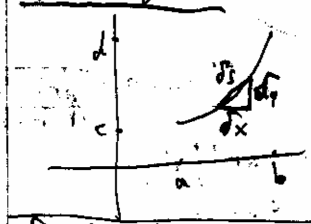
# Arc length (s) 10/5/2007




$$\delta s = \sqrt{(\delta x)^2+(\delta y)^2} \doteq \sqrt{1+\left(\frac{\delta y}{\delta x}\right)^2}\,\delta x = \sqrt{1+\left(\frac{\delta x}{\delta y}\right)^2}\,\delta y$$

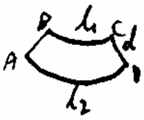
$\therefore$ arc length from $x=a$ to $x=b$ is

$$s = \int_{x=a}^{x=b} ds = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \left(= \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy \text{ if this is easier}\right)$$

P.T.O. for examples.

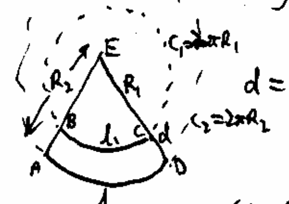
# Surface area of revolution (S.A.) 11/5/2007

Thm (Sector of an annulus)




$\text{Area}(ABCD) = \frac{1}{2}d(l_1+l_2)$.

PF:




$d = R_2 - R_1$

$$\frac{\text{Area}(BEC)}{\pi R_1^2} = \frac{l_1}{C_1} = \frac{l_1}{2\pi R_1} \therefore \text{Area}(BEC) = \tfrac{1}{2}R_1 l_1$$

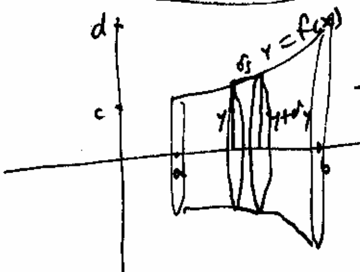
Likewise $\text{Area}(AED) = \frac{1}{2}R_2 l_2$.

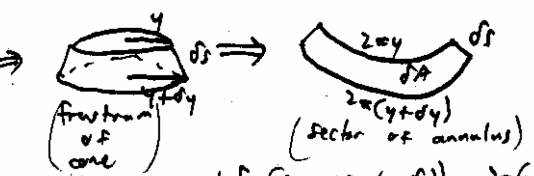
$\therefore \text{Area}(ABCD) = \frac{1}{2}(R_2 l_2 - R_1 l_1)$

Also, by similarity, $\frac{l_1}{R_1} = \frac{l_2}{R_2}$ $\therefore R_2 = \frac{l_2 R_1}{l_1}$ $\therefore \text{Area}(ABCD) = \frac{1}{2}R_1\left(\frac{l_2^2}{l_1} - l_1\right) = \frac{R_1}{2l_1}(l_2^2 - l_1^2)$

$\therefore d = \frac{l_2 R_1}{l_1} - R_1 = \frac{R_1}{l_1}(l_2 - l_1)$ $\therefore R_1 = \frac{l_1 d}{l_2 - l_1}$ $\therefore \text{Area}(ABCD) = \frac{1}{2}\frac{l_1 d (l_2^2 - l_1^2)}{(l_2 - l_1)l_1} = \frac{1}{2}d(l_1+l_2)$ $\square$

Rotation about x-axis



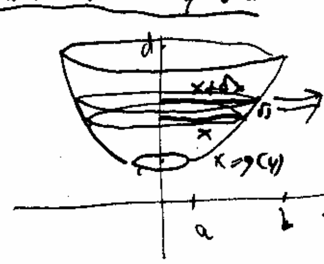

$\therefore \delta A = \frac{1}{2}\delta s\left(2\pi y + 2\pi(y+\delta y)\right) = 2\pi\left(y+\frac{1}{2}\delta y\right)\delta s$

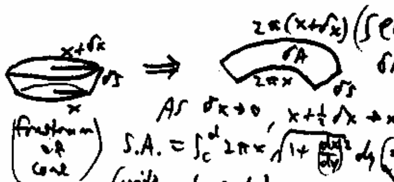
As $\delta y \to 0$, $y+\frac{1}{2}\delta y \to y$ & $\therefore$

$$SA = \int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \left(= \int_c^d 2\pi y\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy \text{ if this is easier}\right)$$

(without ends) To include ends, add $\pi\left((f(a))^2+(f(b))^2\right)$.

Rotation about y-axis




$\delta A = \frac{1}{2}\delta s\left(2\pi x + 2\pi(x+\delta x)\right) = 2\pi\left(x+\frac{1}{2}\delta x\right)\delta s$

As $\delta x \to 0$, $x+\frac{1}{2}\delta x \to x$ & $\therefore$

$$S.A. = \int_c^d 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy \left(= \int_a^b 2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \text{ if this is easier}\right)$$

(without ends)

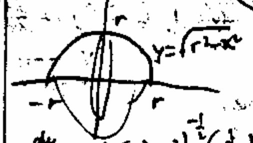
To include ends, add $\pi\left((g(c))^2+(g(d))^2\right)$.

# Surface area of a sphere

$$S.A.\,(\text{sphere}) = 4\pi r^2$$

Pf: $$S.A. = 2\int_0^r 2\pi y\sqrt{1+\left(-x/\sqrt{r^2-x^2}\right)^2}\,dx = \cancel{4\pi}\int_0^r 2\pi\sqrt{r^2-x^2}\sqrt{\frac{r^2-x^2+x^2}{r^2-x^2}}\,dx$$

(by symmetry)




$$\frac{dy}{dx} = \frac{1}{2}(r^2-x^2)^{-\frac{1}{2}}(-2x) = \frac{-x}{\sqrt{r^2-x^2}}$$

$$= 4\pi r\int_0^r dx$$
$$= 4\pi r\,[x]_0^r$$
$$= 4\pi r(r-0)$$
$$= 4\pi r^2. \quad \square$$