdot (V_x - W_x) \cdot (V_z - W_z) \]

General analytical solutions of the flat-fire wind equations are possible by means of quadratures. But a more practical, but not necessarily satisfying, approach would be to determine the effect of one wind component at a time on the projectile’s trajectory. The effect of a vertical wind on a flat-fire trajectory is comparable to the crosswind effect, except that it acts in the vertical direction (Vy) plane or elevation instead of the horizontal direction (Vx or Vz) plane or range/windage.

**Constant Crosswind Effect on Flat-Fire Trajectory:**

Let \( W_x = W_y = 0 \) than equation 2-49a and 2-49b is a constant crosswind, making \( V_y' = 0 \), while \( Z \) is in feet, and \( Z \cdot 12 = \text{inches} \).

Equation 2-48:
\[ V_x = -\mathbf{F} \cdot V_x \]

Equation 2-49:
\[ V_z = -\mathbf{F} \cdot (V_z - W_z) \]

Equation 2-49a:
\[ V_z = W_z \cdot \{1 - \left( V_x \div V_{xo} \right)\} \]

Equation 2-49b:
\[ Z = W_z \cdot \{t - \left( X \div V_{xo} \right)\} \]

Converting wind speed of 8 MPH into feet per second:
\[ W_z = [(8 \cdot 5280) \div 3600] \]
\[ W_z = 11.73 \, \text{fps} \]

Lag Time = \( t - (X \div V_{xo}) \)

\( t \_x \) in seconds is the actual time of bullet flight to range \( X \), \( (X/V_{xo}) \) in seconds is the time it would have taken the bullet flight to range \( X \) in a vacuum, and Lag Time is the difference between the \( T \_x \), the actual time of flight, and \( (X/V_{xo}) \), the time of flight in a vacuum.

Table 2-5
Using The Computation From Table 2-4 Of Range And Time For An 8 MPH Wind
Variable Crosswind Effect on Flat-Fire Trajectory:

Calculate the constant crosswind, $W_z$, starting at range $X_1$, figure 2-3a, where $R > X_1$, given by the equation 2-50.

Equation 2-50:
$$Z(R_1) = W_z \times (t(R) - t(X_1) - \frac{(R - X_1)}{V_{x1}})$$

Then calculate the constant crosswind, $W_z$, starting at range $X_2$, figure 2-3b, where $R > X_2$, given by the equation 2-50a.

Equation 2-50a:
$$Z(R_2) = W_z \times (t(R) - t(X_2) - \frac{(R - X_2)}{V_{x2}})$$

Then subtract $Z(R_2)$ from $Z(R_1)$ to obtain the constant crosswind for that quadrature, figure 2-3c, given by the equation 2-50b. Since any downrange variation of the wind can be approximated by a series of constant winds acting over short intervals provides a general method for calculating the effect of variable crosswinds on the flat-fire trajectories.

Equation 2-50b:
$$Z(R) = W_z \times \{[t(R) - t(X_1)] - \frac{(R - X_1)}{V_{x1}}\} - [t(R) - t(X_2)] - \frac{(R - X_2)}{V_{x2}}$$
Rangewind Effect on Flat-Fire Trajectory:

Let $W_y = W_z = 0$ than equation 2-51 and 2-52 for a constant rangewind. Since there is no crosswind, we can neglect $V_z$.

Equation 2-51:

$$V_x' = \frac{F}{\sqrt{V_x^2}} \left( \frac{2 - (W_x + V_x)}{V_x} \right) - V_x$$

Equation 2-52:

$$V_y' = -\frac{F}{\sqrt{V_x^2}} \left[ 1 - \left( \frac{W_x}{V_x} \right) \right] V_y - \left( \frac{g}{V_x} \right)$$

It can be shown that $V_x$ is at least two orders of magnitude greater than $W_x$ at every point along the trajectory. Thus equation 2-51 and 2-52 can be reduced to equation 2-53 and 2-54 Respectfully without inducing vary much error.

Equation 2-53:

$$V_x' = -\frac{2 F}{\sqrt{V_x^2}} \left( W_x - V_x \right)$$

Equation 2-54:

$$V_y' = -\frac{F}{\sqrt{V_x^2}} V_y - \left( \frac{g}{V_x} \right)$$

Equation 2-55:

$$[V_x] = V_x + \left\{ 2 W_x \left[ 1 - \left( \frac{V_x}{V_{xo}} \right) \right] \right\}$$

Equation 2-56:

$$[V_y] = V_x \cdot \tan \theta$$

Equation 2-57:

$$[t] = t \div \left\{ 1 + \left( 2 W_x \left[ \left( \frac{t}{R} \right) - \left( \frac{1}{V_{xo}} \right) \right] \right) \right\}$$

Equation 2-58:

$$[Y](R) = Y(R) + V_{Y(R)} \cdot \left\{ [t] - t \right\}$$

Elevation Effect of Wind on Flat-Fire Trajectory:

Anemometer measurements of winds near the earth's surface show that vertical wind components are usually much more negligible than winds parallel to the ground. The elevation effect of wind on flat-fire trajectory is so minuscule compared to the horizontal velocity it will not be looked at.

A Special Note on The Ground Effect on Flat-Fire Trajectory:

Shooting near and over flat ground where the projectile is in what is known as ground effect will cause the projectile to shoot a few inches higher than if you are flat shooting across a small crevice or canyon.
The adoption of a "standard bullet," that precise drag measurements could be made, by which all other bullets and their drag decelerations could be compared, is the birth of the ballistic coefficient. The ballistic coefficient of a bullet is usually called "C," and can be defined as: 

\[ C = \frac{\text{drag deceleration of the standard bullet}}{\text{drag deceleration of the actual bullet}}. \]

This statement is true only for bullets that are an exact scaled version of the standard bullet. But, bullets that are similar allow for fairly accurate ballistic computations for the actual bullet.

One of the most useful approximate methods was devised around 1880 by Cornal Francesco Siacci of Italy. Siacci's method for flat-fire trajectories with angles of departure of less than 20 degrees, and it reduces any flat-fire trajectories to easily tabulated quadratures giving distance, time, inclination (flight path angle) and altitude (height) in terms of a "pseudo-velocity". The only restriction being that the angles of departure is sufficiently small to allow for the air density to remain the same and the velocity, \( V \), is very well approximated by \( V_x \sec \theta \) throughout it’s trajectory. The quantity \( V_x \sec \theta \) is the Siacci’s “pseudo-velocity,” and is defined as the product of the horizontal component of velocity and the secant of the angle of departure.

Although the Siacci method was abandoned as impractical for artillery fire by the end of the First World War, its use in direct-fire weapons such as small arms and tank gunnery persisted in the U.S. Army Ordnance until the middle of the twentieth century. The Siacci method is still in almost universal use in the U.S. sporting arms and ammunition industry, and for short-range, flat-fire trajectories of sporting projectiles, its accuracy is sufficient for most practical purposes.

Several countries have carried out tests firing almost worldwide to determine the drag characteristics of a standard bullet adopted by that country. Tests made by Krupp at Meppen in Germany in 1881 and by the Commission d'Experience de Gavre in France from 1873 to 1898 were of particular notice. The Commission d'Experience de Gavre work was very comprehensive and included test firing of velocities up to 6,000 feet per second (FPS) and comprehensive survey of data available from test in other countries. Standard atmospheric conditions for the expression of drag data were adopted a short time after the Commission d'Experience de Gavre work tests were finished. The Krupp test data turned out to be the basis for ballistics tables for small arms, especially sporting and target ammunition, right up to the present time. Krupp used a standard bullet with a flat base design, 3 calibers long and an ogival head of 2 calibers radius. General Mayevski concluded that it was possible to express the retardation as proportional to some power of the velocity in a restricted zone of velocities, using another power of the velocity in the next zone of velocities. He found that the retardation could be expressed in the form \( Av^n \) where both A and n are constant in a restricted velocity zone, but the values of both A and n changes when the velocity zone is passed from one zone to another zone.
• Ingalls' Tables:

In America, the most widely known set of ballistic tables based on Siacci’s method are the Ingalls’ Tables. Colonel James M. Ingalls U.S. Army, in 1884, converted Mayevski's results into English units (left), and based his famous ballistic table on them in 1893. Mayevski’s function is where R is the retardation of the standard bullet in fps. R divided by the proper ballistic coefficient gives the retardation r for any other bullet.

Table 3-1 "Mayevski's values of A and n, converted by Ingalls”

<table>
<thead>
<tr>
<th>Value of n</th>
<th>Velocity limits</th>
<th>Log A equals</th>
</tr>
</thead>
<tbody>
<tr>
<td>R=Cr=Av^2</td>
<td>0 fps to 790 fps</td>
<td>5.6698914 - 10 or -4.3301086</td>
</tr>
<tr>
<td>R=Cr=Av^3</td>
<td>790 fps to 970 fps</td>
<td>2.7734430 - 10 or -7.2265570</td>
</tr>
<tr>
<td>R=Cr=Av^5</td>
<td>970 fps to 1230 fps</td>
<td>6.8018712 - 20 or -13.1981288</td>
</tr>
<tr>
<td>R=Cr=Av^3</td>
<td>1230 fps to 1370 fps</td>
<td>2.9809023 - 10 or -7.0190977</td>
</tr>
<tr>
<td>R=Cr=Av^2</td>
<td>1370 fps to 1800 fps</td>
<td>6.1192596 - 10 or -3.8807404</td>
</tr>
<tr>
<td>R=Cr=Av^1.7</td>
<td>1800 fps to 2600 fps</td>
<td>7.0961978 - 10 or -2.9038022</td>
</tr>
<tr>
<td>R=Cr=Av^1.55</td>
<td>2600 fps to 3600 fps</td>
<td>7.6090480 - 10 or -2.3909520</td>
</tr>
<tr>
<td>Extended from 3600 ft. to 5000 ft.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>R=Cr=Av^1.55</td>
<td>3600 fps to 5000 fps</td>
<td>7.6090480 - 10 or -2.3909520</td>
</tr>
</tbody>
</table>

Table 3-1a "Mayevski's values of A and n, Spherical Projectiles”

<table>
<thead>
<tr>
<th>Value of n</th>
<th>Velocity limits</th>
<th>Log A equals</th>
</tr>
</thead>
<tbody>
<tr>
<td>R=Cr=Av^2</td>
<td>Greater Than 1233 fps</td>
<td>6.3088473 – 10 or -3.6911527</td>
</tr>
<tr>
<td>Value of k = 610.25</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

R=Cr=Av^2(1+(v^2÷k^2)) Less Than 1233 fps | 5.6029333 – 10 or -4.3970667 |

Equation 3-1

R = Cr = (0.284746 * v - 224.221 + Square Root of [(0.234396 * v - 223.754)^2 + 209.043] + [(0.019161 * v * {v - 984.261}) ÷ (371 + {v ÷ 656.174}^10)]) * 0.896

When 0.896 is taken as the coefficient of form of Siacci's projectile compared to Mayevski's, this single formula gives results for the entire range of velocities up to 3,600 foot-seconds that check well with experiment and with Mayevski's discontinuous formula.

• British 1909 Ballistic Table:

The British in 1904 - 1906 carried out test firing experiments and the results, (right), were very similar to that of the Mayevski's results. It was as a result of these firings that the British computed their ballistic Tables of 1909, which are almost identical with those of Ingalls. The British used a standard bullet with an ogival head of 2 calibers radius, is 1 inch in diameter, and weights 1 pound. R = Av^n; also R = Cr = pg; where R is the retardation of the standard bullet in fps, r is the retardation of any other bullet of ballistic coefficient C, v is the velocity of the bullet, m is the index of a power of the velocity in a restricted zone of velocities, A is a constant modifying m, and applying only in a restricted zone of velocities, C is the ballistic coefficient of any bullet as compared with the standard bullet, P is the air resistance in pounds on the nose of the standard bullet, and g is the acceleration due to gravity. The Commission d'Experience de Gavre results were not put into a formula but reduced them to a table showing retardations for each velocity, and this relation became known as the Gavre function. In the United States, a slightly modified form of the Gavre function, usually referred to as the G-function, and later designated by the Ballistic Section of Aberdeen Proving Ground as G_1.

Table 3-2 "British 1909 values of A and m"

<table>
<thead>
<tr>
<th>Limits of velocity</th>
<th>Value of A</th>
<th>Value of m</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 fps to 840 fps</td>
<td>74,422</td>
<td>1.6</td>
</tr>
<tr>
<td>840 fps to 1040 fps</td>
<td>59,939</td>
<td>1.64</td>
</tr>
<tr>
<td>1040 fps to 1190 fps</td>
<td>23,385</td>
<td>6.45</td>
</tr>
<tr>
<td>1190 fps to 1460 fps</td>
<td>95,408</td>
<td>3</td>
</tr>
<tr>
<td>1460 fps to 2000 fps</td>
<td>58,495</td>
<td>1.8</td>
</tr>
<tr>
<td>2000 fps to 2600 fps</td>
<td>15,366</td>
<td>1.67</td>
</tr>
</tbody>
</table>
**Winchester-Western Ballistic Tables:**

In 1965 Winchester-Western published a set of ballistic tables based on four standard drag models for four families of bullets, (right), and defined as below:

- **G₁** drag function - for all bullets except those in the categories below.
- **G₅** drag function - for low base drag bullets (boat tails - 7° 30min Tail Taper w/ 6.19 caliber tangent ogive or tracers).
- **G₆** drag function - for flat base, full patch, sharp nose bullets (6 calibers secant ogive).
- **G₇** drag function - for hollow point and lead nose bullets.

The drag function of G₁, G₅, and G₆ had been developed in earlier research at the U.S. Army Ballistic research laboratories, Aberdeen Proving Ground, Maryland. The G for drag function is in recognition of the extensive work done by the Commission d'Experience de Gavre in France. The drag function G₁ has been and continues to be used by virtually all commercial bullet manufacturers and is almost identical to the drag function used in the Ingalls Tables. Each G function has its own particular standard bullet.

There are also other G-functions:

- **G₇** - for low base drag bullets (boat tails - 7° 30min Tail Taper w/ 10 caliber tangent ogive).
- **G₈** - for round ball - based on 9/16th inch spherical projectile.
- **Rₐ₄** - for .22 Long Rifle, identical to G₁ below 1400 fps.

**Table 3.3**

"1965 Winchester-Western G₁, G₅, G₆, & G₇ Drag Function"

<table>
<thead>
<tr>
<th>u (fps)</th>
<th>G₁(u)</th>
<th>G₅(u)</th>
<th>G₆(u)</th>
<th>G₇(u)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4400</td>
<td>0.4516</td>
<td>0.2324</td>
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</tr>
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<td>0.4418</td>
<td>0.2299</td>
<td>0.1700</td>
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</tr>
<tr>
<td>4200</td>
<td>0.4319</td>
<td>0.2275</td>
<td>0.1694</td>
<td>------</td>
</tr>
<tr>
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<td>0.4222</td>
<td>0.2250</td>
<td>0.1688</td>
<td>------</td>
</tr>
<tr>
<td>4000</td>
<td>0.4125</td>
<td>0.2225</td>
<td>0.1683</td>
<td>------</td>
</tr>
<tr>
<td>3900</td>
<td>0.4030</td>
<td>0.2200</td>
<td>0.1677</td>
<td>------</td>
</tr>
<tr>
<td>3800</td>
<td>0.3935</td>
<td>0.2175</td>
<td>0.1672</td>
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<tr>
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<td>------</td>
</tr>
<tr>
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<td>0.1664</td>
<td>0.6090</td>
</tr>
<tr>
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<td>0.1661</td>
<td>0.5907</td>
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<tr>
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<td>0.1658</td>
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<tr>
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<td>0.3482</td>
<td>0.2050</td>
<td>0.1656</td>
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</tr>
<tr>
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<td>0.1655</td>
<td>0.5330</td>
</tr>
<tr>
<td>3100</td>
<td>0.3320</td>
<td>0.1999</td>
<td>0.1654</td>
<td>0.5129</td>
</tr>
<tr>
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<td>0.3243</td>
<td>0.1973</td>
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<td>0.4927</td>
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<tr>
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<td>0.1947</td>
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<tr>
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<td>0.1654</td>
<td>0.4559</td>
</tr>
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<tr>
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<td>0.2959</td>
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<td>0.1649</td>
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</tr>
<tr>
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<td>0.1643</td>
<td>0.3985</td>
</tr>
<tr>
<td>2400</td>
<td>0.2829</td>
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<td>0.1635</td>
<td>0.3792</td>
</tr>
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<td>0.1789</td>
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<td>0.2627</td>
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<td>0.1586</td>
<td>0.3215</td>
</tr>
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<td>0.1696</td>
<td>0.1560</td>
<td>0.3023</td>
</tr>
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<td>1900</td>
<td>0.2473</td>
<td>0.1651</td>
<td>0.1527</td>
<td>0.2831</td>
</tr>
<tr>
<td>1800</td>
<td>0.2385</td>
<td>0.1594</td>
<td>0.1488</td>
<td>0.2639</td>
</tr>
<tr>
<td>1700</td>
<td>0.2284</td>
<td>0.1525</td>
<td>0.1442</td>
<td>0.2446</td>
</tr>
<tr>
<td>1600</td>
<td>0.2169</td>
<td>0.1444</td>
<td>0.1392</td>
<td>0.2247</td>
</tr>
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<tr>
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<td>0.0348</td>
<td>0.0361</td>
</tr>
<tr>
<td>700</td>
<td>0.0294</td>
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<td>0.0294</td>
</tr>
<tr>
<td>600</td>
<td>0.0249</td>
<td>0.0194</td>
<td>0.0263</td>
<td>0.0249</td>
</tr>
<tr>
<td>500</td>
<td>0.0212</td>
<td>0.0168</td>
<td>0.0223</td>
<td>0.0212</td>
</tr>
<tr>
<td>400</td>
<td>0.0176</td>
<td>0.0139</td>
<td>0.0183</td>
<td>0.0176</td>
</tr>
</tbody>
</table>

Equation: \( G₁(u) = Aₙu^{kₙ} \);
Equation: \( G₅(u) = Aₙu^{kₙ} \);
Equation: \( G₆(u) = Aₙu^{kₙ} \);
Equation: \( G₇(u) = Aₙu^{kₙ} \);
Sierra Bullet Company's Ballistic Tables:

In 1971 Sierra Bullet Co. retested all their bullets with test firings and found that G5 drag function was not the best for the boat tail bullets but the G1 drag function was the best instead. With these findings, the G1 drag function was made the standard drag model for all Sierra bullets. This is fortunate, since the entire Sporting and Firearms Industries bullet's Ballistic Coefficient is based on the same table and can then be compared to one another for a true analysis. The Sierra Reloading Manual 4th Edition says their Ballistic coefficients for the Sierra bullets can be used with the Ingalls Tables without error. The Ingalls Tables has 7 velocity zones and originally it has a top velocity of 3600 fps. Later this was extended to 5000 fps but without any change in the value of n, 1.55, and the value of log A, 7.6090480 - 10. While Sierra bullets calculated their ballistic coefficient on a 9 velocity zones with a maximum velocity of 4400 fps.

The Standard Bullet & The Ballistic Coefficient:

The ballistic coefficient of any standard bullet is always 1.0 and when a bullet being compared is less then 1.0 then it means that bullet will slow down faster then the standard bullet.

A mathematical definition for C is: \( C = \frac{w}{i \cdot d^2} \) * \( \frac{P_0}{P} \); Where w is the weight of the bullet in pounds, d is the diameter of the bullet in inches, i is what is called the form factor, \( P_0 \) is the standard sea-level air density measured in lb/ft³, and P is the local air density also measured in lb/ft³. If the standard sea-level and the local air density is the same the equation reduces to \( C = \frac{w}{i \cdot d^2} \). The formula also shows that it is related to sectional density. Sectional density is equal to the bullet's weight in pounds divided by the square of the bullet's diameter; \( SD = \frac{w}{d^2} \). As you can see from this formula the ballistic coefficient (C) increases as the weight of the bullet increases and decreases as the square of the bullet's diameter increases. We can write C as the sectional density divided by the form factor; \( C = SD + i \). As the sectional density increases the ballistic coefficient will also increase. Here we can see that a bullet with the same form factor but a different sectional density is like moving the graph back and forth along the X-axis of an X-Y plot. The bullet will be the same as for the standard bullet just a little more amount of slowing down will change. But if we change the form factor that is like moving the graph along the Y-axis and no matter what you do with the sectional density the plot will never be like the original. This is why the ballistic coefficient depends on the muzzle velocity of the bullet under test with even the slightest variation on the form factor.

Bullets that are an identical copy, except for scale, to the standard bullet will have the same ballistic coefficient no matter what the initial velocity of the bullet is at the time it is fired. But, if the bullet is different from the standard bullet, even though it is similar, its ballistic coefficient will very depending on what the bullet's initial velocity is.

Drag always acts to slow the bullet down. Drag is actually a vector tangent to the bullet path but against the bullet and as the path curves downward as the range increases, the vector tips upward and part of the vector's drag component will be in the vertical direction, parallel to but opposed to gravity. This vertical component is very small but it will act to keep the bullet from dropping as far as if gravity was acting on the bullet alone.

The drop of a bullet at any range is almost proportional to the square of the time of flight. A bullet with a high ballistic coefficient will have a shorter time of flight, everything else being equal. But a lighter
bullet with a lower ballistic coefficient may have less of a drop at the same range then a heavier bullet with a higher ballistic coefficient because the lighter bullet can have a greater muzzle velocity then a heavier bullet can.

Chronographs measure's the velocity of a bullet by measuring the time the bullet takes to travel a known distance. The formula is: $\text{Velocity} = \frac{\text{distance}}{\text{time}}$. This velocity is really the velocity that the bullet was traveling at a point halfway through the distance. The larger the distance and the more accurate the timer the more accurate the velocity can be determined. All distances and timers used needs to be as accurate as possible or errors will occur. You can calculate ballistic coefficient in three ways but two of the ways are from test firings.

- **First Way To Find BC, "The Form Factor i"**:  

  The first way is the Form Factor, shown in table 3-5 and figure 3-1; designated by i. There is no absolute form factor for any bullet. The form factor is always for a particular ballistic table, and is determined by comparison of the retardation of the bullet in question with the retardation of the standard bullet for which the ballistic table was made. You can estimate the form factor fairly close from the bullet shape chart. Edgar Bugless and Wallace H. Coxe, Ballistic Engineers for the DuPont Co. made the first one as part of a series of ballistic charts called "A Short Cut to Ballistics" published by the DuPont Co. For a given bullet; to estimate the coefficient of form, i. 1) Lay the bullet on the group of ogive curves, (below), marked with the proper caliber for the bullet. 2) Slide the bullet along until it matches one of the ogive curves as closely as possible. 3) Refer to the instruction page and read off the coefficient of form.  

  The Form Factor way of calculating ballistic coefficient is theoretical or the ideal Ballistic Coefficient. It does not take into consideration any other variables that are encountered in the real world as a bullet is fired from a firearm and in flight to the target.

<table>
<thead>
<tr>
<th>Table 3-5</th>
<th>Value - Of - i</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Bullet Form</strong></td>
<td><strong>Normal</strong></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>Blunt Bullet, Cylindrical........</td>
<td>2.30</td>
</tr>
<tr>
<td>Blunt Bullet, Taper Side 0.9 Cal.</td>
<td>1.85</td>
</tr>
<tr>
<td>Blunt Bullet, Taper Side 0.8 Cal.</td>
<td>1.50</td>
</tr>
<tr>
<td>Blunt Bullet, Taper Side 0.7 Cal.</td>
<td>1.30</td>
</tr>
<tr>
<td>Blunt Bullet, Taper Side 0.6 Cal.</td>
<td>1.10</td>
</tr>
<tr>
<td>Head Radius of 0.5 Cal.............</td>
<td>1.10</td>
</tr>
<tr>
<td>Head Radius of 1.0 Cal.............</td>
<td>1.10</td>
</tr>
<tr>
<td>Head Radius of 1.5 Cal.............</td>
<td>0.95</td>
</tr>
<tr>
<td>Head Radius of 2.0 Cal.............</td>
<td>0.85</td>
</tr>
<tr>
<td>Head Radius of 3.0 Cal. M. V.</td>
<td>0.70</td>
</tr>
<tr>
<td>2000 - 3500 f.s.</td>
<td></td>
</tr>
<tr>
<td>under 2000 f.s..............</td>
<td>0.75</td>
</tr>
<tr>
<td>Head Radius of 4.0 Cal. M. V.</td>
<td>0.60</td>
</tr>
<tr>
<td>2000 - 3500 f.s..............</td>
<td>0.60</td>
</tr>
<tr>
<td>under 2000 f.s..............</td>
<td>0.70</td>
</tr>
<tr>
<td>Head Radius of 6.0 Cal. M. V.</td>
<td>0.55</td>
</tr>
<tr>
<td>2000 - 3500 f.s..............</td>
<td>0.55</td>
</tr>
<tr>
<td>under 2000 f.s..............</td>
<td>0.65</td>
</tr>
<tr>
<td>Head Radius of 8.0 Cal. M. V.</td>
<td>0.49</td>
</tr>
<tr>
<td>2000 - 3500 f.s..............</td>
<td>0.49</td>
</tr>
<tr>
<td>under 2000 f.s..............</td>
<td>0.60</td>
</tr>
<tr>
<td>Head Radius of 10.0 Cal. M. V.</td>
<td>0.44</td>
</tr>
<tr>
<td>2000 - 3500 f.s..............</td>
<td>0.44</td>
</tr>
<tr>
<td>under 2000 f.s..............</td>
<td>0.55</td>
</tr>
<tr>
<td>Balls with M. V. under 1000 f.s.</td>
<td>2.00</td>
</tr>
<tr>
<td>Balls with M. V. between 1000 -</td>
<td>1.70</td>
</tr>
<tr>
<td>13000 f.s..................</td>
<td></td>
</tr>
<tr>
<td>Balls with M. V. over 1300 f.s...</td>
<td>1.40</td>
</tr>
</tbody>
</table>
An example, we have a .30 caliber bullet with a blunt point of 0.06” in diameter and a flat base. We put that bullet on the column marked .30. Then slide that bullet down so it best matches the diagram. Our bullet is too slim for the ogive 4 and too fat for the ogive 6, so the ogive evidently has a 5-caliber radius. We see that for velocities over 2000 fps an 4 caliber ogive with a 0.2 caliber flat point has a coefficient of form of 0.70, and one with a 6 caliber ogive with a 0.2 caliber flat point has a coefficient of form of 0.65. So, one with a 5 caliber ogive will have a coefficient of form equal to the average of these two, or about 0.675.

The British Textbook of Small Arms, 1909, gives a formula for calculating the form for pointed bullets, such as the .30-06 spitzer, which gives close results. It is \( i = \frac{2}{n} \times \sqrt{\frac{4n - 1}{7}} \); where \( i \) is the form factor, and \( n \) is the radius of the ogive expressed in calibers. A .30-06 bullet has an ogive with a radius of 2.1 inches, or 7 calibers, and the form factor computed by that formula,

\[
\begin{align*}
\text{Equation 3-2} \\
i &= \frac{2}{7} \times \sqrt{\frac{28 - 1}{7}} \\
i &= \frac{2}{7} \times \sqrt{\frac{27}{7}} \\
i &= 0.285714285714286 \times 1.96396101212393 \\
i &= 0.561131717749695 \text{ or comes out to be } 0.56
\end{align*}
\]

For ogival pointed bullets the radius of the ogive, \( n \), can be determined rather easily by first measuring the diameter and the length of the head of the bullet. Then using the formula \( n = \frac{4L^2 + 1}{4} \); where \( L \) is the length of the head of the bullet expressed in calibers.

If the ballistic coefficient is found by test firing the results must be converted to sea level standard atmospheric conditions. So you must note the altitude, height above sea level, temperature, barometric pressure, and relative humidity and we will put it together after we have learned the two ways of finding the ballistic coefficient from test firing.

- The Creation Of A Ballistic Table, Second Way To Find BC, "The Three Sky Screen Method" - Time of Flight:

The second way is by test firing and we need to find out the Muzzle Velocity, Time of Flight, and Total Range, this is the method used by Sierra Bullets. This takes two chronographs, one to measure the Muzzle Velocity and the other to measure the Time of Flight, but only three sky screens. It will not work if you fire a bullet to measure muzzle velocity then fire another bullet to measure time of flight. The muzzle velocity and time of flight must be measured at the same time for each bullet fired. This requires three screens. The first screen to trigger both chronographs, the second screen to...
stop the first chronograph to calculate the muzzle velocity, a good distance is 10 feet. That chronograph
basically calculates the velocity by the formula $MV = \text{distance} \div \text{time}$. This is the measured velocity half
way between the first and second screens. The third screen stops the second chronograph for the total time
of flight. But we need to take the Time of Flight starting at the muzzle velocity to screen three and that
would be total time of flight minus (Muzzle Velocity time of flight divided by 2). The range is the distance
between the muzzle velocity and the third screen and that is half way between screen one and screen two to
screen third, $\text{Range} = \text{total distance (from screen one to screen three)} \text{minus (muzzle velocity distance (from
screen one to screen two) divided by 2)}$.

Let say we are using a new bullet that we just designed. It is a .277 diameter, 100-grain BTHP with a
sectional density of 0.186.

$$D_1 = 10 \text{ feet};$$
$$D_2 = 155 \text{ feet};$$
$$T_1 = 0.002890 \text{ seconds};$$
$$T_2 = 0.116859 \text{ seconds};$$

Equation 3-3
Range = $D_2 - (D_1 \times \frac{1}{2})$
Range = 155 - (10 \times \frac{1}{2})
Range = 155 - 5 = 150 feet (Range must be in feet);

Equation 3-4
Muzzle Velocity = $D_1 \div T_1$
Muzzle Velocity = 10 $\div$ 0.002890
Muzzle Velocity = 3460.20761245675 or ~ 3460 feet per second;

Equation 3-5
Time of Flight = $T_2 - (T_1 \times \frac{1}{2})$
Time of Flight = 0.116859 - (0.002890 \times \frac{1}{2})
Time of Flight = 0.116859 - 0.001445
Time of Flight = 0.115414 sec.

We need to create a distance and timetable of the standard bullet and we will use the Mayevski's or
Ingalls formula. The calculations for a distance table must be taken in small units and is:

Equation 3-6
Average velocity = $(\text{upper velocity} + \text{lower velocity}) \times \frac{1}{2}$
Average velocity = $(3460 + 3440) \times \frac{1}{2}$
Average velocity = 3450 fps

Then we find the retardation of the standard bullet for this velocity. From the chart of Mayevski's
values of Log A and n, we see that 3450 fps falls between 2600 fps to 5000 fps velocity zone, Log A =
7.6090480 - 10 and n = 1.55.

Equation 3-7
$r = \log(((\log \text{Average velocity}) \times n) + \log A)$
$r = \log((\log 3450) \times 1.55) + 7.6090480 - 10)$
$r = \log(3.53781909507327 \times 1.55) + 7.6090480 - 10)$
$r = \log(5.48361959736357 + 7.6090480 - 10)$
$r = \log(13.0926675973636 - 10)$
$r = \log 3.0926675973636 \text{ ('log is the inverse log, (not the negative of the log), But 10^x) or}$
Now we find the distance the standard bullet travels while it is slowing down from 3460 fps to 3440 fps by:

Equation 3-8
\[
\text{distance table} = \frac{(\text{upper velocity}^2 - \text{lower velocity}^2)}{(2 \times r)}
\]
\[
\text{distance table} = \frac{(3460^2 - 3440^2)}{2 \times 1237.84879252468}
\]
\[
\text{distance table} = \frac{(11971600 - 11833600)}{2474.96958504936}
\]
\[
\text{distance table} = 138000 ÷ 2474.96958504936
\]
\[
\text{distance table} = 55.7582609635374
\]

Now we find the time it took the standard bullet to travel the distance of 55.7582609635374 feet by:

Equation 3-9
\[
\text{time table} = \frac{\text{distance}}{\text{average velocity}}
\]
\[
\text{time table} = \frac{55.7582609635374}{3450}
\]
\[
\text{time table} = 0.016161814772040
\]

Now we do it all over again with the next set of velocity range. In this case, it would be the upper velocity is equal to 3440 and the lower velocity is equal to 3420 or it can be closer to 3440 as long as it is not equal to the upper velocity. It is not a good idea to go greater than 20 fps below the upper velocity at a time for accuracy. Each time you go through this cycle you add the new distance to the old distance to keep a running tab of the total distance, this it a distance table, and add the new time to the old time to keep a running tab of the total time, this is a timetable. You continue to repeat this cycle until the added total time equals or gets as close as you can come to the Time of Flight. Then you divide the total distance the standard bullet traveled in the time your bullet took to travel, in this example 0.115414 sec. the Time of Flight, in to the Range, 150 feet, and that will give you the Ballistic Coefficient. The standard bullet, from Ingalls table, will have traveled about 391.1940127 feet in 0.115414 sec.

Equation 3-10
\[
C = \frac{\text{Range}}{\text{total distance the standard bullet traveled}}
\]
\[
C = \frac{150}{391.1940127}
\]
\[
C = 0.383441451377817 \text{ or } 0.3834 \text{ rounding to the fourth place passed the decimal point is by far good enough.}
\]

- The Creation Of A Ballistic Table, Third Way To Find BC, *"The Four Sky Screen Method"* - Terminal Velocity:

The third way is by test firing also but this time our two chronographs are going to tell us the Muzzle Velocity and the Terminal Velocity over a known Range. This will use four screens, the first two is for the muzzle velocity and the last two is for the terminal velocity. But this time the range is taken at a point half way between the muzzle velocity screens and half way between the terminal velocity screens. So if the screens for the muzzle velocity are 10 feet apart and the screens for the terminal velocity are 10 feet apart and screen one is 210 feet from screen four then the Range will be 200 feet. The distance between screen one and screen four minus (Muzzle velocity distance * \(\frac{1}{2}\)) minus (Terminal velocity distance * \(\frac{1}{2}\)).

Figure 3-3
Let use a different bullet design. It is a 0.277 diameter, 130 grain FMJ with a sectional density of 0.242.

\[ D_1 = 10 \text{ feet}; \]
\[ D_2 = 210 \text{ feet}; \]
\[ T_1 = 0.003045 \text{ sec.}; \]
\[ T_2 = 0.003220923117; \]

Equation 3-11
\[ \text{Range} = R - (D_1 \times \frac{1}{2}) - (D_2 \times \frac{1}{2}) \]
\[ \text{Range} = 210 - (10 \times \frac{1}{2}) - (10 \times \frac{1}{2}) \]
\[ \text{Range} = 210 - 5 - 5 \]
\[ \text{Range} = 200 \text{ feet} \] (Range must be in feet);

Equation 3-12
\[ \text{Muzzle Velocity} = D_1 \div T_1 \]
\[ \text{Muzzle Velocity} = 10 \div 0.003045 \]
\[ \text{Muzzle Velocity} = 3284.07224958949 \text{ or } \approx 3284 \text{ feet per second}; \]

Equation 3-13
\[ \text{Terminal Velocity} = D_2 \div T_2 \]
\[ \text{Terminal Velocity} = 10 \div 0.003220923117 \]
\[ \text{Terminal Velocity} = 3104.6999995809 \text{ or } \approx 3105 \text{ feet per second}. \]

This is a much more direct approach then using Time of Flight but not as accurate. You can use Ingalls table or you can create a distance table as you did in the Time of Flight approach as above. Let's use Ingalls table for our example this time. Look up the muzzle velocity and terminal velocity and write down the associated \( S(u) \) function values. The large "\( V \)" as in \( S(V) \) is for Beginning velocity, in this case it is the muzzle velocity, and a small "\( v \)" as in \( S(v) \) is for ending velocity, in this case it is the terminal velocity. To calculate Ballistic Coefficient or "\( C \)" the formula is: \( C = \text{Range} \div (S(v) - S(V)) \). I have interpolated the terminal velocity of 3104.7 fps.

Equation 3-14
\[ C = \text{Range} \div (S(v) - S(V)) \]
\[ C = 200 \div (s(3104.7) - S(3284)) \]
\[ C = 200 \div (1403.8 - 882.2) \]
\[ C = 200 \div 521.6 \]
\[ C = 0.383435582822086 \text{ rounding to the fourth place we have a ballistic coefficient of 0.3834} \]

Now let's work this out with the terminal velocity rounded up to 3105 fps.

Equation 3-15
\[ C = \text{Range} \div (S(v) - S(V)) \]
\[ C = 200 \div (s(3105) - S(3284)) \]
\[ C = 200 \div (1402.9 - 882.2) \]
\[ C = 200 \div 520.7 \]
\[ C = 0.384098329172268 \text{ rounding to the fourth place we have a ballistic coefficient of 0.3841}. \text{ In most reloading manuals the difference would only be 0.001.} \]

- **Fourth Way To Find BC, "Target Zeroing Method":**

There is a fourth way of finding the ballistic coefficient of a bullet. By zeroing your firearm at say 100 yards for a rifle bullet and chronographing it's velocity and finding out how much it drops at say 200 yards. The muzzle velocity and the amount of drop can be calculated to give you the BC of the bullet. Now with
this method there are so many errors that can be introduced it doesn't lend itself as being very viable, but it is theoretically possible with the muzzle velocity and a ruler.

- Converting To Standard Atmospheric Conditions:

<table>
<thead>
<tr>
<th>Table 3-6 &quot;Standard Altitude &amp; Atmospheric Conditions&quot;</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Altitude</strong></td>
</tr>
<tr>
<td>Sea level</td>
</tr>
<tr>
<td>Barometric pressure</td>
</tr>
<tr>
<td>Temperature</td>
</tr>
<tr>
<td>Relative humidity</td>
</tr>
<tr>
<td>Air density</td>
</tr>
<tr>
<td>Speed of sound</td>
</tr>
</tbody>
</table>

Now that we have found the ballistic coefficient we must convert it to sea level standard atmospheric conditions and from there we can convert that to any altitude or atmospheric conditions that we would want to. Changes in altitude and atmospheric conditions affect a bullet's drag in two different ways. One is by changes in air density, and the other is by a change in the speed of sound. As air density decreased with altitude, or barometric pressure drops below sea level standard pressure, or air temperature rises above sea level standard temperature, or the relative humidity increases, or any combination, bullet drag decreases because the air is thinner and that effectively will increase the ballistic coefficient of a bullet. Likewise if the opposite occurs bullet drag increases because the air is thicker or more density and that will effectively decrease the ballistic coefficient of a bullet. The effect on drag of a change in the speed of sound is more complex, because the effect depends on the bullet velocity relative to the speed of sound, as well as on altitude and atmospheric conditions. Generally, as we increase in altitude, the speed of sound decreases, and this tends to further decrease drag. As you increase in altitude the air density decreases and the ballistic coefficient increases making the bullet shoot flatter and have less Time of Flight then at altitudes closer to sea level. But, this may not always be the case because the muzzle velocity developed by a firearm depends on the temperature of the powder in each round. Higher powder temperature causes higher muzzle velocities, and vice versa. If a shooter develops a load in nice weather at low altitude and takes this same load to a higher altitude it should shoot flatter due to the lower bullet drag. But as you go up in altitude the temperature will drop causing a lower muzzle velocity thereby offsetting the improvement due to the lower drag.

This is where we use the altitude, temperature, barometric pressure, and relative humidity that we recorded at the beginning of our test firings. The formula for this is: $C(\text{converted to sea level}) = C(\text{measured}) \div (Fa^{(1.0 + Ft - Fp)} \cdot Frh)$; where Fa is conversion number for the altitude at test firing, Ft is the temperature at test firing, Fp is the barometric pressure at test firing, and Frh is the relative humidity at test firing. The following two tables will be used to find the standard temperature, standard pressure, and Fa at altitudes from sea level to 15,000 feet, and the vapor pressure of water for the true temperature at the time and location of the test firings.
### Table 3-7
"Standard Atmospheric Conditions versus Altitude"

<table>
<thead>
<tr>
<th>Altitude (ft)</th>
<th>P std (mm Hg)</th>
<th>inches Hg</th>
<th>Fa</th>
<th>T std (° F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sea Level</td>
<td>59.0</td>
<td>750.0</td>
<td>29.53</td>
<td>1.000</td>
</tr>
<tr>
<td>1,000</td>
<td>55.4</td>
<td>722.7</td>
<td>28.45</td>
<td>1.031</td>
</tr>
<tr>
<td>2,000</td>
<td>51.9</td>
<td>696.3</td>
<td>27.41</td>
<td>1.062</td>
</tr>
<tr>
<td>3,000</td>
<td>48.3</td>
<td>670.9</td>
<td>26.41</td>
<td>1.095</td>
</tr>
<tr>
<td>4,000</td>
<td>44.7</td>
<td>646.4</td>
<td>25.45</td>
<td>1.128</td>
</tr>
<tr>
<td>5,000</td>
<td>41.2</td>
<td>622.7</td>
<td>24.52</td>
<td>1.163</td>
</tr>
<tr>
<td>6,000</td>
<td>37.6</td>
<td>599.8</td>
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<td>577.8</td>
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<td>536.1</td>
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</tr>
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<td>18.85</td>
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<td>18.16</td>
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<td>444.0</td>
<td>17.48</td>
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<tr>
<td>15,000</td>
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<td>427.6</td>
<td>16.83</td>
<td>1.573</td>
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### Table 3-8
"Vapor Pressure of Water"

<table>
<thead>
<tr>
<th>Temp (° F)</th>
<th>Vapor-Pressure (mm Hg)</th>
<th>Vapor-Pressure (inches Hg)</th>
</tr>
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<tbody>
<tr>
<td>0</td>
<td>1.0</td>
<td>0.04</td>
</tr>
<tr>
<td>2</td>
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<td>0.08</td>
</tr>
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<td>2.4</td>
<td>0.09</td>
</tr>
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</tr>
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<td>22</td>
<td>2.9</td>
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<td>24</td>
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<td>26</td>
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</tr>
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<tr>
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<td>0.23</td>
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<td>40</td>
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<td>0.25</td>
</tr>
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<td>42</td>
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<td>0.27</td>
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<tr>
<td>44</td>
<td>7.3</td>
<td>0.29</td>
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<tr>
<td>46</td>
<td>7.9</td>
<td>0.31</td>
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<tr>
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<td>9.2</td>
<td>0.36</td>
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</tr>
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<td>58</td>
<td>12.3</td>
<td>0.49</td>
</tr>
<tr>
<td>60</td>
<td>13.3</td>
<td>0.52</td>
</tr>
</tbody>
</table>
Let's say our shooting range is at 2000 feet altitude on a balmy 85° F with a barometric pressure of 29.93 inches of Hg and a relative humidity of 74%. The formula to find $F_t$ is $(T - T_{std}) ÷ (459.6 + T_{std})$; where $T$ is the true temperature at time of firing, $T_{std}$ is the standard atmospheric temperature for that altitude.

Equation 3-16

$$F_t = (T - T_{std}) ÷ (459.6 + T_{std})$$
$$F_t = (85 - 51.9) ÷ (459.6 + 51.9)$$
$$F_t = 33.1 ÷ 511.5$$
$$F_t = 0.064711632453568$$

The formula to find $F_p$ is $(P - P_{std}) ÷ P_{std}$; where $P$ is the true barometric pressure at time of firing, $P_{std}$ is the standard pressure for that altitude. Both pressure value may be in either millimeters or inches of Hg, as long as both are in the same units.

Equation 3-17

$$F_p = (P - P_{std}) ÷ P_{std}$$
$$F_p = (29.93 - 27.41) ÷ 27.41$$
$$F_p = 2.52 ÷ 27.41$$
$$F_p = 0.091937249179132$$

The formula to find $F_{rh}$ is $0.9950 * (P ÷ (P - 0.3783 * rh * VP_w))$; where $P$ is the true barometric pressure at time of firing, $rh$ is the relative humidity at time of firing and is expressed as a decimal fraction, 45% becomes .45, and $VP_w$ is the vapor pressure of water at the temperature at the time of firing. Both $P$ and $VP_w$ may be in either millimeters or inches of Hg, as long as both are in the same units.

Equation 3-18

$$F_{rh} = 0.9950 * (P ÷ (P - (0.3783 * rh * VP_w)))$$
$$F_{rh} = 0.9950 * (29.93 ÷ (29.93 - (0.3783 * .74 * 1.215)))$$
$$F_{rh} = 0.9950 * (29.93 ÷ (29.93 - (0.3783 * .8991)))$$
$$F_{rh} = 0.9950 * (29.93 ÷ (29.93 - .34012953))$$
$$F_{rh} = 0.9950 * 1.01149479617847$$
$$F_{rh} = 1.00643732219758$$

Now we use our formula $C_{(converted to sea level)} = C_{(measured)} ÷ (F_a^{(1.0 + F_t - F_p)} *) F_{rh})$.

Equation 3-19

$$C_c = C_m ÷ (F_a^{(1.0 + F_t - F_p)} *) F_{rh})$$
$$C_c = 0.3834 ÷ (1.062^{(1.0 + 0.064711632453568 - 0.091937249179132) * 1.00643732219758})$$
$$C_c = 0.3834 ÷ (1.062^{(0.97277438327436 * 1.00643732219758)})$$
$$C_c = 0.3834 ÷ (1.06026215668409 * 1.00643732219758)$$
$$C_c = 0.3834 ÷ 1.06708740580057$$
$$C_c = 0.359295778317577 or 0.3593$$

The value 0.3593 is the ballistic coefficient of our bullet, in the first case of test firing, shot at a muzzle velocity of 3460 fps and, in the second case of test firing, at a muzzle velocity of 3284 fps if we had shot them at sea level standard atmospheric conditions.
Converting To A Change In Altitude & Conditions:

Now, let's say we want to take our same load and bullet to 6500 ft altitude with a temperature of 51° F, relative humidity of 40%, and barometric pressure of 29.20. We use the same formulas as before except this time the equation to find our new ballistic coefficient looks like this: $C_{ep} = C_{c} \times (F_a(1.0 + F_t - F_p) \times F_{rh})$; where $C_{ep}$ is the expected ballistic coefficient at the new altitude and $C_{c}$ is the ballistic coefficient at sea level standard atmospheric conditions.

Equation 3-20
$F_t = \left( \frac{T - T_{std}}{459.6 + T_{std}} \right)$
$F_t = \left( \frac{51 - 35.85}{459.6 + 35.85} \right)$
$F_t = 15.15 \div 495.45$
$F_t = 0.030578262185892$

Equation 3-21
$F_p = \left( \frac{P - P_{std}}{P_{std}} \right)$
$F_p = \left( \frac{29.20 - 23.185}{23.185} \right)$
$F_p = 6.015 \div 23.185$
$F_p = 0.259434979512616$

Equation 3-22
$F_{rh} = 0.9950 \times \left( \frac{P}{P - (0.3783 \times rh \times V_{PW})} \right)$
$F_{rh} = 0.9950 \times \left( \frac{29.20}{29.20 - (0.3783 \times 0.40 \times 0.375)} \right)$
$F_{rh} = 0.9950 \times \left( \frac{29.20}{29.20 - 0.05402124} \right)$
$F_{rh} = 0.9950 \times 1.00185347146668$
$F_{rh} = 0.996844204109342$

Equation 3-23
$C_{ep} = C_{c} \times (F_a(1.0 + F_t - F_p) \times F_{rh})$
$C_{ep} = 0.3593 \times \left( \frac{1.2175(1.0 + 0.030578262185892 - 0.259434979512616) \times 0.996844204109342}{29.20 - 23.185} \right)$
$C_{ep} = 0.3593 \times (1.21750.259434979512616 \times 0.996844204109342)$
$C_{ep} = 0.3593 \times (1.16388165251810 \times 0.996844204109342)$
$C_{ep} = 0.3593 \times 1.16020867958187$
$C_{ep} = 0.416862978573765 \text{ or } 0.4169$

Working with the Form Factor i:

Now that we have the ballistic coefficient let's see what else we can do with it. Remember the form factor $i$, we can calculate $i$ for any bullet that we know the muzzle velocity, Range, Time of Flight, or terminal velocity. We just calculate the ballistic coefficient then by using $C = \frac{w}{i \times d^2}$ we can find $i$ by rearranging this formula to look like this: $i = \frac{w}{C \times d^2}$. Let's use the data from the first test firing. We have already found $C$ to be equal to 0.3834 and our bullet diameter is 0.277 and it weighs 100 grains, that's all we need to know to find $i$. But we need to convert 100 grains into pounds, so we divide 7000 into 100 and we get 0.014285714285714 lbs.

Equation 3-24
$i = \frac{w}{C \times d^2}$
$i = 0.014285714285714 \div (0.3834 \times 0.277^2)$
$i = 0.014285714285714 \div (0.3834 \times 0.076729)$
$i = 0.014285714285714 \div 0.0294178986$
$i = 0.48561300995578 \text{ or } 0.49$
Let's try the bullet from the second test firing. That too has a C of 0.3834, a diameter of 0.277, and weighs 130 grain. Divide 130 by 7000 and we get 0.018571428571429 lbs.

Equation 3-25
\[ i = \frac{w}{(C \times d^2)} \]
\[ i = 0.018571428571429 \div (0.3834 \times 0.277^2) \]
\[ i = 0.018571428571429 \div (0.3834 \times 0.076729) \]
\[ i = 0.018571428571429 \div 0.0294178986 \]
\[ i = 0.631296912942279 \text{ or } 0.63 \]

By the way, you can also make the formula look like this: \( i = \frac{w}{(C \times d^2 \times 7000)} \); where w, the weight of the bullet, is in grains now.

• Siacci Equations & Ballistic Tables:

In the Siacci method there are some assumptions made in the creation of these equations. First is the trajectory is everywhere close to the line of sight that the air density and temperature can be assumed to be constant. And secondly is the value of the velocity, \( V \), is well approximated by \( V_x \sec \theta_0 \) along the entire trajectory. The drag is dependent on the Mach number and may be replaced with the dependence on velocity without incurring significant error. A modern Siacci calculation makes for corrections to temperature (speed of sound). Siacci’s quantity \( V_x \sec \theta_0 \) is called the “pseudo-velocity” and is defined as the product of the horizontal component of velocity and the secant of the angle of departure.

Equation 3-26
\[ \hat{V} \cos \theta_0 = -\sum \Sigma F \times V^2 \times \cos \theta_0 \]

Equation 3-27
\[ \frac{dv}{dt} = -\sum \Sigma F \times V^2 \]

Equation 3-28
\[ S = \left( \frac{\pi}{4} \right) \times d^2 \]

Equation 3-29
\[ \sum F = \left[ (p \times \pi \times d^2 \times C_D) \div (8 \times m) \right] \]
\[ = \left[ (p \times \pi \times C_D) \div (8 \times C) \right] \]

Equation 3-30
\[ C = m \div (i \times d^2) \times [P_o \div P], \text{ American practice, } C \text{ has the unit of lb./in.}^2 \text{ When both } [P_o \div P] \text{ and } i \text{ are unity then; } \]
\[ C = m \div d^2 \]

Equation 3-31
\[ G(V) = \left[ (p \times \pi \times C_D \times V) \div 8 \right] \]

Equation 3-32
\[ \Sigma F = G(V) \div (C \times V) \]

The working expressions for computing the ballistic functions. Since both \([P_o \div P]\) and \(i\) are taken as unity in treating the experimental data for determining these constants, but it will be seen by the manner in which \( C \) and \( A_o \) enter into the various expressions for the ballistic functions, that the results will be the same if we make \( A_o \) constant and give to \([P_o \div P]\) and \(i\) suitable values determined by experiment for different kinds of projectiles and atmospheric conditions. By this means ballistic tables computed for standard conditions can be used for all types of projectiles and for any condition of the atmosphere. The following are the equations to calculate the Siacci Inclination function, 3-I, Time of Flight function, 3-T, Space or
Distance function, 3-S, and Altitude function, 3-A from the Mayevski's values of A and n, converted by Ingalls for each velocity zones:

Extended Velocities up to 5000 FPS; Velocities Between 3600 fps and 2600 fps.

Equation 3-S(u)
\[ S(u) = 21780.89 - (10^{2.7377395} \times u^{0.45}) \]

Equation 3-S(V)
\[ S(V) = 21780.89 - (10^{2.7377395} \times V^{0.45}) \]

Equation 3-I(u)
\[ I(u) = 10^{4.0089663} \times (1 ÷ u^{1.55}) \]

Equation 3-I(V)
\[ I(V) = 10^{4.0089663} \times (1 ÷ V^{1.55}) \]

Equation 3-A(u)
\[ A(u) = [10^{6.3585256} \times (1 ÷ u^{1.1})] - 276.636 \]

Equation 3-A(V)
\[ A(V) = [10^{6.3585256} \times (1 ÷ V^{1.1})] - 276.636 \]

Equation 3-T(u)
\[ T(u) = [10^{2.6505893} \times (1 ÷ u^{0.55})] - 4.9502 \]

Equation 3-T(V)
\[ T(V) = [10^{2.6505893} \times (1 ÷ V^{0.55})] - 4.9502 \]

Equation 3-M(u)
\[ M(u) = [10^{5.9644118} \times (1 ÷ u^{2.55})] \]

Equation 3-B(u)
\[ B(u) = [10^{5.0531445} \times (1 ÷ u^{2.1})] - 3.845 \]

Velocities Between 2600 fps and 1800 fps.

Equation 3-S(u)
\[ S(u) = 31227.10 - (10^{3.4266809} \times u^{0.3}) \]

Equation 3-S(V)
\[ S(V) = 31227.10 - (10^{3.4266809} \times V^{0.3}) \]

Equation 3-I(u)
\[ I(u) = [10^{4.4816993} \times (1 ÷ u^{1.7})] + 0.00452 \]

Equation 3-I(V)
\[ I(V) = [10^{4.4816993} \times (1 ÷ V^{1.7})] + 0.00452 \]

Equation 3-A(u)
\[ A(u) = [10^{7.2393734} \times (1 ÷ u^{1.4})] - (10^{1.0817886} \times u^{0.3}) - 39.264 \]

Equation 3-A(V)
\[ A(V) = [10^{7.2393734} \times (1 ÷ V^{1.4})] - (10^{1.0817886} \times V^{0.3}) - 39.264 \]

Equation 3-T(u)
\[ T(u) = [10^{7.05587042} \times (1 ÷ u^{0.7})] - 3.688 \]
Equation 3-T(V)
T(V) = \[10^{3.0587042 \times (1 \div V^{0.7})}\] - 3.688

Equation 3-M(u)
M(u) = \[10^{6.4724384 \times (1 \div u^{2.7})}\] + 0.000102

Equation 3-B(u)
B(u) = \[10^{0.9960294 \times (1 \div u^{2.4})}\] - (10^{-0.5625952 \times u^{0.3}}) + 0.356

Velocities Between 1800 fps and 1370 fps.

Equation 3-S(u)
S(u) = 62875.33 - \[10^{4.2429561 \times \log (u)}\]

Equation 3-S(V)
S(V) = 62875.33 - \[10^{4.2429561 \times \log (V)}\]

Equation 3-I(u)
I(u) = \[10^{5.3880564 \times (1 \div u^{2.0})}\]

Equation 3-I(V)
I(V) = \[10^{5.3880564 \times (1 \div V^{2.0})}\]

Equation 3-A(u)
A(u) = \[10^{8.9677668 \times (1 \div u^{2.0})}\] - \[10^{2.4923277 \times \log (u)}\] + 1051.999

Equation 3-A(V)
A(V) = \[10^{8.9677668 \times (1 \div V^{2.0})}\] - \[10^{2.4923277 \times \log (V)}\] + 1051.999

Equation 3-T(u)
T(u) = \[10^{3.8807404 \times (1 \div u)}\] - 1.8837

Equation 3-T(V)
T(V) = \[10^{3.8807404 \times (1 \div V)}\] - 1.8837

Equation 3-M(u)
M(u) = \[10^{7.4036191 \times (1 \div u^{2.0})}\] + 0.000581

Equation 3-B(u)
B(u) = \[10^{1.0872383 \times (1 \div u^{2.0})}\] - \[10^{1.007207 \times \log (u)}\] + 35.112

Velocities Between 1370 fps and 1230 fps.

Equation 3-S(u)
S(u) = \[10^{7.0190977 \times u}\] + 365.69

Equation 3-S(V)
S(V) = \[10^{7.0190977 \times V}\] + 365.69

Equation 3-I(u)
I(u) = \[10^{8.3503224 \times (1 \div u^{3.0})}\] + 0.06083

Equation 3-I(V)
I(V) = \[10^{8.3503224 \times (1 \div V^{3.0})}\] + 0.06083

Equation 3-A(u)
\[ A(u) = [10^{14.7673601} \times (1 \div u^{4.0})] + [10^{5.8032141} \times (1 \div u)] - 57.984 \]

Equation 3-A(V)
\[ A(V) = [10^{14.7673601} \times (1 \div V^{4.0})] + [10^{5.8032141} \times (1 \div V)] - 57.984 \]

Equation 3-T(u)
\[ T(u) = [10^{6.7180677} \times (1 \div u^{2.0})] + 0.879 \]

Equation 3-T(V)
\[ T(V) = [10^{6.7180677} \times (1 \div V^{2.0})] + 0.879 \]

Equation 3-M(u)
\[ M(u) = [10^{10.4700377} \times (1 \div u^{4.0})] + 0.003016 \]

Equation 3-B(u)
\[ B(u) = [10^{16.7371654} \times (1 \div u^{5.0})] + [10^{4.4985002} \times (1 \div u)] - 6.145 \]

Velocities Between 1230 fps and 970 fps.
Equation 3-S(u)
\[ S(u) = [10^{12.7210075} \times u^{3.0}] + 6034.48 \]

Equation 3-S(V)
\[ S(V) = [10^{12.7210075} \times V^{3.0}] + 6034.48 \]

Equation 3-I(u)
\[ I(u) = [10^{14.3075048} \times (1 \div u^{5.0})] + 0.109117 \]

Equation 3-I(V)
\[ I(V) = [10^{14.3075048} \times (1 \div V^{5.0})] + 0.109117 \]

Equation 3-A(u)
\[ A(u) = [10^{26.6025436} \times (1 \div u^{8.0})] + [10^{11.7589003} \times (1 \div u^{3.0})] + 329.618 \]

Equation 3-A(V)
\[ A(V) = [10^{26.6025436} \times (1 \div V^{8.0})] + [10^{11.7589003} \times (1 \div V^{3.0})] + 329.618 \]

Equation 3-T(u)
\[ T(u) = [10^{12.5960688} \times (1 \div u^{4.0})] + 2.6089 \]

Equation 3-T(V)
\[ T(V) = [10^{12.5960688} \times (1 \div V^{4.0})] + 2.6089 \]

Equation 3-M(u)
\[ M(u) = [10^{16.4199775} \times (1 \div u^{6.0})] + 0.006834 \]

Equation 3-B(u)
\[ B(u) = [10^{28.638638} \times (1 \div u^{9.0})] + [10^{10.5556825} \times (1 \div u^{3.0})] + 12.394 \]

Velocities Between 970 fps and 790 fps.
Equation 3-S(u)
\[ S(u) = [10^{7.226557} \times (1 \div u)] - 5571.36 \]

Equation 3-S(V)
\[ S(V) = [10^{7.226557} \times (1 \div V)] - 5571.36 \]
Equation 3-I(u)
\[ I(u) = \left(10^{8.5577817} \times \left(1 \div u^{3.0}\right)\right) - 0.050275 \]

Equation 3-I(V)
\[ I(V) = \left(10^{8.5577817} \times \left(1 \div V^{3.0}\right)\right) - 0.050275 \]

Equation 3-A(u)
\[ A(u) = \left(10^{15.1822787} \times \left(1 \div u^{4.0}\right)\right) - \left(10^{9.279108} \times \left(1 \div u\right)\right) + 624.046 \]

Equation 3-A(V)
\[ A(V) = \left(10^{15.1822787} \times \left(1 \div V^{4.0}\right)\right) - \left(10^{9.279108} \times \left(1 \div V\right)\right) + 624.046 \]

Equation 3-T(u)
\[ T(u) = \left(10^{6.923527} \times \left(1 \div u^{2.0}\right)\right) - 1.8801 \]

Equation 3-T(V)
\[ T(V) = \left(10^{6.923527} \times \left(1 \div V^{2.0}\right)\right) - 1.8801 \]

Equation 3-M(u)
\[ M(u) = \left(10^{10.624497} \times \left(1 \div u^{4.0}\right)\right) - 0.009169 \]

Equation 3-B(u)
\[ B(u) = \left(10^{7.132084} \times \left(1 \div u^{5.0}\right)\right) - \left(10^{5.188837} \times \left(1 \div u\right)\right) + 106.425 \]

Velocities Between 790 fps and 0 fps.

Equation 3-S(u)
\[ S(u) = 158436.8 - \left(10^{4.6923243} \times \log(u)\right) \]

Equation 3-S(V)
\[ S(V) = 158436.8 - \left(10^{4.6923243} \times \log(V)\right) \]

Equation 3-I(u)
\[ I(u) = \left(10^{5.8374246} \times \left(1 \div u^{2.0}\right)\right) - 0.419591 \]

Equation 3-I(V)
\[ I(V) = \left(10^{5.8374246} \times \left(1 \div V^{2.0}\right)\right) - 0.419591 \]

Equation 3-A(u)
\[ A(u) = \left(10^{8.8665032} \times \left(1 \div u^{2.0}\right)\right) + \left(10^{4.3151504} \times \log(u)\right) - 68192.39 \]

Equation 3-A(V)
\[ A(V) = \left(10^{8.8665032} \times \left(1 \div V^{2.0}\right)\right) + \left(10^{4.3151504} \times \log(V)\right) - 68192.39 \]

Equation 3-T(u)
\[ T(u) = \left(10^{4.3301086} \times \left(1 \div u\right)\right) - 15.4595 \]

Equation 3-T(V)
\[ T(V) = \left(10^{4.3301086} \times \left(1 \div V\right)\right) - 15.4595 \]

Equation 3-M(u)
\[ M(u) = \left(10^{7.8529873} \times \left(1 \div u^{3.0}\right)\right) - 0.045608 \]

Equation 3-B(u)
\[ B(u) = \left(10^{7.7059747} \times \left(1 \div u^{3.0}\right)\right) + \left(10^{3.3513634} \times \log(u)\right) - 7165.846 \]
Mayevski's Drift Functions:

Mayevski, following Desparre's method, which is founded upon the hypothesis that the angle made at any instant by the axis of the projectile with the tangent to the trajectory, is very small, and which holds true for direct fire, has deduced an expression for the drift, which, when modified for direct fire is:

\[
D = \left[ \frac{\pi \cdot \mu}{n} \right] \cdot \left\{ \frac{\gamma \cdot h}{X} \cdot \left[ \frac{g \cdot C \cdot V}{\cos^3 \theta} \right] \cdot \left\{ \frac{[B(u) - B(V)]}{[S(u) - S(V)]} - M(V) \right\} \cdot \left[ \frac{X}{10000} \right] \right\}
\]

\[
\mu = \frac{k^2}{R^2}
\]

where \(k\) is the radius of gyration of the projectile with reference to its axis, and \(R\) its radius. Mayevski gives for the mean value of \(\mu\) for cored shot of the modern type, and \(n\) is the ogive in calibers.

\[
\mu = 0.53
\]

Equation 3-35

Length of head (ogive) = 0.5 * square root of \((4 \cdot n - 1)\)

Equation 3-36

\[
F(n) = \left[ \text{square root of } (4 \cdot n - 1) \div 30 \right] \cdot \left\{ (840 \cdot n^4) - (760 \cdot n^3) + (238 \cdot n^2) - (24 \cdot n) + 3 \right\} \cdot \left\{ 4 \cdot n^2 \cdot (2 \cdot n - 1) \right\} \cdot \left\{ (7 \cdot n^2) - (4 \cdot n) + 1 \right\} \cdot \sin^{-1} \left[ \text{square root of } (4 \cdot n - 1) \div (2 \cdot n) \right]
\]

Equation 3-37

\[
F_1(n) = \left( 1 \div 3 \right) \cdot \left\{ (12 \cdot n^2) - (4 \cdot n) + 1 \right\} \cdot \left[ \text{square root of } (4 \cdot n - 1) \right] \cdot \left\{ 4 \cdot (2 \cdot n) \cdot n^2 \sin^{-1} \left[ \text{square root of } (4 \cdot n - 1) \div (2 \cdot n) \right] \right\}
\]

Equation 3-38

\[
k_1^2 = R^2 \cdot \left[ F(n) \div F_1(n) \right]
\]

or, in calibers,

Equation 3-40

\[
k_1 = \left( 1 \div 4 \right) \cdot \left[ F(n) \div F_1(n) \right]^{1/2}
\]

Table 3-9

| n    | F(n)  | F_1(n)  | |F(n) - F_1(n)| |\text{1/2} |
|------|-------|---------|-----------------|-----------------|
| 0.5  | 0.2667| 0.6667  | 0.632           |
| 1.0  | 0.3921| 1.0074  | 0.624           |
| 1.5  | 0.4862| 1.2586  | 0.622           |
| 2.0  | 0.5647| 1.4674  | 0.621           |
| 2.5  | 0.6336| 1.6499  | 0.620           |
| 3.0  | 0.6937| 1.8141  | 0.619           |
Equation 3-41
\[ a = L - \sqrt{4n - 1}, \text{where } L \text{ is the total length of the projectile, and } a \text{ is the cylindrical part of the projectile.} \]

Equation 3-42
\[ \mu = \frac{k^2}{R^2} = \frac{(F(n) + a)}{(F_1(n) + (2 \cdot a))} \]

**Siacci Equations of Primary Functions:**

The utility of the Siacci method is based on the use of numerical quadrature to tabulate four primary functions which are the solutions of the Siacci equations for Time of Flight function, 3-T, Space or Distance function, 3-S, Inclination function, 3-I, Altitude function, 3-A. The integral is from \( V_{\text{max}} \) to \( V \).

**Equation 3-T**
\[ T(V) = \int dV \div [V \cdot G(V)] \]

**Equation 3-S**
\[ S(V) = \int dV \div G(V) \]

**Equation 3-I**
\[ I(V) = \int [(2 \cdot g \cdot dV) \div (V^2 \cdot G(V))] \]

**Equation 3-A**
\[ A(V) = \int [I(V) \cdot dV] \div G(V) \]

Integrating the four Siacci functions give the following four results.

**Equation 3-tsec**
\[ t = C \cdot [T(u) - T(V)] \]

**Equation 3-Xsec**
\[ X = (C \cdot \cos(\theta_o) \cdot [S(u) - S(V)] \]

**Equation 3-Øsec**
\[ \tan(\phi) = \tan(\theta_o) - \left(\frac{C \cdot X}{2 \cdot \beta \cdot \cos^2(\theta_o)}\right) \cdot [I(u) - I(V)] \]

**Equation 3-Ysec**
\[ Y = Y_o + (X \cdot \tan(\theta_o)) - \left(\frac{C \cdot X \cdot \sec(\theta_o)}{2 \cdot \beta \cdot \cos^2(\theta_o)}\right) \cdot \left\{\left[(A(u) - A(V)) \div (S(u) - S(V))\right] - I(V)\right\} \]

The four equations below are the most generalize equations showing what Col. Ingalls described as the factor \( \beta \), as an “integrating factor” that was intended to compensate for the errors introduced by several approximations made in Siacci’s method. Various choices may be made for the “\( \beta \)” factor. The \( \beta = \sec(\theta_o) \) will give us the same results as 3-T, 3-S, 3-I, and 3-A, while Col. Ingalls chose to use \( \beta = \sqrt{\sec(\theta_o)} \), which is an excellent all-around value, and for short ranges and very flat-fire trajectories where \( \theta_o < 5 \text{ degrees}, \cos(\theta_o) \approx 1, \sec(\theta_o) \approx 1, \text{ and } \beta = 1 \), are used in calculating short range trajectories for the ammunition and firearms industries.

**Equation 3-tß**
\[ t = \left[\frac{C}{(\beta \cdot \cos(\theta_o))}\right] \cdot [T(u) - T(V)] \]

**Equation 3-Xß**
\[ X = (C \div \beta) \cdot [S(u) - S(V)] \]

**Equation 3-Øß**
\[ \tan(\phi) = \tan(\theta_o) - \left(\frac{C \cdot X}{2 \cdot \beta \cdot \cos^2(\theta_o)}\right) \cdot [I(u) - I(V)] \]
Equation 3-Y\(\beta\)
\[ Y = Y_o + (X \times \tan (\theta_o)) - \left[ (C \times X) \div (2 \times \beta \cos^2 (\theta_o)) \right] \times \left[ \frac{(A(u) - A(V))}{(S(u) - S(V))} - I(V) \right] \]

The following approximations are with \(\beta = 1\) are therefore generally used in calculating short range trajectories for sporting use:

Equation 3-t
\[ t = C \times [T(u) - T(V)] \]

Equation 3-X
\[ X = C \times [S(u) - S(V)] \]

Equation 3-\(\theta\)
\[ \tan (\theta) = \tan (\theta_o) - \{0.5 \times C \times [I(u) - I(V)]\} \]

Equation 3-Y
\[ Y = Y_o + (X \times \tan (\theta_o)) - \{0.5 \times C \times X \times \left[ \frac{(A(u) - A(V))}{(S(u) - S(V))} - I(V) \right] \} \]

• Working with Tables:

If we are given the Range, muzzle velocity, and \(C\); we can find the remaining velocity. We have a Range of 200 yards, muzzle velocity of 3200 fps, and a \(C\) of 0.1819. Now you can build the distance table as we did before and the remaining velocity will be at the point when the distance table reaches 600 feet. Or we can look up in the Ingalls' table and plug the numbers into the following formula:

Equation 3-43
\[ S(v) = S(V) + (X \div C) \]
\[ S(v) = S(3200) + (600 \div 0.1819) \]
\[ S(v) = 1124.4 + 600 \div 0.1819 \]
\[ S(v) = 1124.4 + 3298.51566794942 \]
\[ S(v) = 4422.91566794942 \text{ or } 4422.9 \]
\[ v = 2180 \text{ fps} \]

Lets take this a bit farther. We have a muzzle velocity of 3200 fps, the terminal velocity of 2660 fps, and the Range of 100 yards; we need to find the remaining velocity at 200 yards, 300 yards, 400 yards, and 500 yards. The formula for this is very easy, it is: \(S(v)\) for 200 yards = \(S(V) + 2 \times (S(v) - S(V))\), \(S(v)\) for 300 yards = \(S(V) + 3 \times (S(v) - S(V))\), \(S(v)\) for 400 yards = \(S(V) + 4 \times (S(v) - S(V))\), and \(S(v)\) for 500 yards = \(S(V) + 5 \times (S(v) - S(V))\), and so on.

Equation 3-43a
\[ S(v) \text{ for } 100 \text{ yards} = S(V) + 1 \times (S(v) - S(V)) \]
\[ S(v) = S(3200) + 1 \times (S(2660) - S(3200)) \]
\[ S(v) = 1124.4 + 1 \times (2772.9 - 1124.4) \]
\[ S(v) = 1124.4 + 1648.5 \]
\[ S(v) = 1124.4 + 1648.5 \]
\[ S(v) = 2772.9 \]
\[ v = 2660 \text{ fps remaining velocity at } 100 \text{ yards}, \text{ as we already know.} \]

Equation 3-43b
\[ S(v) \text{ for } 200 \text{ yards} = S(V) + 2 \times (S(v) - S(V)) \]
\[ S(v) = S(3200) + 2 \times (S(2660) - S(3200)) \]
\[ S(v) = 1124.4 + 2 \times (2772.9 - 1124.4) \]
\[ S(v) = 1124.4 + 2 \times 1648.5 \]
\[ S(v) = 1124.4 + 3297 \]

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\[ S(v) = 4421.4 \]
\[ v = 2180.155695 \approx 2180 \text{ fps remaining velocity at 200 yards.} \]

Equation 3-43c
\[ S(v) \text{ for 300 yards} = S(V) + 3 \times (S(v) - S(V)) \]
\[ S(v) = S(3200) + 3 \times (S(2660) - S(3200)) \]
\[ S(v) = 1124.4 + 3 \times (2772.9 - 1124.4) \]
\[ S(v) = 1124.4 + 3 \times 1648.5 \]
\[ S(v) = 1124.4 + 4945.5 \]
\[ S(v) = 6069.9 \]
\[ v = 1764.562859 \approx 1765 \text{ fps remaining velocity at 300 yards, and so on.... you get the picture.} \]

If we are given the initial velocity V of 3200 fps, C of 0.1819, and a Range of 500 yards; we can find the Time of Flight, the formula is: \( \text{ToF} = C \times (T(v) - T(V)) \). You could build a time table as we have done above but lets use Ingalls' table again. First find the remaining velocity for 500 yards and that is 1163 fps.

Equation 3-44
\[ \text{ToF} = C \times (T(v) - T(V)) \]
\[ \text{ToF} = 0.1819 \times (T(1163) - T(3200)) \]
\[ \text{ToF} = 0.1819 \times (4.7655 - 0.331) \]
\[ \text{ToF} = 0.1819 \times 4.4345 \]
\[ \text{ToF} = 0.80663555 \text{ or 0.81 seconds} \]

All bullets arc to their target and we would like to find out what the highest point is on that arc to a known range. We must first know the muzzle velocity, 3200-fps, the range, 300 yards, and the C, 0.1819. Now we use the formula \( H \text{ (in feet)} = 4 \times T^2 \) or \( H \text{ (in inches)} = 48 \times T^2 \). First find the remaining velocity at 300 yards, I'll let you do the math on that one. We have a remaining velocity of 1765 fps. Now find the Time of Flight and that is found to be 0.380 seconds.

Equation 3-45
\[ H = 4 \times T^2 \]
\[ H = 4 \times 0.38^2 \]
\[ H = 4 \times 0.1444 \]
\[ H = 0.5776 \text{ feet or 6.9312 inches. This is the highest point at which this bullet will gain for 300 yards} \]

Now if we know the height at which a bullet will gain we can find the Time of Flight for that range by the formula: \( T = \text{Square root of} \ (H \text{ (in feet)} \times \frac{1}{2}) \) or \( T = \text{Square root of} \ (H \text{ (in inches)} \div 3) \times \frac{1}{4} \). Given a range of 300 yards mid range trajectory is 7 inches, we will find the approximate Time of Flight.

Equation 3-46
\[ T = \text{Square root of} \ (H \div 3) \times \frac{1}{4} \]
\[ T = \text{Square root of} \ (7 \div 3) \times \frac{1}{4} \]
\[ T = \text{Square root of} \ 2.3104 \times \frac{1}{4} \]
\[ T = 1.52 \times \frac{1}{4} \]
\[ T = 0.38 \text{ seconds.} \]
Trajectories, Part 4
"Atmosphere:"

The "Point-Mass" Trajectory:
Numerical Integration Method

By Donna Cline
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• **The Numerical Integration Method for The Flat-Fire Point-Mass Trajectories:**

  The widespread availability of computers since the late 1940's has led to an explosion in the development and use of numerical integration methods by both amateur and professional ballisticians.

  There are a number of numerical integration methods that are available for solving ordinary differential equations but only few of the more commonly used methods will be discussed. The most basic approach is known as the Euler’s method, which solves the differential equation by a stepwise procedure, in which a new value is found from the sum of the previous value plus the product of the local derivative and the integration step size. Euler’s method is simple, but it is not particularly accurate unless one uses extremely small values of the step size are used. A modified form of the Euler’s method can be used, known as the Heun’s method. This uses the derivatives at both the beginning and end of the integration step, to calculate an average value over the step. The Heun’s method can also include as iteration loop to further refine the average value of the derivative. The Euler method is known as a “first order” integration and the Heun method is a “second order” integration.

  A more powerful numerical integration method for solving ordinary differential equations is the category known as Runge-Kutta (R-K) method. There are second, third, and fourth order R-K systems, plus a number of variations of each and they all work very well for particular problems. The second order R-K method is equivalent to Heun’s method, with a single corrector step (no iteration). There is even a fifth order R-K system that is known as “Butcher’s method,” which has been used extensively in calculations of high precision orbital trajectories. The accuracy of numerical integration increases and the computational efficiency decreases with increasing order of the Runge-Kutta methods.

  There are other multi-step predictor-corrector methods, such as the Newton-Cotes method, Milne’s method, and the Adams method have been used to calculate exterior ballistic trajectories.

  Normally, transforming the independent variable in the differential equations from downrange time to downrange distance facilitated the approximate analytical solutions of the flat-fire point-mass trajectory. But through the numerical integration method no transforming of the independent variable is required to exact a solution from the differential equations. Either time or distance could be used, and the accuracy of the solution would not be affected but the independent variable being in distance is more convenient for printed outputs at fixed range intervals than if fixed time intervals are used. Also, if distance is used as the independent variable we can avoid the need for numerical interpolation.
Equation 4-1 through 4-3 are transformed with equation 4-4 to produce the following three equations 4-5 through 4-7 with the independent variable being downrange distance, X. Also see "Effects of Wind in The Calculations of The Flat-Fire Trajectory" section in "Trajectory Part 2"

Equation 4-0:
\[
\Sigma F = \left( \frac{p \cdot S \cdot C_D}{2 \cdot m} \right) = \left( \frac{p \cdot \pi \cdot C_D \cdot d^2}{8 \cdot m} \right)
\]

Equation 4-1:
\[
\dot{V}_x = -\Sigma F \cdot \nabla \cdot (V_x - W_x)
\]

Equation 4-2:
\[
\dot{y} = -\Sigma F \cdot \nabla \cdot (V_y - W_y) - g
\]

Equation 4-3:
\[
\dot{z} = -\Sigma F \cdot \nabla \cdot (V_z - W_z)
\]

Equation 4-4:
\[
\nabla = \text{square root of } \left[ (V_x - W_x)^2 + (V_y - W_y)^2 + (V_z - W_z)^2 \right]
\]

Equation 4-5:
\[
V_x = -\Sigma F \cdot (\nabla \cdot V_x) \cdot (V_x - W_x)
\]

Equation 4-6:
\[
V_y = -\Sigma F \cdot (\nabla \cdot V_x) \cdot (V_y - W_y) - (g \div V_x)
\]

Equation 4-7:
\[
V_z = -\Sigma F \cdot (\nabla \cdot V_x) \cdot (V_z - W_z)
\]

The observed values of the vertical wind component, Wy, are very small near the surface of the earth and the vertical wind component can therefore be neglected in the calculations.
### Table 4-1 "U.S. Standard Atmospheres"

<table>
<thead>
<tr>
<th></th>
<th>Army Standard Metro (ASM)</th>
<th>International Civil Aviation Organization (ICAO)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Air Temperature:</strong></td>
<td>59° F</td>
<td>59° F</td>
</tr>
<tr>
<td></td>
<td>15° C</td>
<td>15°</td>
</tr>
<tr>
<td><strong>Decrease in air temperature with increasing altitude:</strong></td>
<td>$T(Y) = \left[T_0(°F) + 459.67\right]e^{-KY} - 459.67$</td>
<td>$T(Y) = \left[T_0(°F) + 459.67\right]e^{-KY} - 459.67$</td>
</tr>
<tr>
<td></td>
<td>$T_0(°F)$ = air temperature at firing site.</td>
<td>$T_0(°F)$ = air temperature at firing site.</td>
</tr>
<tr>
<td></td>
<td>$Y$ = altitude above firing site (feet).</td>
<td>$Y$ = altitude above firing site (feet).</td>
</tr>
<tr>
<td></td>
<td>$T(Y)$ = air temperature at altitude Y(°F).</td>
<td>$T(Y)$ = air temperature at altitude Y(°F).</td>
</tr>
<tr>
<td></td>
<td>$K$ = temperature-altitude decay factor (1/feet).</td>
<td>$K$ = temperature-altitude decay factor (1/feet).</td>
</tr>
<tr>
<td><strong>Temperature-altitude decay factor:</strong></td>
<td>$K = 6.015 * 10^{-6}$ (1/feet)</td>
<td>$K = 6.858 * 10^{-6} + (2.776 * 10^{-11}Y)$ (1/feet)</td>
</tr>
<tr>
<td><strong>Speed of sound:</strong></td>
<td>$a_o = 49.19$ square root of $[T(Y) + 459.67]$</td>
<td>$a_o = 49.0223$ square root of $[T(Y) + 459.67]$</td>
</tr>
<tr>
<td><strong>Decrease air density with increasing altitude:</strong></td>
<td>$p(Y) = p_0e^{-hY}$</td>
<td>$p(Y) = p_0e^{-hY}$</td>
</tr>
<tr>
<td></td>
<td>$p_0$ = air density at firing site (lb./foot^3).</td>
<td>$p_0$ = air density at firing site (lb./foot^3).</td>
</tr>
<tr>
<td></td>
<td>$Y$ = altitude above firing site (feet).</td>
<td>$Y$ = altitude above firing site (feet).</td>
</tr>
<tr>
<td></td>
<td>$p(Y)$ = air density at altitude Y (lb./foot^3).</td>
<td>$p(Y)$ = air density at altitude Y (lb./foot^3).</td>
</tr>
<tr>
<td></td>
<td>$h$ = air density-altitude decay factor (1/feet).</td>
<td>$h$ = air density-altitude decay factor (1/feet).</td>
</tr>
<tr>
<td><strong>Standard sea level air density:</strong></td>
<td>$p_0 = 0.0751265$ lbs./foot^3</td>
<td>$p_0 = 0.0764742$ lbs./foot^3</td>
</tr>
<tr>
<td></td>
<td>$h = 3.158 * 10^{-5}$ (1/feet)</td>
<td>$h = 2.926 * 10^{-5} + (1.0 * 10^{-10}Y)$ (1/feet)</td>
</tr>
<tr>
<td>For altitudes up to 20,000 ft above sea level, the above equations given exact results for the variation in air temperature and air density with increasing altitude. At 35,000 ft altitude, the errors increase to about one percent. At altitudes above 40,000 ft the errors grow rapidly with increasing altitude and the above equations should not be used where the summit of the trajectory exceeds 40,000 ft in altitude.</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Air density from the state of an ideal gas:</strong></td>
<td>$(p + p_o) = (P + P_o)\left(\frac{518.67}{T_0(°F) + 459.67}\right)$</td>
<td>$(p + p_o) = (P + P_o)\left(\frac{518.67}{T_0(°F) + 459.67}\right)$</td>
</tr>
<tr>
<td></td>
<td>$(p + p_o)$ = ratio of air density at firing site to standard air density.</td>
<td>$(p + p_o)$ = ratio of air density at firing site to standard air density.</td>
</tr>
<tr>
<td></td>
<td>$P$ = barometric pressure at firing site.</td>
<td>$P$ = barometric pressure at firing site.</td>
</tr>
<tr>
<td></td>
<td>$P_o$ = standard barometric pressure.</td>
<td>$P_o$ = standard barometric pressure.</td>
</tr>
<tr>
<td></td>
<td>$T_0(°F)$ = air temperature at firing site.</td>
<td>$T_0(°F)$ = air temperature at firing site.</td>
</tr>
<tr>
<td><strong>Standard sea level barometric pressure:</strong></td>
<td>$P_o = 29.53$ inches of Hg = 750 mm of Hg</td>
<td>$P_o = 29.92$ inches of Hg = 760 mm of Hg</td>
</tr>
<tr>
<td><strong>Humidity correction at sea level for air density ratio:</strong></td>
<td>$\mathcal{f}<em>{pRH} = 1 - 0.00378(R_H - 78)(P</em>{WV}/29.53)$</td>
<td>$\mathcal{f}<em>{pRH} = 1 - 0.00378(R_H - 78)(P</em>{WV}/29.92)$</td>
</tr>
<tr>
<td></td>
<td>$\mathcal{f}_{pRH}$ = humidity correction factor to the air density ratio.</td>
<td>$\mathcal{f}_{pRH}$ = humidity correction factor to the air density ratio.</td>
</tr>
<tr>
<td></td>
<td>$R_H$ = relative humidity (percent).</td>
<td>$R_H$ = relative humidity (percent).</td>
</tr>
<tr>
<td></td>
<td>$P_{WV}$ = water vapor pressure at the local temperature (Inches of Mercury (Hg)).</td>
<td>$P_{WV}$ = water vapor pressure at the local temperature (Inches of Mercury (Hg)).</td>
</tr>
<tr>
<td><strong>Corrected air density ratio:</strong></td>
<td>$C(p + p_o) = \mathcal{f}_{pRH} \cdot (p + p_o)$</td>
<td>$C(p + p_o) = \mathcal{f}_{pRH} \cdot (p + p_o)$</td>
</tr>
<tr>
<td><strong>Humidity correction at sea level for the speed of sound:</strong></td>
<td>$\mathcal{f}<em>{aRH} = 1 + 0.0014(R_H - 78)(P</em>{WV}/29.53)$</td>
<td>$\mathcal{f}<em>{aRH} = 1 + 0.0014(R_H - 78)(P</em>{WV}/29.92)$</td>
</tr>
<tr>
<td></td>
<td>$\mathcal{f}_{aRH}$ = humidity correction factor to the speed of sound in air.</td>
<td>$\mathcal{f}_{aRH}$ = humidity correction factor to the speed of sound in air.</td>
</tr>
<tr>
<td><strong>Corrected speed of sound:</strong></td>
<td>$C(a_o) = \mathcal{f}_{aRH} \cdot a_o$</td>
<td>$C(a_o) = \mathcal{f}_{aRH} \cdot a_o$</td>
</tr>
</tbody>
</table>
In general, an increasing humidity causes a slight decrease in air density, because the density of water vapor is less than that of dry air. On the other hand, increasing humidity causes a slight increase in the speed of sound. For air temperature below 70° Fahrenheit, the changes in both air density and speed of sound for a 100 percent change in humidity, are less than 1 percent, and may therefore be neglected for all practical purposes. At 100° Fahrenheit, the air density for saturated air is 2.5 percent below that of dry air; the difference has increased to 6.1 percent at 130° Fahrenheit. Thus for air temperature above 70° Fahrenheit, the humidity correction to the air density is small, but not negligible. The small correction for the humidity effect on the speed of sound should also be made at temperature above 70° Fahrenheit, but it is actually important only when the projectile flight velocity is near the speed of sound, where a small change in the Mach number causes a relatively large change in the drag coefficient.

- Comparison of Point-Mass and Siacci Trajectories:

As we recall from trajectory Part 3, while the Siacci method to the flat-fire point-mass trajectory has been used for ordnance trajectory calculations for 80 years, since the 1880's, the cannon artillery had abandoned the Siacci method by the end of World War I, but Siacci method has become the small arms standard of the U.S. Sporting and Ammunition Industry to the present day. While a comparison of the Siacci method with the numerical integration method of the point-mass differential equations needs to be examined, the maximum range of most small arms projectiles occur for gun elevation angles in the neighborhood of twenty-eight to thirty-five degrees, it would behoove us also to examine the behavior of Siacci trajectories up to these limits.

The first concern in long-range Siacci trajectories, at high angles of gun elevations, is the behavior of the ballistic coefficient over a large velocity or Mach number range. For the long range calculations of a typical maximum range of small arms trajectory where the projectile starts off at a moderately to high supersonic velocity, then coasts down through the transonic region and ends up at a relatively low subsonic velocity, the choosing of a standard drag function must be re-examined.

For example in figure 4-1 we see the drag coefficient versus Mach number for the M80 projectile, that is a .308 caliber boattail ball bullet with a weight of 147 grains. Figure 4-2 shows the variation of the M80
ballistic coefficient of $C_1$ with Mach number, relative to the $G_1$ drag function; the variation of the ballistic coefficient of $C_7$ with Mach number, relative to the $G_7$ drag function can be seen in figure 4-3. Since the accuracy of the Siacci method depends on having the ballistic coefficient be nearly constant for the entire trajectory, it is obvious that the $G_7$ drag function is superior to $G_1$ for the M80 projectile for this example.

Figure 4-1

Figure 4-2
Three cases will be considered using the G7 drag function with the appropriate average ballistic coefficient to compare the Siacci method with modern numerical integration method for the .30 caliber ball M80 bullet. The first case will be for the flat-fire trajectory out to 1000-yard range. The second case assumes a gun elevation angle of 15 degrees above the horizontal, which is generally considered as the upper limit for the Siacci method. And the third case is for a gun elevation of 30 degrees, which is essentially a maximum range for this bullet and which clearly violates the Siacci method’s assumptions.

We will use an average ballistic coefficient of 0.198 lb./in² for the M80 bullet out to 1000-yard range. The point-mass trajectory will use the Army Standard Metro and the sectional density of 0.221 lb./in² as its ballistic coefficient value, since the actual measured M80 drag coefficient versus Mach number curve is used.

Table 4-3
Comparison of Point-Mass and Siacci Trajectory for the M80 Bullet:

<table>
<thead>
<tr>
<th>Range (Yards)</th>
<th>Point-Mass Trajectory</th>
<th>Siacci (G7) Trajectory</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Hight (Inches)</td>
<td>Velocity (FPS)</td>
</tr>
<tr>
<td>0</td>
<td>0.0</td>
<td>2810</td>
</tr>
<tr>
<td>200</td>
<td>83.2</td>
<td>2371</td>
</tr>
<tr>
<td>400</td>
<td>141.3</td>
<td>1959</td>
</tr>
<tr>
<td>600</td>
<td>162.6</td>
<td>1580</td>
</tr>
<tr>
<td>800</td>
<td>127.0</td>
<td>1244</td>
</tr>
<tr>
<td>1000</td>
<td>0.0</td>
<td>1024</td>
</tr>
</tbody>
</table>
The results are illustrated in table 4-3 and figure 4-4, that shows at 1000-yards the modern point-mass method and the Siacci method are very close in agreement. The gun elevation angle was less than one degree and the flat-fire Siacci equations thus give very good results.

The second and third cases under consideration, as described earlier, for higher gun elevation angles, the terminal or striking velocity of the M80 bullet at ground impact is down 400 fps and the average ballistic coefficient ($C_7$) used for the Siacci trajectories were 0.191 lb./in$^2$ for the 15 degree elevation case and 0.190 lb./in$^2$ for the 30 degree elevation case. A comparison of the results of the modern point-mass method with the Siacci method is illustrated in table 4-4 and in figure 4-5 and 4-6.

Table 4-4

<table>
<thead>
<tr>
<th>Gun Elevation Angle (Degrees)</th>
<th>Point-Mass Range (Yards)</th>
<th>Siacci Range (Yards)</th>
<th>Siacci Range (Yards)</th>
<th>Siacci Range (Yards)</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>3735</td>
<td>3790</td>
<td>3625</td>
<td>3675</td>
</tr>
<tr>
<td>30</td>
<td>4430</td>
<td>4900</td>
<td>4085</td>
<td>4295</td>
</tr>
</tbody>
</table>
For gun elevation angle above 5 degrees, equation 4-8 through 4-11, which contain \( \beta \), and \( \cos(\Theta_o) \) term, should give more accurate answer than equation 4-12 through 4-15, which drop the trigonometric terms in the flat-fire approximation. To provide the most accurate comparison of the older method with modern point-mass trajectories, the Siacci method was run three ways, using equations 4-8 and 4-10 with \( \beta = \sec(\Theta_o) \), and with \( \beta = \text{square root (sec}(\Theta_o)) \), and a third run using the flat-fire equation 4-12 and 4-14.

Equation 4-8:
\[
X = \left( C + \beta \right) \left[ S(v) - S(V) \right]
\]

Equation 4-9:
\[
t = \left[ C \div \left( \beta \cdot \cos(\Theta_o) \right) \right] \left[ T(v) - T(V) \right]
\]

Equation 4-10:
\[
Y = Y_o + \left[ X \cdot \tan(\Theta_o) \right] - \left[ \left( C \cdot X \right) \div \left( 2 \cdot \beta \cdot \cos^2(\Theta_o) \right) \right] \cdot \left\{ \left[ \left( A(v) - A(V) \right) \div \left( S(v) - S(V) \right) \right] - I(V) \right\}
\]

Equation 4-11:
\[
\tan(\Theta) = \tan(\Theta_o) - \left[ \left( C \cdot X \right) \div \left( 2 \cdot \beta \cdot \cos^2(\Theta_o) \right) \right] \left[ \left( I(v) - I(V) \right) \right]
\]

Equation 4-12:
\[
X = C \cdot \left[ S(v) - S(V) \right]
\]

Equation 4-13:
\[
t = C \cdot \left[ T(v) - T(V) \right]
\]

Equation 4-14:
\[
Y = Y_o + \left[ X \cdot \tan(\Theta_o) \right] - \left\{ \left[ \left( A(v) - A(V) \right) \div \left( S(v) - S(V) \right) \right] - I(V) \right\}
\]
Equation 4-15:
\[ \tan(\theta) = \tan(\theta_0) - (0.5 \times C) \times [(I(v) - I(V))] \]

Figure 4-5 graphically illustrates the 15 degree gun elevation case. Compared with the modern point-mass method (numerical integration), the Siacci method using equations 4-8 and 4-10 both under predict the level-ground impact range. If \( \beta \) is set equal to sec \( (\theta_0) \) equivalent to using equations 4-16 and 4-17, the range error is –110 yards; if \( \beta \) is set equal to Col. Ingalls’ value, square root [sec \( (\theta_0) \)], the range error is –60 yards. The flat-fire Siacci equations 4-12 through 4-14, (equivalent to setting \( \beta = 1 \) and \( \cos(\theta_0) = 1 \)) overestimate the level-ground range by +55 yards for the 15 degree gun elevation case.

Equation 4-16:
\[ X = (C \times \cos(\theta_0)) \times [S(v) - S(V)] \]

Equation 4-17:
\[ Y = Y_0 + [X \times \tan(\theta_0)] - (0.5 \times C \times X \times \sec(\theta_0)) \times \{(A(v) - A(V)) \div (S(v) - S(V))\} - I(V) \]

The Siacci solutions deteriorate significantly at 30 degrees gun elevation angle, as we would expect. The results are shown in figure 4-6. Equations 4-8 through 4-10 again under predict the range: for \( \beta = \text{sec} (\theta_0) \), the range error is –345 yards; if \( \beta = \text{square root} [\text{sec} (\theta_0)] \), the level-ground range error is –135 yards. The flat-fire Siacci approximation equations 4-12 and 4-14 over predict the range by 470 yards. Of the three different Siacci approximations considered, Ingalls’ value, \( \beta = \text{sqroot} [\text{sec} (\theta_0)] \) appears to be better than the other two choices at 30 degree gun elevation angle.

The above calculation compares numerical integration for the spark-range measured M80 drag coefficient curve with a Siacci calculation using a close \((G_7)\) but not exact drag function. This corresponds to what would be done in practice. However, a more exact comparison of the two methods should use the same drag function for both calculations. If \( G \), is used for both calculations, with \( \beta = \text{square root} (\text{sec} (\theta_0)) \), at 15 degrees gun elevation, the Siacci range error is –129 yards. The differences between the more exact comparison are smaller as expected, but are still unacceptably large.

The comparison of the modern point-mass trajectories with the Siacci method, the conclusion we have seen, is that the Siacci method is still useful for flat-fire trajectories (gun elevation angle below 5 degrees), provided a drag function is available such that the ballistic coefficient remains nearly constant over the velocity range of the intended trajectory. For higher gun elevation angles, the error in the Siacci method grows rapidly with the increase in range, and a modern point-mass trajectory will prove to be a much more satisfactory method.
Table 4-5 Here is a computer program in Q-Basic called "MCTRAJ" from "Modern Exterior Ballistics" by Robert L. McCoy.

10 REM PROGRAM [MCTRAJ.BAS], MAR 1987. [REVISED 07/90; 02/93;05/94]
15 REM [Q-BASIC VERSION -- OCTOBER 1994]
20 CLS
30 KEY OFF
40 COLOR 7,1,8: CLS
50 KEY ON
60 DEFDBL A-H,M,O-Z
70 REM POINT MASS TRAJECTORIES FOR SMALL ARMS.
80 REM THE PROGRAM REQUIRES AN INPUT TABLE OF
90 REM DRAG COEFFICIENT (CD) VERSUS MACH NUMBER (M).
100 REM ADDITIONAL REQUIRED INPUT ARE:
110 REM MUZZLE VELOCITY (FT/SEC); BALLISTIC COEFFICIENT
120 REM HEIGHT OF SIGHT LINE ABOVE BORE CENTERLINE
130 REM GUN ELEVATION ANGLE (MINUTES); RATIO OF AIR
140 REM DENSITY TO
150 REM STANDARD DENSITY; AIR TEMPERATURE (DEG F); RANGE
160 REM PRINT
170 REM INTERVAL (YARDS/METERS); RANGE TO TERMINATE
180 REM TRAJECTORY (YARDS/METERS); RANGE WIND SPEED (MPH--POSITIVE IF WIND BLOWS
190 REM FROM GUN TO
200 REM أحد خيار للعجلة في المكعب
210 REM TRAJECTORY
220 REM REM THE PROGRAM ALSO PROVIDES AN OPTION TO ADJUST
230 REM THE TRAJECTORY
240 REM TO PASS THROUGH A SPECIFIED POINT IN SPACE,
250 REM DENOTED BY
260 REM 210 REM RMATCH (YARDS/METERS), AND HMATCH (INCHES). THE
270 REM GUN ELEVATION ANGLE
280 REM IS ADJUSTED UNTIL THE TRAJECTORY PASSES THROUGH
290 REM THE POINT
300 REM (RMATCH, HMATCH). IF NO ADJUSTMENT IS DESIRED,
310 REM INPUT ZEROS
320 REM FOR RMATCH AND HMATCH, AND THE TRAJECTORY WILL
330 REM BE RUN WITH
340 REM THE INPUT ELEVATION ANGLE.
350 REM THE PROGRAM SOLVES THE TRAJECTORY BY NUMERICAL
360 REM INTEGATION
370 REM USING THE HEUN METHOD, WHICH IS AN ITERATIVELY
380 REM APPLIED
390 REM SECOND ORDER PREDICTOR-CORRECTOR TECHNIQUE.
400 REM
410 REM LINE OF SIGHT (INCHES); DEFORMATION (INCHES); TOTAL VELOCITY
420 REM (FT/SEC); TIME OF FLIGHT (SECONDS); RANGE COMPONENT OF
430 REM VELOCITY (FX--FT/SEC); VERTICAL COMPONENT OF
440 REM VELOCITY (FY--FT/SEC); AND HORIZONTAL COMPONENT OF
450 REM VELOCITY (FZ--FT/SEC).
460 REM REM DEFINE PROGRAM CONSTANTS.
470 REM
470 CLS
480 PRINT
490 PRINT
500 PRINT
510 PRINT
520 PRINT
530 PRINT
540 PRINT
550 PRINT
560 PRINT
570 END
1310 PRINT
1320 PRINT "ENTER HEIGHT OF SIGHT LINE ABOVE BORE LINE (INCHES):";
1330 INPUT H0
1340 PRINT
1350 PRINT "ENTER GUN ELEVATION ANGLE (MINUTES):";
1360 INPUT P0
1370 PRINT
1380 PRINT "ENTER RATIO OF AIR DENSITY OF SEA LEVEL STANDARD:";
1390 INPUT R0
1400 PRINT
1410 PRINT "ENTER AIR TEMPERATURE (DEGREES, FAHRENHEIT):";
1420 INPUT TDF
1430 PT0=INT(10*TDF)/10
1440 PRINT
1450 PRINT "ENTER RANGE PRINT INTERVAL";UR$;":";
1460 INPUT P2
1470 PRINT
1480 PRINT "ENTER RANGE TO TERMINATE TRAJECTORY";UR$;":";
1490 INPUT R3
1500 N1=INT(R3/P2+1.5)
1510 IF N1<=101 THEN 1580
1520 PRINT "THIS PRINT INTERVAL GIVES OVER 100 LINES OF OUTPUT!"
1530 PRINT "INCREASE PRINT STEP?(ENTER Y FOR YES, N FOR NO:)";
1540 INPUT PPI$
1550 PRINT
1560 IF PPI$="Y" THEN 1450
1570 PRINT "ENTER RANGE WIND SPEED (MILES/HOUR):";
1580 INPUT W1
1590 PRINT "ENTER CROSS WIND SPEED (MILES/HOUR):";
1600 INPUT W3
1610 PRINT "THE FOLLOWING TWO INPUT SPECIFY THE COORDINATES";
1620 PRINT "OF A POINT THROUGH WHICH THE TRAJECTORY MUST PASS."
1630 PRINT "IF THIS OPTION IS NOT DESIRED, INPUT ZERO FOR BOTH";
1640 PRINT "THE MATCH RANGE AND THE MATCH HEIGHT.
1650 PRINT "ENTER THE TRAJECTORY MATCH RANGE, RMATCH";UR$;":";
1660 PRINT "ENTER THE TRAJECTORY MATCH HEIGHT, HMATCH";UR$;":";
1670 PRINT "REM INITIALIZE INPUT VALUES FOR TRAJECTORY CALCULATION.
1680 N=2
1690 J=0
1700 P1=W1
1710 P3=W3
1720 H0=(-H0)
1730 R4=D3*P2
1740 R5=D3*R3
1750 R6=D3*R8
1760 PR4=P2
1770 PR5=R3
1780 H3=H0/12
1790 W1=(22*W1)/15
1800 V3=V0*COS(P0/3437.74677#)
1810 V4=V0*SIN(P0/3437.74677#)
1820 V5=0
1830 R1=0
1840 PR1=0
1850 H1=H0/12
1860 D1=0
1870 T1=0
1880 PX3=PR4
1890 REM CHECK FOR TRAJECTORY MATCH VALUES.
1900 IF R6<0 THEN 2060
1910 IF R6<0 THEN 2060
1920 IF R6<0 THEN 2060
1930 IF R6<0 THEN 2060
1940 IF R6<0 THEN 2060
1950 IF R6<0 THEN 2060
1960 IF R6<0 THEN 2060
1970 IF R6<0 THEN 2060
1980 IF R6<0 THEN 2060
1990 IF R6<0 THEN 2060
2000 PX3=PR4
2010 REM PRINT HEADERS AND FIRST LINE OUTPUT.
2020 CLS
2030 PRINT
2040 IF SI$="S" THEN PRINT "ARMY STANDARD METRO"
2050 IF SI$="I" THEN PRINT "ICAO STANDARD ATMOSPHERE"
2060 PRINT 
2070 PRINT "DRAG FUNCTION:";K$
2080 PRINT "PROJECTILE IDENTIFICATION:";K2
2090 PRINT "MUZ VEL", "C", "H0", "ELEV", "DENSITY"
2100 PRINT "(FT/SEC)", "(LB/IN^2)", "(INCHES)", "(MINUTES)", "RATIO"
2110 PRINT
2120 IF SI$="S" THEN PRINT "ARMY STANDARD METRO"
2130 IF SI$="I" THEN PRINT "ICAO STANDARD ATMOSPHERE"
2140 PRINT
2150 PRINT "TEMP", "RANGEWIND", "CROSSWIND", "RMATCH",
2160 PRINT "HMATCH";UR$;"(INCHES)"
2170 PRINT
2180 PRINT PT0,P1,P3,R8,H8
2190 PRINT
2200 PRINT
2210 PRINT
2220 PRINT
2230 PRINT
2240 PRINT
2250 PRINT
2260 PRINT "RMATCH",UR$;":";
2270 PRINT "HMATCH",UR$;":";
2280 PRINT
2290 PRINT
2300 PRINT
2310 PRINT
2320 PRINT
2330 PRINT
2340 PRINT
2350 PRINT
2360 PRINT
2370 PRINT
2380 PRINT
2390 PRINT
2400 PRINT
2410 PRINT
2420 PRINT
2430 PRINT
2440 PRINT USING U1$; Q(1), R(1), T(1), V(1), W(1), X(1), Y(1), Z(1)
2450 REM BEGIN TRAJECTORY CALCULATION.
2460 REM APPLY EULER PREDICTOR FORMULA
2470 REM APPLY ITERATIVE HEUN CORRECTOR FORMULA
2480 REM APPLY EULER PREDICTOR FORMULA
2490 REM APPLY EULER PREDICTOR FORMULA
2500 REM APPLY EULER PREDICTOR FORMULA
2510 REM APPLY EULER PREDICTOR FORMULA
2520 X1=1
2530 Gosub 4010
2540 C1=2
2550 C4=(C3*C1*B1*EXP((RH1+RH2*H1)*H1))/V3
2560 A1=C4*(V3-W1)
2570 A2=C4*V4-G/V3
2580 A3=C4*(V5-W3)
2590 REM APPLY EULER PREDICTOR FORMULA
2600 R2=R1+D3
2610 PR2=PR1+DINT
2620 V6=V3+A1*D3
2630 V7=V4+A2*D3
2640 V8=V5+A3*D3
2650 B2=SQR((V6-W1)^2+V7^2+(V8-W3)^2)
2660 REM APPLY ITERATIVE HEUN CORRECTOR FORMULA
2670 U=B2
2680 T0=(TDF+459.67#)*EXP((TK1+TK2*H1)*H1+459.67#)
2690 V1=V6*SQR(T0+459.67#)
2700 X1=B1-SQR((V3-W1)^2+V4^2+(V5-W3)^2)
2710 V6=V3+A1*D3
2720 Gosub 4010
2730 Gosub 4010
2740 C5=(C3*C2*B2*EXP((RH1+RH2*H1)*H1))/V6
2740 A4=C5*(V6-W1)
2750 A5=C5*V7-G/V6
2760 A6=C5*(V8-W3)
2770 V6=V3+.5*(A1+A4)*D3
2780 V7=V4+.5*(A2+A5)*D3
2790 V8=V5+.5*(A3+A6)*D3
2800 B2=SQR((V6-W1)^2+V7^2+(V8-W3)^2)
2810 E2=ABS((B2-U)/B2)
2820 IF E2>E1 THEN 2670
2830 REM COMPUTE VALUES AT R2.
2840 H2=H1+((V4+V7)/(V3+V6))*D3
2850 D2=D1+((V5+V8)/(V3+V6))*D3
2860 T2=T1+(2*D3)/(V3+V6)
2870 V2=SQR(V6^2+V7^2+V8^2)
2880 REM RESET CONDITIONS AT R1 TO NEW CONDITIONS AT R2.
2890 R1=R2
2900 PR1=PR2
2910 H1=H2
2920 D1=D2
2930 T1=T2
2940 V3=V6
2950 V4=V7
2960 V5=V8
2970 A1=A4
2980 A2=A5
2990 A3=A6
3000 P5=PR2
3010 P6=12*H2
3020 P7=12*D2
3030 REM CHECK STATUS OF PRINT CONDITIONS.
3040 IF L=1 THEN 3190
3050 IF PR2 < PX3 THEN 3190
3060 Q(N)=P5
3070 R(N)=P6
3080 T(N)=P7
3090 V(N)=V2
3100 W(N)=T2
3110 X(N)=V6
3120 Y(N)=V7
3130 Z(N)=V8
3140 PRINT USING U1$; Q(N); R(N); T(N); V(N); W(N); X(N); Y(N); Z(N)
3150 N=N+1
3160 REM INCREMENT PRINT RANGE.
3170 PX3=PX3+PR4
3180 REM CHECK FOR CONDITIONS TO STOP TRAJECTORY CALCULATION.
3190 IF PR2>=PR7 THEN 3210
3200 GOTO 2600
3210 IF L=0 THEN 3420
3220 REM ITERATION TO ADJUST ELEVATION ANGLE.
3230 E3=ABS((H2-H3)
3240 IF E3< .00001# THEN 3370
3250 J=J+1
3260 E(J)=P0
3270 H(J)=H2
3280 IF J>=2 THEN 3330
3290 E(J+1)=P0+2#
3300 P0=E(J+1)
3310 GOTO 1920
3320 REM ALGORITHM TO ADJUST ELEVATION ANGLE.
3330 IF H(J-1)-H(J) THEN 4100
3340 E(J)=E(J-1)+(E(J-1)-E(J))/(H(J-1)-H(J))
3350 IF J>20 THEN 3300
3360 REM RUN FINAL TRAJECTORY WITH PRINTS.
3370 PR6=0
3380 GOTO 1920
3390 REM [CHECK FOR HARD COPY OF OUTPUT]
3400 REM [SAVE OUTPUT FOR HARD COPY, IF DESIRED]
3410 REM (N2=NO. OF TRAJECTORY LINES IN OUTPUT)
3420 N2=N-1
3430 PRINT
3440 PRINT
3450 PRINT "DO YOU WANT HARD COPY OF THIS OUTPUT?"
3460 PRINT "ENTER Y FOR YES, N FOR NO:"
3470 INPUT K1$
3480 IF K1$="N" THEN 3900
3490 IF K1$="Y" THEN 3510
3500 GOTO 3440
3510 REM ECHO INPUT DRAG COEFFICIENT TABLE.
3520 LPRINT
3530 LPRINT "DRAG FUNCTION:";K$
3540 LPRINT
3550 IF LPDC=1 THEN 3610
3560 LPRINT "MACH NO.", "CD"
3570 FOR i=1 TO JJ
3580 LPRINT M(I),D(I)
3590 NEXT I
3600 LPDC=1
3610 LPRINT
3620 IF SI$="S" THEN LPRINT "ARMY STANDARD METRO";
3630 IF SI$="I" THEN LPRINT "ICAO STANDARD ATMOSPHERE"
3640 LPRINT
3650 LPRINT "PROJECTILE IDENTIFICATION:";K2$
3660 LPRINT
3670 LPRINT "MUZ VEL", "C", "H0", "ELEV", "DENSITY"
3680 LPRINT "(FT/SEC)", "(LB/IN^2)", "(INCHES)"
3690 LPRINT "(MINUTES)", "RATIO"
3700 LPRINT
3710 LPRINT V0, C, P4, PR0, RO
3720 LPRINT
3730 LPRINT "TEMP", "RANGE", "CROSSWIND", "RMATCH"
3740 LPRINT "(DEG,F)", "(MPH)", "(MPH)", UR$;
3750 LPRINT "(INCHES)"
3760 LPRINT
3770 LPRINT PT0, P1, P3, R8, H8
3780 LPRINT
3790 LPRINT
3800 LPRINT PT0; TAB(1); "RANGE"; TAB(1); "HEIGHT"; TAB(23);
3810 "DEPL."; TAB(33); "VEL"; TAB(41); "TIME"; TAB(51); "VX";
3820 "VY"; TAB(68); "VZ"
3830 FOR N=1 TO N2
3840 LPRINT USING U1$; Q(N); R(N); T(N); V(N); W(N); X(N); Y(N); Z(N)
3850 NEXT N
3860 REM CHECK FOR ADDITIONAL CASES.
3870 LPRINT
3880 LPRINT
3890 LPRINT
3900 PRINT
3910 PRINT
3920 PRINT "RUN ANOTHER CASE? (ENTER Y FOR YES, N FOR
3930 NO):"
3940 INPUT K1$
3950 IF K1$="N" THEN 4170
3960 PRINT
3970 PRINT "USE SAME DRAG COEFFICIENT TABLE?"
3980 PRINT "ENTER Y FOR YES, N FOR NO:";
3990 INPUT K1$
4000 GOTO 880
4010 REM SUBROUTINE FOR INTERPOLATION IN DRAG TABLE
4020 I=1
4030 IF (I+1)>MMI THEN 4130
4040 IF X1< M(I+1) THEN 4070
4050 I=I+1
4060 GOTO 4030
4070 S=(D(I+1)-D(I))/(M(I+1)M(I))
4080 X2=D(I)+S*(X1-M(I))
4090 RETURN
4100 PRINT
4110 PRINT
4120 PRINT "ELEVATION ANGLE ITERATION DID NOT CONVERGE.
4130 GOTO 3880
4140 PRINT
4150 PRINT "TRAJECTORY CANNOT REACH THE SPECIFIED
4160 MAXIMUM RANGE."
4170 PRINT "TRY A SHORTER MAXIMUM RANGE."
4180 GOTO 3880
4190 END