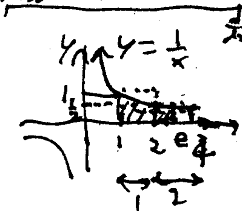
Some Results about e

The derivative of $\log_e x$ or $\ln x$. (natural logarithm)

$\frac{d}{dx}\ln x = \frac{1}{x}$:



There exists a number e such that

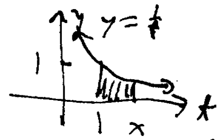
$$\int_1^e \frac{1}{x}\,dx = 1 \qquad 2 < e < 4$$

because

$$\frac{1}{2} < \int_1^2 \frac{1}{x}\,dx < 1$$

$$1(\tfrac{1}{2})+2(\tfrac{1}{4}) = 1 < \int_1^4 \frac{1}{x}\,dx < 2 = 1(1)+2(\tfrac{1}{2})$$

Let $L(x) = \int_1^x \frac{1}{t}\,dt$ for $x>0$.



$L(x) > 0$ for $x > 1$

$L(x) = 0$ for $x = 1$

$L(x) < 0$ for $x < 1$

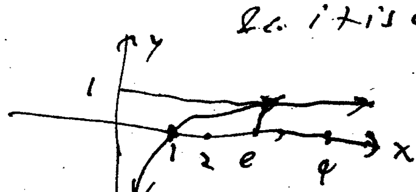
$\frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x}$ by the fundamental theorem of calculus

& ∴ $L'(x) = \frac{1}{x}$ for $x > 0$.

$\frac{1}{2} < L(2) < 1$ & $1 < L(4) < 2$.

$L''(x) = \frac{-1}{x^2} < 0$ for all $x > 0$

i.e. it is concave down.



$L(e) = 1$.

As $x \to 0^+$, $L'(x) \to \infty$

Approximation As $x \to \infty$, $L'(x) \to 0$:

methods can be used to approximate e as $2.718281828\ldots$ & it can be shown that $e \notin \mathbb{Q}$.

$$\frac{d}{dx}L(x^a) = \frac{d}{dx^a}L(x^a)\cdot\frac{dx^a}{dx} \text{ by chain rule}$$

$$= \frac{1}{x^a}\cdot a x^{a-1}$$

$$= \frac{a}{x}$$

$$= a\,L'(x)$$

∴ $L(x^a) = a L(x) + C$

But $L(1^a) = L(1) = 0 = aL(1) + C = a(0) + C$ & ∴ $C = 0$

Hence $L(x^a) = aL(x)$

$\therefore L(x) = L(e^{\log_e x})$

$= (\log_e x) L(e)$

$= \log_e x \, (1)$

$= \log_e x$

$= \ln x$.

Hence $\boxed{\dfrac{d}{dx} \ln x = \dfrac{1}{x}}$ $\left(\text{f.i. } \int \frac{1}{x} \, dx = \ln x + C \right)$

$\boxed{\dfrac{d}{dx} \ln f(x) = \dfrac{f'(x)}{f(x)}}$ – by chain rule

$\Rightarrow \boxed{\int \dfrac{f'(x)}{f(x)} dx = \ln f(x) + C}$

$\boxed{\dfrac{d e^x}{dx} = e^x}$

Pf: Let $y = e^x$

$x = \ln y$

$\dfrac{dx}{dy} = \dfrac{1}{y}$

$\therefore \dfrac{dy}{dx} = y = e^x$ $\square$ $\Rightarrow \boxed{\int e^x dx = e^x + C}$

$\dfrac{d}{dx} e^{f(x)} = f'(x) e^{f(x)}$ $\Rightarrow \boxed{\int f'(x) e^{f(x)} dx = e^{f(x)} + C}$

– by chain rule.

$\dfrac{d}{dx}(x \ln x - x)$

$= x \cdot \frac{1}{x} + \ln x \cdot 1 - 1$

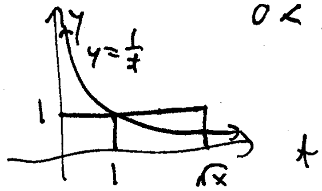
$= \ln x$

$\therefore \boxed{\int \ln x \, dx = x \ln x - x + C}$

$\dfrac{d}{dx} a^x = \dfrac{d}{dx} e^{x \ln a} = (\ln a) e^{x \ln a} = (\ln a) \cdot a^x \therefore \boxed{\dfrac{d}{dx} a^x = \ln a \cdot a^x}$

$\Rightarrow \boxed{\int a^x dx = \dfrac{1}{\ln a} a^x + C}$ Also, if $y = \log_a x$, $x = a^y \therefore \dfrac{dx}{dy} = a^y \ln a = x \ln a$

$\therefore \boxed{\dfrac{d}{dx} \log_a x = \dfrac{1}{x \ln a}}$



$$0 < \int_1^{\sqrt{x}} \frac{1}{t}\,dt < 1\cdot\sqrt{x}$$
$$\therefore 0 < [\ln t]_1^{\sqrt{x}} < \sqrt{x}$$
$$\therefore 0 < \ln\sqrt{x} - \ln 1 < \sqrt{x}$$
$$\therefore 0 < \tfrac{1}{2}\ln x < \sqrt{x}$$
$$\therefore 0 < \frac{\ln x}{x} < \frac{2}{\sqrt{x}}$$

As $x \to \infty$, $\frac{2}{\sqrt{x}} \to 0$ $\therefore$ $\boxed{\lim_{x\to\infty} \frac{\ln x}{x} = 0}$ Let $u = \frac{1}{x}$ $\therefore \ln u = -\ln x$

$u \to 0^+$ as $x \to \infty$

$$\therefore \lim_{u\to 0^+} -u\ln u = \lim_{x\to\infty}\frac{\ln x}{x} = 0$$

Hence $\boxed{\lim_{x\to 0^+} x\ln x = 0}$

Likewise,

Let $v = x^k$ $\therefore$ $\lim_{x\to\infty}\frac{\ln x}{x^k} = \lim_{v\to\infty}\frac{\ln v^{\frac{1}{k}}}{v} = \frac{1}{k}\lim_{v\to\infty}\frac{\ln v}{v} = \frac{0}{k} = 0$

$$\lim_{x\to 0^+} x^k \ln x = \lim_{v\to 0^+} v\ln v^{\frac{1}{k}} = \frac{1}{k}\lim_{v\to 0^+} v\ln v = \frac{0}{k} = 0$$

$\therefore$ $\boxed{\lim_{x\to\infty}\frac{\ln x}{x^k} = 0}$ & $\boxed{\lim_{x\to 0^+} x^k\ln x = 0}$.

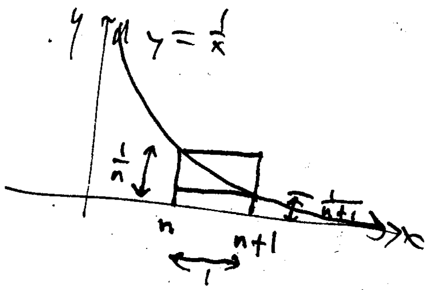
$$y = \ln x \Rightarrow y' = \lim_{h\to 0}\frac{\ln(x+h) - \ln x}{h} = \lim_{h\to 0}\left(\frac{1}{h}\ln\left(1+\frac{h}{x}\right)\right)$$
$$= \lim_{h\to 0}\ln\left(1+\frac{h}{x}\right)^{\frac{1}{h}} = \frac{1}{x}$$

Let $n = \frac{1}{h}$ & $u = \frac{1}{x}$ $\Rightarrow$ $\lim_{n\to\infty}\ln\left(1+\frac{u}{n}\right)^n = \lim_{h\to 0}\ln\left(1+\frac{h}{x}\right)^{\frac{1}{h}} = \frac{1}{x} = u$

$$\therefore \lim_{n\to\infty}\left(1+\frac{u}{n}\right)^n = e^u$$

Let $u = 1$ $\therefore$ $\boxed{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e}$

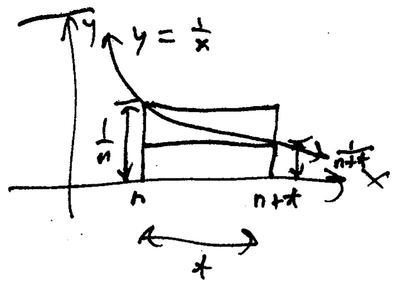
Another method:



$$1\cdot\frac{1}{n+1} < \int_n^{n+1}\frac{1}{x}\,dx < 1\cdot\frac{1}{n}$$
$$\therefore \frac{n}{n+1} < n\,[\ln x]_n^{n+1} < 1$$
$$\therefore \frac{n}{n+1} < n\ln\frac{n+1}{n} < 1$$

$\therefore \frac{n}{n+1} < \ln\left(1+\frac{1}{n}\right)^n < 1$. But as $n\to\infty$, $\frac{n}{n+1} = \frac{1}{1+\frac{1}{n}} \to 1$

$\therefore \ln\left(1+\frac{1}{n}\right)^n \to 1$ $\therefore \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$.



$$t \cdot \frac{1}{n+t} < \int_n^{n+t} \frac{1}{x}\,dx < t \cdot \frac{1}{n}.$$

$$\therefore \frac{tn}{n+t} < n\,[\ln x]_n^{n+t} < t$$

$$\therefore \frac{t}{1+\frac{t}{n}} < n \ln \frac{n+t}{n} < t$$

$t > 0 \leftarrow$

$$\therefore \frac{t}{1+\frac{t}{n}} < \ln\left(1+\frac{t}{n}\right)^n < t$$

As $n \to \infty$, $\frac{t}{1+\frac{t}{n}} \to t \quad \therefore \ln\left(1+\frac{t}{n}\right)^n \to t$

Hence $\boxed{\lim_{n\to\infty}\left(1+\frac{t}{n}\right)^n = e^t}$

$0! = 1$

$n! = 1 \cdot \ldots \cdot n, \ n > 0$

If $\frac{d\,f(x)}{dx} = f(x) \Rightarrow \frac{d\,f(x)}{f(x)} = dx \Rightarrow x = \int \frac{d\,f(x)}{f(x)} = \ln f(x) + c$

Let $x = 0 \ \therefore \ \ln f(0) + c = 0$

$\therefore \ c = -\ln f(0)$

$\therefore \ x = \ln \frac{f(x)}{f(0)} \ \therefore \ f(x) = f(0)\,e^x.$

i.e. $f(x) = a e^x$ for a constant a.

$$\frac{d}{dx} \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=1}^{\infty} \frac{n}{n!} x^{n-1} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} x^{n-1}$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} \ \therefore \ \sum_{n=0}^{\infty} \frac{x^n}{n!} = a e^x$$

for a constant a.

Let $x = 0 \ \therefore \ a e^0 = a = 1$

$$\therefore \boxed{\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x}$$

Let $x = 1 \ \therefore \ \boxed{\sum_{n=0}^{\infty} \frac{1}{n!} = e.}$

$y = \ln x = f(x)$ Grad. at $x = 1$.

$y'(1) = \left.\frac{1}{x}\right|_{x=1} = 1$.

But, $y'(1) = \lim\limits_{x\to 1} \frac{\ln x - \ln 1}{x-1} = \lim\limits_{x\to 1} \frac{\ln x}{x-1} = 1$.

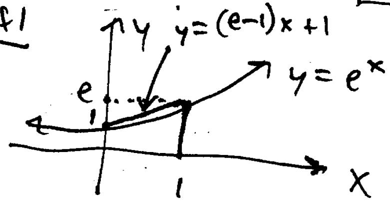
$\therefore \lim\limits_{u\to 1} \frac{\ln u}{u-1} = 1$. Let $h = \ln u$ $\therefore u = e^h$

$\therefore u \to 1 \Rightarrow h \to 0$ & $\lim\limits_{h\to 0} \frac{h}{e^h - 1} = 1$ $\therefore \lim\limits_{h\to 0} \frac{e^h-1}{h} = 1$.

Another method: $\frac{d}{dx} e^x = \lim\limits_{h\to 0} \frac{e^{x+h} - e^x}{h} = e^x$ Let $x = 0$ $\therefore \boxed{\lim\limits_{h\to 0} \frac{e^h - 1}{h} = 1}$

$\boxed{e<3}$ – more elegant proof:

Proof 1



$$\frac{d^2 e^x}{dx^2} = e^x > 0 \quad \forall x \in \mathbb{R}$$

$\Rightarrow y = e^x$ is concave up $\forall x \in \mathbb{R}$

$\therefore$ for $0 < x < 1$, $e^x < (e-1)x + 1$

$$\therefore \int_0^1 e^x\,dx < \int_0^1 ((e-1)x+1)\,dx$$

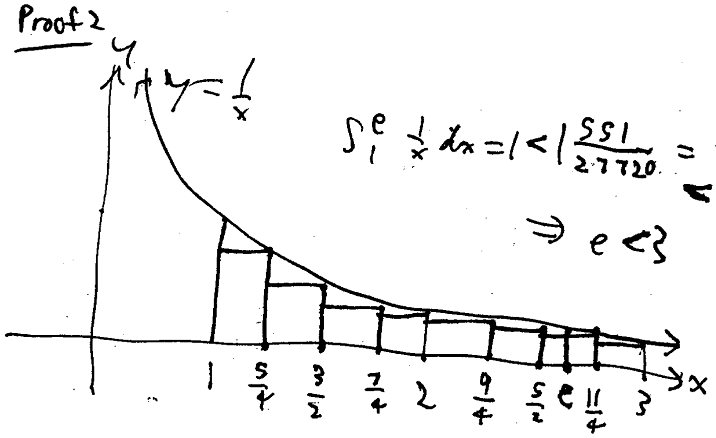
$$[e^x]_0^1 < [\tfrac{1}{2}(e-1)x^2 + x]_0^1$$

$$e - 1 < \tfrac{1}{2}(e-1) + 1$$

$$2e - 2 < e - 1 + 2$$

$$e < 3 \qquad \square.$$

Proof 2



$$\int_1^e \frac{1}{x}\,dx = 1 < 1\tfrac{551}{27720} = \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} < \int_1^3 \frac{1}{x}\,dx$$

$\Rightarrow e < 3$

$e<3$ Proof 3

$$\frac{1}{n!} = \frac{1}{1.2.3.\cdots.n} < \frac{1}{1.\underbrace{2.2\cdots 2}_{n-1 \text{ lots of } 2}} = \left(\frac{1}{2}\right)^{n-1} \quad \text{for } n \geq 3$$

$$\begin{aligned}
\therefore\ e = \sum_{n=0}^{\infty} \frac{1}{n!} &= 1+1+\frac{1}{2}+\sum_{n=3}^{\infty}\frac{1}{n!} \\
&= 2\tfrac{1}{2} + \sum_{n=3}^{\infty}\frac{1}{n!} \\
&< 2\tfrac{1}{2} + \sum_{n=3}^{\infty}\left(\frac{1}{2}\right)^{n-1} \\
&= 2\tfrac{1}{2} + \frac{1}{4}\cdot\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{n} \\
&= 2\tfrac{1}{2} + \frac{1}{4}\cdot\frac{1}{1-\frac{1}{2}} \\
&= 2\tfrac{1}{2} + \frac{1}{2} \\
&= 3
\end{aligned}$$

$$\Rightarrow e<3$$

$$\boxed{e = 2 + \cfrac{1}{1+\cfrac{1}{2+\cfrac{2}{3+\cfrac{3}{\ddots}}}}}$$

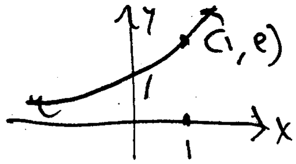
Observe:

Using a calculator,

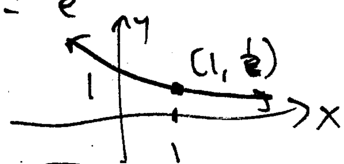
$$\begin{aligned}
e &= 2.718281828\ldots \\
&= 2 + 0.718281828\ldots \\
&= 2 + \frac{1}{1.39221119\ldots} \\
&= 2 + \frac{1}{1+0.39221119\ldots} \\
&= 2 + \cfrac{1}{1+\cfrac{1}{2.54964677\ldots}} \\
&= 2 + \cfrac{1}{1+\cfrac{1}{2+0.54964677\ldots}} \\
&= 2 + \cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1.81935024\ldots}}}
\end{aligned}$$

$$\begin{aligned}
&= 2 + \cfrac{1}{1+\cfrac{1}{2+\cfrac{2}{3.63870048\ldots}}} \\
&= 2 + \cfrac{1}{1+\cfrac{1}{2+\cfrac{2}{3+0.63870048\ldots}}} \\
&= 2 + \cfrac{1}{1+\cfrac{1}{2+\cfrac{2}{3+\cfrac{1}{1.56567909\ldots}}}} \\
&= 2 + \cfrac{1}{1+\cfrac{1}{2+\cfrac{2}{3+\cfrac{3}{\vdots}}}}
\end{aligned}$$

$y = e^x$ $\qquad e > 1 \Rightarrow y = e^x$ is increasing



$y = e^{-x}$



$\boxed{\lim_{x\to-\infty} x e^x = 0}$ & $\boxed{\lim_{x\to\infty} x e^{-x} = 0}$ — (e^x dominates over x)

Pf: $\lim_{u\to 0^+} u \ln u = 0$ Let $u = e^x$ $\therefore x = \ln u$

$u \to 0^+ \Rightarrow x \to -\infty$ $\therefore \lim_{x\to-\infty} x e^x = 0.$

$\lim_{u\to\infty} \frac{\ln u}{u} = 0$ Let $u = e^x$ $\therefore x = \ln u$

As $u\to\infty$, $x\to\infty$

$\therefore \lim_{x\to\infty} x e^{-x} = 0.$

Likewise, $\boxed{\lim_{x\to-\infty} x^k e^x = 0}$ & $\boxed{\lim_{x\to\infty} x^k e^{-x} = 0}$

Pf:

$\lim_{u\to-\infty} u e^u = 0$ Let $u = \frac{x}{k}$ $\therefore u\to-\infty \Rightarrow x\to-\infty$ $(k>0)$

$e^u = e^{\frac{x}{k}}$

$\therefore \lim_{x\to-\infty} \frac{x}{k} e^{\frac{x}{k}} = 0$

$\therefore \lim_{x\to-\infty} x e^{\frac{x}{k}} = 0$

$\therefore \lim_{x\to-\infty} x^k e^x = 0$

Replace x with $-x$: $\lim_{x\to\infty} (-x)^k e^{-x} = 0$ $\therefore \lim_{x\to\infty} x^k e^{-x} = 0.$

Lemma: $e = \sum_{r=0}^{\infty} \frac{1}{r!}$ (proved earlier)

$\boxed{e \text{ is irrational}}$

Proof. $1 + \sum_{r=0}^{\infty} \frac{1}{\prod_{i=0}^{r}(n+2+i)} < \sum_{r=0}^{\infty} \frac{1}{2^r} = \frac{1}{1-\frac{1}{2}} = 2.$

Hence $\frac{1}{(n+1)!}\left(1 + \sum_{r=0}^{\infty} \frac{1}{\prod_{i=0}^{r}(n+2+i)}\right) = \sum_{r=1}^{\infty} \frac{1}{(n+r)!} < \frac{2}{(n+1)!}$

$\therefore$ from the **Lemma**, for any $n \in \mathbb{Z}^+$, $e = \sum_{r=0}^{\infty} \frac{1}{r!} = \sum_{r=0}^{n} \frac{1}{r!} + R_n$ where $R_n := \sum_{r=1}^{\infty} \frac{1}{(n+r)!}$.

Now we have that $0 < n!R_n < n! \cdot \frac{2}{(n+1)!} = \frac{2n!}{n!(n+1)} = \frac{2}{n+1}$.

If $n > 1$, $\frac{2}{n+1} < 1$ and so $0 < n!R_n < \frac{2}{n+1} < 1$.

Therefore $n!R_n \notin \mathbb{Z}^+$ .... $(*)$.

Notice that $n!, \frac{n!}{2!}, \ldots, \frac{n!}{n!} \in \mathbb{Z}^+ \;\therefore\; \sum_{r=0}^{n} \frac{n!}{r!} = n! \sum_{r=0}^{n} \frac{1}{r!} \in \mathbb{Z}^+$.

Now assume $e \in \mathbb{Q}$, i.e., that e is rational.

$e \in \mathbb{Q} \Rightarrow$ there exists $a, b \in \mathbb{Z}^+$ such that $e = \frac{a}{b}$ since $e > 0$

$\Rightarrow$ if $n > b$ such that $b|n!$ so that $\frac{n!}{b} \in \mathbb{Z}^+$ then we have that

$\frac{n!a}{b} = n!e = n!\left(\sum_{r=0}^{n} \frac{1}{r!} + R_n\right) = n! \sum_{r=0}^{n} \frac{1}{r!} + n!R_n \in \mathbb{Z}^+$ since $a \in \mathbb{Z}^+$

$\Rightarrow n!R_n \in \mathbb{Z}^+$ since $n! \sum_{r=0}^{n} \frac{1}{r!} \in \mathbb{Z}^+$. But this contradicts $(*)$.

Hence the assumption that $e \in \mathbb{Q}$ is false. Hence $e \notin \mathbb{Q}$ $\square$

$\boxed{\text{If } i=\sqrt{-1},\ e^{i\theta} = \cos\theta + i\sin\theta}$

Pf: $f(\theta) = \cos\theta + i\sin\theta \qquad \therefore f(0) = \cos 0 + i\sin 0 = 1$

$f'(\theta) = -\sin\theta + i\cos\theta = i(\cos\theta + i\sin\theta) = i f(\theta)$

$\frac{f'(\theta)}{f(\theta)} = i \qquad \therefore \int \frac{f'(\theta)}{f(\theta)}\, d\theta = \ln f(\theta) = \int i\, d\theta = i\theta + c$

$\therefore \ln f(0) = \ln 1 = 0 = i(0) + c = c \;\therefore c = 0$

$\therefore \ln f(\theta) = i\theta$

$\therefore f(\theta) = e^{i\theta}$

$\therefore e^{i\theta} = \cos\theta + i\sin\theta \qquad \square$

$\boxed{e^{i\pi} = -1}$

Pf: $e^{i\pi} = \cos\pi + i\sin\pi = -1 + i(0) = -1$

$\boxed{e \text{ is transcendental}}$

e will not satisfy any polynomial equation with integer coefficients — Proof: see Ian Stewart's Galois Theory

Lagarias equivalence to the Riemann Hypothesis

Let $n > 1$ be an integer

$$h_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$$

σ_n = sum of positive divisors of n.

$$f_n = h_n + e^{h_n} \ln h_n$$

Then $\boxed{f_n > \sigma_n}$

Observe with a calculator:

n	σ_n	h_n	f_n	$f_n > \sigma_n$?
2	3	3/2	3.32	✓
3	4	11/6	5.62	✓
4	7	25/12	7.98	✓
5	6	137/60	10.38	✓
6	12	147/60	12.83	✓
		etc.		

A \$1,000,000 prize exists on www.claymath.org to prove it for all $n \geq 1$