A Coupled FEM-BEM Formulation in Structural Acoustics for Imaging a Material Inclusion

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A rectangular plate, point supported at four corners, is irradiated with a normally incident acoustic wave. The plate has an inclusion embedded in it supposedly at an unknown location. A structural-acoustic coupled formulation using FEM and BEM is developed and the inclusion is imaged (located) from the surface displacement response of the plate. The efficacy of the formulation is demonstrated by varying the location, size, geometry, and material properties. For variations in location, size, and geometry of the inclusion, the method is seen to give satisfactory results. The frequency of the incident wave is limited to 4775 Hz due to the available computing power. This limits the smallest size and clarity of the image for certain cases. For some cases where the material properties are widely different, and where one would expect the results to distinguish the two materials clearly, the image clarity is less than satisfactory. A list of factors is presented for possible future work.

1. INTRODUCTION

The Finite Element Method (FEM) has evolved into a standardised technique for simulating structural dynamic systems. However, it can also be used in solving a large class of partial differential equations. This is well demonstrated in standard textbooks of FEM such as Zienkowicz, Cook, etc. The advent of the Boundary Element Method (BEM) as a generalised numerical procedure for partial differential equations is of comparatively recent origin. BEM has been used extensively in the areas of electromagnetics, electrostatics, geophysics, and acoustics, to name a few. Each of the above methods (FEM and BEM) has its own unique advantages. For example, FEM generates sparse matrices which are computationally efficient. However, since the entire domain is discretised, the size of the problem is potentially large. In problems where the domain of interest extends to infinity, (exterior problem) the standard FEM formulation cannot be used. On the other hand, in BEM, only the boundary of the domain needs to be discretised and hence the size of the problem is smaller. Thus, the exterior problem becomes tractable. However, the matrices are full and evaluation of the matrix requires significant computational effort.

In the area of acoustics, of the several problems which can be tackled using BEM, we will be focussing on solving steady state radiation/scattering problems. In this context, Shenck was one of the earliest to present a formulation of boundary integral equations in acoustic radiation problems. He presented the Surface Helmholtz Integral Formulation and the Interior Helmholtz Integral Formulation. However, he showed that for each of these formulations there occur frequencies where the solution becomes non-unique. The frequencies at which this non-uniqueness occurs are different for the two formulations. Thus, he recommended the use of both these formulations in tandem to arrive at an overdetermined system of equations which may be solved in a least squared sense. Burton and Miller also studied the non-uniqueness problem associated with boundary integrals in acoustics and suggested the use of a linear combination of the Surface Helmholtz Equation (SHE) along with its differentiated form to circumvent this problem. Most later researchers have used either Shenck’s or Burton-Miller’s method in their work and have had their reasons to offer for doing so. Meyer et al. used the boundary integral equation to solve for the acoustic radiation field from vibrating bodies of arbitrary shape. They developed an accurate and efficient numerical procedure to evaluate the singular oscillatory kernels appearing in the integral equations presented by previous workers. Seybert et al. showed that for problems involving axisymmetric geometry and boundary conditions, it is possible to have a simplified model representation which eases the computational effort. Seybert and Soenarko solved the problem of radiation and scattering of acoustic waves in a half space. For the case of the infinite boundary problem, a modified Green’s function of the Helmholtz equation was developed by placing an image point for each source point symmetrically behind the half plane. Extending notions of FEM, Seybert et al. formulated an isoparametric boundary element to solve the Helmholtz equation and used this method in predicting sound radiated from vibrating bodies. Jiang et al. carried some studies on error and convergence with relation to wavelength of the acoustic wave and characteristic dimensions of the boundary element chosen to represent the geometry of the vibrating structure. All these works did not consider the effect of elasticity of the structure in affecting its acoustic radiation.

As is well known (see Fahy, Cremer and Heckl, Junger and Feit) the finite elasticity of the radiating structure affects its radiation specially in cases such as underwater acoustics. The structure, in the presence of acoustic loading, not only radiates, it also scatters its own radiated field, leading to structural acoustic coupling. This phenomenon was studied by many researchers by coupling the FEM based structural model and the BEM based acoustic model. One of the earliest works found on this topic was by Wilton. The method proposed finding the acoustic loading force on the structure from the acoustic pressures and also enforcing con-
tinity of velocities across the two domains. He demonstrated
the method for the case of a plane wave incident on an elastic
sphere for various wavenumbers. Zienkowicz et al.\textsuperscript{14} also
demonstrated that in a generalised case, it is possible to cou-
ple the finite element and boundary element procedures to
better simulate certain phenomena. Mathews\textsuperscript{15} studied the
acoustic radiation from a three-dimensional arbitrary struc-
ture using a coupled finite element-boundary element ap-
proach. The finite elements used were isoparametric. The
shape functions used for modelling the surface of the struc-
ture and for interpolating the acoustic variables over the radi-
ating surface were the same. Ben Hariem et al.\textsuperscript{16} presented a
variational formulation for the fluid structure problem which
leads to symmetric algebraic equations. As an extension to
their work, Jeans and Mathew\textsuperscript{17} implemented higher order
isoparametric elements and evolved better techniques to ex-
pedite the solution at multiple frequencies. Everstine et al.\textsuperscript{18}
reported the coupling of NASTRAN finite elements with a
solver for the boundary integral equations, for studies of
fluid-structure interaction.

In this study, we use the coupled BEM-FEM formulation in
a novel application for the case of an inhomogenous struc-
ture interacting with an incident acoustic wave. We study the
case where the structure is comprised of an inclusion embed-
ded in a bulk material which is impinged on by an acoustic
wave. Both the bulk material and the inclusion are taken to
possess finite elasticity. The results show that displacement
fields close to the surface throw considerable light on the
shape, size, and properties of the inclusion.

2. FORMULATION

The standard FEM and BEM formulations are presented
in detail in the Appendix. We present the structural-acoustic
coupling formulation here and, for the sake of continuity, we
introduce the formulation with the final BEM and FEM equa-
tions. The BEM formulation for the acoustic pressure on
the surface of the radiator as a function of the surface velocities
and the incident pressure wave is given by (Eq. (35) in the
Appendix)

\[
[A][\tilde{P}] = [B][V] + \{p_{inc}\}, \tag{1}
\]

where \(\tilde{P}\) and \(V\) are column vectors whose \(j\)-th components
are the pressure and the normal velocity on the \(j\)-th element.
\(p_{inc}\) is a vector of incidence acoustic pressures at the field
points. \(A\) and \(B\) are matrices. Similarly, the FEM formulation
results in the following equation (Eq. (42) in the Appendix)

\[
[M][d] + [K][d] = \{F_{ext}\}, \tag{2}
\]

where \(d\) is a nodal displacement vector and \(F_{ext}\) is an external
force vector which includes the loading on the surface due to
the acoustic pressures. These two equations are used in the
structural-acoustic coupling formulation in the following sec-
ction.

2.1. Structural-Acoustic Coupling

The steady state response of an elastic structure is mod-
elled using the FEM equations as

\[
-(\omega^2[M] + [K])[d] = \{F_{ext}\} = \{F_{rad}\} + \{F_D\}, \tag{3}
\]

where \(M\) and \(K\) are mass and stiffness matrices with dimen-
sions \(m \times m\). \(\{F_{rad}\}\) is a \(m \times 1\) load vector correspon-
ding to the acoustic loading of the fluid external to the structure,
\(\{F_D\}\) is a \(m \times 1\) load vector corresponding to the other exter-
nal forces which drive the structure, and \(\{d\}\) is a \(m \times 1\) col-
umn vector of nodal displacements throughout the structure.
As evident in the above equation, \(\{F_{ext}\}\) has been divided
into a contribution from the acoustic loading \(\{F_{rad}\}\) and all
other external forces \(\{F_D\}\).

The acoustic loading obtained from the BEM formulation
is in terms of pressures at the centroids of the triangular sur-
f ace elements. These triangular surface elements appear at
the boundary because of the solid tetrahedral elements mod-
elling the structure. The elemental centroidal acoustic
pressure obtained from the BEM is multiplied by the elemental
triangular area and a third of it is apportioned to each node of
that triangular element. Thus, each of the surface nodes gets
a contribution of this force based on the number of elements
to which it is common. The load vector thus obtained is
\(\{F_{rad}\}\). To further elaborate, the node numbering in the BEM
and the FEM has been done in such a way that if the node
numbers in BEM start from 1 and end at \(N\), the FEM node
numbers for the structural nodes, which coincide with these
BEM nodes, are exactly the same. It should be mentioned,
however, that since FEM is used to discretise the structure,
the FEM nodes are greater in number. For example, Fig. 1
shows a plate with two surface triangular boundary elements,
and the direction of the incident acoustic wave which is not
aligned with the global coordinate system. The node numbers
are the same, i.e., one to four for BEM as well as FEM.

![Figure 1. A representative plate with two surface BEM elements to
illustrate the transformation between the acoustic pressure and the
load vector.](image-url)

Let a unit pressure act on each of the triangles. Let \(l_1, l_2, l_3\)
be the direction cosines of the outward normal of the triangu-
lar elements with respect to the global coordinate system.
Let \(a_1\) and \(a_2\) be the areas of the two triangles, then the net
forces in the direction of the inward normal on each of the
triangles due to the acoustic pressure is \(a_1\) and \(a_2\). These
have \(-a_1 l_i\) and \(-a_2 l_i, (i = 1, 2, 3)\) as the components in the
global directions. Next, a third of this force is apportioned to
each of the nodes of the element. In this particular example,
nodes 2 and 4 are common to both the elements so that they
get contributions from both the element acoustic loads. In
FEM, the external load vector \( \{ F_{ext} \} \) has the dimension \( m \times 1 \)
and the corresponding acoustic pressure vector in BEM has
the dimension \( N \times 1 \). Thus we need a transform of dimension
\( m \times N \) between the acoustic load on the surface and the FEM
load vector as it appears in Eq. (3). The transform matrix ac-
commodates not only the directionality but also the influence
of the true force amplitude on each element (by accounting
for the area),

\[
D_j = \begin{bmatrix}
\frac{3r-2 \times 3s-1 \times 3t}{a_1 l_1^3} & \frac{3r-2 \times 3s-1 \times 3t}{a_2 l_2^3} & \frac{3r-2 \times 3s-1 \times 3t}{a_3 l_3^3} \\
0 & 0 & 0
\end{bmatrix} \rightarrow
\begin{bmatrix}
\frac{3r-2 \times 3s-1 \times 3t}{a_1 l_1^3} & \frac{3r-2 \times 3s-1 \times 3t}{a_2 l_2^3} & \frac{3r-2 \times 3s-1 \times 3t}{a_3 l_3^3} \\
0 & 0 & 0
\end{bmatrix}^T,
\]

where \( r, s, t \) correspond to the node numbers of the \( i \)-th ele-
ment. For the example given in Fig. 1, \( i \) takes values 1 and 2. Thus, we have for the pressure vector \( \{ P \} \)

\[
\{ P^{rad} \} = -[D][\dot{P}].
\]  

Next, since on the boundary elements piecewise con-
stancy of variables has been assumed, this holds true even for
the surface normal velocity. Hence, again, we need a trans-
form from the centroidal normal surface velocities to the
nodal displacements in the structure. This is found as follows.
For a boundary element \( \Delta \Gamma_j \), a normal structural velocity
vector is defined at its centroid. The value for this is obtained
by averaging the vector velocity distribution over the element
and may be written as

\[
\begin{bmatrix}
v_{xj} \\
v_{yj} \\
v_{zj}
\end{bmatrix} = \frac{1}{A} \int_{\Delta \Gamma_j} \text{rot} \frac{1}{ho} \begin{bmatrix}
u_x \\
u_y \\
u_z
\end{bmatrix} d\Gamma_j,
\]

where \( v_{xj}, v_{yj}, \) and \( v_{zj} \) are the velocity components at the cen-
troid of the \( j \)-th surface element. \( u_x, u_y, \) and \( u_z \) are the compo-
tents of the finite element structural displacement field on
the boundary element \( \Delta \Gamma_j \). The \( u_x, u_y, \) and \( u_z \) are related to
the nodal displacement vector \( \{ d_i \} \) of the \( j \)-th finite element in
the following way (the \( i \)-th finite element corresponds to
the \( j \)-th boundary element)

\[
\begin{bmatrix}
u_{xj} \\
u_{yj} \\
u_{zj}
\end{bmatrix} = \frac{1}{A} \int_{\Delta \Gamma_j} \text{rot} \frac{1}{ho} \begin{bmatrix}
u_x \\
u_y \\
u_z
\end{bmatrix} d\Gamma_j,
\]

where \( N_i \) is the local shape function matrix that represents
the surface element. Since \( N_i \) is a function of the three local
coordinates, one of the coordinates is set so that it represents
the surface of the plate. Thus, \( u_x, u_y, \) and \( u_z \) are functions
over the boundary element area. \( \{ d_i \} \) is the local nodal dis-
placement vector in the \( j \)-th finite element given by

\[
\{ d_i \} = \begin{bmatrix}
u_{x1} \\
u_{y1} \\
u_{z1} \\
u_{x2} \\
u_{y2} \\
u_{z2} \\
u_{x3} \\
u_{y3} \\
u_{z3}
\end{bmatrix}^T,
\]

where \( u_{xj}, u_{yj} \) and \( u_{zj} \) are the \( i \)-th node numbers \( i \) are the \( x, y, z \) displacements
at the local node numbers \( i \). Thus,

\[
\{ v_{xj} v_{yj} v_{zj} \} = [C_i] \{ d_i \},
\]

where

\[
[C_i] = \frac{1}{A_j} \int_{\Delta \Gamma_j} \text{rot} \frac{1}{ho} [N_i] d\Gamma_j.
\]  

It should be noted that \( \{ d_i \} \) is a constant vector and hence
is taken out of the integration. Next, to obtain the normal ve-
focity for the element \( \Delta \Gamma_j \), the velocity expression in Eq. (9)
is dotted with the unit element normal,

\[
v_n = [l_{a1} l_{a2} l_{a3}][v_x v_y v_z]^T
= [l_{a1} l_{a2} l_{a3}][C_i] \{ d_i \} = [L][C_i][\dot{d_i}],
\]

The above equation is for a single surface element. The
total surface velocity vector is obtained by assembling the in-
dividual element matrices accounting for the connectivities.
The final velocity vector is given by

\[
[V] = [L][C][\dot{d}].
\]

Finally, the four equations needed for presenting the cou-
pled structural acoustic formulation are

\[
[A][\dot{P}] = [B][V] + \{ P \}^{inc};
\]

\[
-\omega^2 [M] + [K]\{d\} = \{ F \};
\]

and

\[
[V] = [L][C][\dot{d}].
\]

The final equation for \( \dot{d} \) is given by

\[
(-\omega^2 [M] + [K] + [DA]^{-1} [B][L][C])\{ d \} = -[D][A]^{-1} \{ P \}^{inc}.
\]

2.2. Plate

In this section, a \( 2 \times 2 \times 0.2 \) m steel plate, submerged in
water and having zero displacements at the four corners is ir-
radiated with a normally incident sound wave. The in-plane
coordinates of the plate are \( x \) and \( y \), and \( z \) is the normal coor-
dinate in the thickness direction. The point \( x = 0, y = 0, z = 0 \)
is at the centre of the plate. The plate has an inclusion of a
certain size and at a certain location. The response of the
plate to the total sound wave (blocked and scattered) is com-
puted using the formulation given above. The location and
extent of the inclusion is estimated from the response contour
across the plate. Within this general framework, several para-
metric studies are conducted. The degree of accuracy is ex-
amined with which the inclusion can be imaged as a function
of its size, shape, location, frequency of the incident wave,
the ratios of Young’s moduli, and densities of the parent and the
inclusion material. The density of water is taken as
\( \rho = 1500 \text{ kg/m}^3 \), and the speed of sound \( c = 1500 \text{ m/s} \).

The FEM mesh for the plate is created in ANSYS using
tetrahedral elements (Solid 72). The resulting triangular mesh
on the outer surface of the plate itself is taken as the BEM
mesh. The average element size is 0.1 m. Here we mention that
the size of the mesh is dictated strongly by the comput-
ing power. The mesh density imposes a limitation on the highest frequency of the incident sound wave. For the current mesh, this frequency is found to be 4775 Hz. Given a sonic velocity of 1500 m/s in water, the corresponding wavenumber is 20 m\(^{-1}\). In several of the cases to be presented below, a higher incident wave frequency for a more refined mesh would have given much better results, but due to the reasons given above, we were limited to 4775 Hz. The FE mesh was a 3-D tetrahedron mesh with 4737 elements and 1365 nodes. The 2-D BE triangular mesh had 2132 elements and 1068 nodes. A simple validation test was made where the parent and inclusion material were taken to be the same. The results are presented in the Appendix.

**Figure 2.** Real displacement on the surface \(z = 0.1\) m, for a steel plate with a square lead inclusion of dimension on each side 0.9 m, when irradiated with an acoustic wave of \(k = 20\) m\(^{-1}\).

First, we demonstrate the accuracy of the method in locating the inclusion. Figure 2 shows the results for the square steel plate with a square lead inclusion, 0.9 \(\times\) 0.9 m, at the centre of the plate. \((E_{\text{steel}} = 2 \times 10^{11} \text{ N/m}^2, \rho_{\text{steel}} = 7840 \text{ kg/m}^3, E_{\text{lead}} = 6.67 \times 10^9 \text{ N/m}^2, \rho_{\text{lead}} = 13000 \text{ kg/m}^3)\). The contours for the real displacement of the plate are presented for an incident wavenumber of 20 m\(^{-1}\). From the results plotted, the parent material and the distinct inclusion regions can be seen clearly. The actual response magnitude is not seen to be of significance and hence the contour scale is not given. In Figs. 2 through 10, and 14 through 18, the axes represent the coordinates on the plate with the origin at the centre. The contour represents the surface displacement.

Next, we examine the efficacy of the method in discerning shapes of inclusions keeping all other parameters the same. Figure 3 shows the results obtained for a rhombohedral inclusion, located at the centre. It is clear from the figure that the shape is identified. Clearly there would exist a limitation to the minimum orientation difference which would be adequately captured in the structural acoustic response. Quantifying such resolution related issues are left for future work.

The simulations described up to this point have the inclusion in the centre. The computational technique should be capable of picking up the inclusion located in any part of the parent material. The next simulation was carried out by moving the inclusion to the corner centred at (-0.5,0.5). The rest of the parameters were kept the same. The real displacement of the top surface of the plate is shown in Fig. 4. The figure approximately indicates the shape and location of the embedded inclusion. It is reasonable to conclude that any inclusion of dimension 0.9 m embedded in the parent material of size 2 m can be adequately captured, regardless of its location. Figures 2 and 4 represent extreme cases when the inclusion is at the centre and when it is at a corner. It is expected that any intermediate location would give a reasonably good match.

**Figure 3.** Real displacement on the surface \(z = 0.1\) m, for a steel plate with a rhombus-shaped lead inclusion of dimension on each side 0.9 m, when irradiated with an acoustic wave of \(k = 20\) m\(^{-1}\).

**Figure 4.** Real displacement on the surface \(z = 0.1\) m, for a steel plate with a square-shaped lead inclusion of dimension on each side 0.9 m, centred at the corner (-0.5,0.5) when irradiated with an acoustic wave of \(k = 20\) m\(^{-1}\).
The essential feature in all these results is that the parent material has an overall low response compared to the inclusion material (due to the material constants chosen). As one moves from the parent material to the inclusion, there should be a steep change in response amplitude. The inclusion region in this case will show high frequency differences in displacement compared with the parent material. Thus, the gradient of the displacement field can also be used for imaging the inclusion. Figure 5 represents the gradient of the real displacement field shown in Fig. 2. Similarly, Figs. 6 and 7 represent the gradient of the real displacement fields plotted in Figs. 3 and 4 respectively.

Each of Figs. 5, 6, and 7 indicates the presence of the embedded inclusion with the approximate size and location. A notable difference between the contour plots of displacement and the gradient of displacement is that in the latter case, triangular patches are seen to have a constant colour. This is expected, as we have used linear triangular elements for the FEM modelling.
of resolution is reached. The image can be improved with a higher frequency excitation. As mentioned above, due to memory size, we were restricted to a certain upper incident frequency and hence this limitation.

Next, the effect of wavenumber of the incident wave on the response was studied. The same sets of simulations were carried out for a lower wavenumber of \( k = 1 \) m\(^{-1}\). Figure 9 shows the response for this case. On comparison with Fig. 2, it can be seen that the former image possesses rounded features for the inclusion. This is expected at the lower frequency. Figure 10 shows the results for a rhombohedral inclusion. Here again, compared with Fig. 4 we find rounded features for the lower frequency case. However, it is worth noting that even in the lower frequency region the inclusion-shapes (square and rhombus) are sufficiently different for correct identification. Thus, if we are content in sacrificing sharpness, we can still detect the presence of the inclusion even with a longer wavelength. This appreciably saves computational effort.

2.3. Image Processing

The results of the previous sections showed that under suitable circumstances the raw image, in the form of a surface plot of the real displacement, has appreciable information about the embedded inclusion. This section discusses the methods developed to enhance this information.

It is noted that the figures mainly show the parent material region to be predominantly in black (having a lower response) and the inclusion region having all shades of grey spilled into it. Ideally, what we desire is that each of the parent material and the inclusion region be represented by a different shade of grey. To this end, we devised the following algorithms.

To test the capability of the method in imaging a small inclusion, a 0.4 × 0.4 m lead inclusion was placed in the parent matrix. Figure 8 shows the response of the plate. The inclusion is square shaped and located at the centre of the parent steel plate. Here also, the incident wavenumber is 20 m\(^{-1}\). Figure 8 does show a distinct change in response around the central region of the plate. However, it is difficult to obtain the shape and size of the inclusion from the plot. This suggests that with inclusion dimensions less than about 0.4 × 0.4 m (which is less than 5% of the parent material area) the limit of resolution is reached. The image can be improved with a higher frequency excitation. As mentioned above, due to memory size, we were restricted to a certain upper incident frequency and hence this limitation.

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For surface plots, MATLAB, by default, divides the field into seven levels and assigns a colour to each of these levels. Our aim is to divide the entire region into two distinct shades. With the knowledge that the bulk material has a predominantly low response, we replaced the values corresponding to the lowest level with an arbitrary value of 0. All other values were arbitrarily assigned a value of 100. This procedure is called Level 1 processing. However, since the response within the inclusion region has both high and low values, the inclusion region will have sub-regions with 0 value and those with 100. The parent region will have all zero values.

This drawback is addressed in the Level 2 image processing. In this procedure we scan each row of the image first from left to right. On finding a value of 100, the location is noted as say \( x_1 \), and then the scanning for the same row is done from right to left, and the first 100 value is again noted, denoted by \( x_2 \). All intermediate values between \( x_1 \) and \( x_2 \) are then assigned a value of 100. Thus, the discontinuities associated with 0 values are removed within the inclusion region. This procedure is repeated in the vertical direction, converting the entire inclusion region to values of 100.

These simple image processing algorithms were found to enhance the image quality as we required. The enhanced images are shown in Figs. 11-13. It may be recalled that these images were developed taking the inputs from Figs. 2, 3, and 4, respectively. The axes in each of these cases are the pixel numbers.

A limitation of these algorithms is that the inclusion region near the interface having a zero value is not rectified.
This may require more innovative algorithms for processing in order to enhance the image and can form a good area for future work.

3. IMAGING WITH DIFFERENT RATIOS OF YOUNG’S MODULUS AND DENSITY

It seems that the response in the parent plate and in the inclusion is dominantly flexural. The first longitudinal mode in steel in the thickness direction occurs above 12 kHz, whereas the driving frequency is at 4775 Hz. Hence, the parent plate carries flexure only. The entire plate moves up and down in the thickness direction unison. In the lead inclusion, the first longitudinal mode occurs at approximately 2 kHz and the next mode approximately at 4 kHz (using a simple pure longitudinal wave calculation). These two modes are uniform across the entire lead surface. Hence, for the flexure of the inclusion surface, a uniform displacement value will either be added or subtracted. This displacement value is small because the inclusion is stiff in compression. Based on the displacement variations within the lead, flexure is the dominant mode of vibration, even within the lead itself.

The results presented above are for the particular ratios of $E_1$ and $E_2$, and $\rho_1$ and $\rho_2$. The value of $E_1$ was 30 times that of $E_2$ and $\rho_2$ was twice that of $\rho_1$. One would like to have the proposed method be useful in as general a case as possible. To this end, simulations were conducted with different $E_1$ and $E_2$ ratios, and $\rho_1$ and $\rho_2$ ratios. The success in locating

Figure 11. Imaging a square inclusion embedded at the centre.
the inclusion for a general case has been mixed. For the case where the parent material was five times stiffer than the inclusion, the image clarity was poor. A ratio of five seems to be the lower end for proper resolution of the materials.

Similarly, even when the inclusion was 30 times stiffer than the parent material, the results were not satisfactory (see Fig. 15). (Since the case with the parent material 30 times stiffer gave reasonable results, it seems that the problem does not have bi-directional symmetry.) Figures 16 and 17 present cases where the inclusion density was 60 times the parent material density and vice versa, respectively. The same inadequacy is evident here also. It should be mentioned that despite this inadequacy, certain features related to wavespeeds are visible in the response. When the inclusion is stiffer, the wavespeed is higher within the inclusion and hence the wavelengths are longer than in the parent material. Thus, a greater number of amplitude variations will be seen in the parent material than in the inclusion (as in Fig. 15). Similarly, when the inclusion is denser than the parent material, the wavespeed is very low within the inclusion and so is the wavelength. So, more amplitude variations are present in the inclusion material (as in Fig. 16).

In order to understand this, a 1-D case of a cantilevered beam with an inclusion was studied numerically. The inclusion was placed at about half the length of the beam. Both the beam and the inclusion have the same cross sectional dimensions. For the sake of brevity we just summarise the results. Specifically, density ratios of 2, 1/2, 3, and 1/3 were used and
the response along the beam was computed. These above ratios were sufficient to produce an appreciably different response between the two materials. However, in the case of Young’s moduli, ratios of 9, 1/9, 25, and 1/25 were required to cause an observable difference in response between the two materials. These results can be explained approximately as follows.

The driving point impedance for a point force in an infinite beam is given by

$$Z = \frac{2EI}{(EI)^{1/4} \omega^{1/2} (\rho bh)^{3/4} (1 + j)},$$  \hspace{1cm} (15)$$

where the symbols are in accordance with common convention. If at the driving point, the left side is made of one material and the right side made of another, then approximately the ratio of velocities on either side will be the ratio of the driving point impedances given by the above formula, and computed using the different material properties. This is so because the geometry remains same, and the materials are different on each side. This ratio is given by

$$\frac{V_1}{V_2} = \left(\frac{E_1 \rho_1^3}{E_2 \rho_2^3}\right)^{rac{1}{4}},$$  \hspace{1cm} (16)$$

where by $V_1$ and $V_2$ we imply the peak amplitudes within each material to the left and right of the driving point. From the above equation, it can be seen that the response is less sensitive to Young’s modulus variations compared to density.

**Figure 13.** Imaging a rhombus-shaped inclusion embedded at the centre.
Figure 14. Real displacement on the surface $z = 0.1 \text{ m}$, for a square plate with a square inclusion of dimension on each side 0.9 m embedded at the centre, when irradiated with an acoustic wave of $k = 20 \text{ m}^{-1}$, $E_{\text{parent}} = 5E_{\text{inclusion}}$.

Figure 15. Real displacement on the surface $z = 0.1 \text{ m}$, for a square plate with a square inclusion of dimension on each side 0.9 m embedded at the centre, when irradiated with an acoustic wave of $k = 20 \text{ m}^{-1}$, $E_{\text{inclusion}} = 30E_{\text{parent}}$.

Figure 16. Real displacement on the surface $z = 0.1 \text{ m}$, for a square plate with a square inclusion of dimension on each side 0.9 m embedded at the centre, when irradiated with an acoustic wave of $k = 20 \text{ m}^{-1}$, $\rho_{\text{inclusion}} = 60\rho_{\text{parent}}$.

Figure 17. Real displacement on the surface $z = 0.1 \text{ m}$, for a square plate with a square inclusion of dimension on each side 0.9 m embedded at the centre, when irradiated with an acoustic wave of $k = 20 \text{ m}^{-1}$, $\rho_{\text{parent}} = 60\rho_{\text{inclusion}}$.

variations. Since this behaviour has been observed in the beam example, the above equation can be taken as a reasonable model for the phenomena observed. One would expect a similar expression to apply for the plate case also. However, the plate geometry does not lend itself to a 2-D wave solution. Hence, the expression has to be arrived at through an empirical method. To complicate matters, the plate will have localised twisting of the surface, complications due to corner effects, multiple surfaces from which waves can get reflected and the presence of a nearfield at every reflection, and the possibility of coupling of longitudinal and flexure due to the 3-D nature of the model. A large number of simulations will be needed to ascertain the physics of the phenomena. The response will depend strongly on geometry, and a rhombus will respond differently from a rectangle. These are some of the possible reasons why even with widely varying densities, the materials were not commensurately distinguishable. In addition, the decision based on the visual image created by the response may not be the best parameter to use. The existing response may already carry enough information about the in-
clusion (as stated with regard to wavespeeds) but may need data processing to bring it out. The above mentioned issues are left for future investigations. However, the ratios chosen in this manuscript have shown the formulation to be sufficiently accurate. Even in the cases where the image quality was not satisfactory, the wave speed behaviour within the materials captured the physics of the situation.

However, the first longitudinal mode in steel in the thickness direction occurs above 12 kHz. Hence, the parent plate carries flexure only. The entire plate thickness moves up and down in unison. In the lead inclusion, the first longitudinal mode occurs at approximately 2 kHz and the next mode approximately at 4 kHz (using a simple pure longitudinal wave calculation). These two modes are uniform across the entire lead surface. Hence, to the flexure on the inclusion surface, a uniform displacement value will either be added or subtracted. This displacement value is small because the inclusion is stiff in compression. Based on the displacement variations within the lead we can conclude that flexure is the dominant mode even within the lead inclusion.

4. CONCLUSIONS

A rectangular plate, point supported at four corners, was irradiated with an acoustic wave at normal incidence. The plate had an inclusion embedded in it supposedly at an unknown location. The parent material was chosen to be steel which was 30 times stiffer than the inclusion material. It was also half as dense as the inclusion. These ratios gave responses with which the two materials could be distinguished from one another. A structural-acoustic coupled formulation using FEM and BEM was developed and the inclusion imaged (located) from the surface displacement response of the plate. The usefulness of the formulation was demonstrated by varying the location, size, geometry, and material properties of both materials. For variations in location, size, and geometry of the inclusion, the method gave satisfactory results. The frequency of the incident wave was limited to 4775 Hz due to the available computing power. The frequency range limited the smallest size of the inclusion and clarity of the image for certain cases. The raw image could be easily enhanced with a simple image cleaning algorithm to provide better pictures of the inclusion. For some cases where the material properties were widely different and where one would expect the results to distinguish the two materials clearly, the image clarity left something to be desired. We have listed the factors which could be responsible for such lack of clarity. These issues will be part of future work.

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APPENDIX

Boundary Element Method

The BEM formulation for the acoustic simulation is described briefly here. The governing differential equation in acoustics is the Helmholtz equation which is given by

$$\nabla^2 \phi + k^2 \phi = 0 \quad \text{in} \, \Omega ,$$  

(17)
where $\phi$ is the acoustic potential in which

$$\rho \frac{\partial \phi}{\partial t} = \vec{p} \quad \text{and} \quad \frac{\partial \phi}{\partial n_{q}} = v_{n},$$

(18)

$k, \rho, \vec{p}, v_{n}$ are the wavenumber, density, acoustic pressure, and normal velocity, respectively. The Green’s function ($G$) is the solution of the equation

$$(\sigma^{2} + k^{2})\phi = \delta(r_{i}) \quad \text{in} \quad \Omega,$$

(19)

where $\delta(r_{i})$ is the dirac delta function at point $r_{i}$. For a time dependence of $e^{j\omega t}$, the Green’s function is given in three dimensional spatial coordinates $(x, y, z)$ as

$$G(r, r_{i}) = \frac{1}{4\pi} \frac{e^{j\|r-r_{i}\|}}{|r-r_{i}|},$$

(20)

where $|r-r_{i}|$ is the distance from the point $r_{i}$ to point $r$. The detailed derivation may be found in reference$^{b}$. Equation (17), after suitable manipulation, can be expressed as an integral equation over the boundary of the domain $\Gamma$.

This is called the Surface Helmholtz Equation (SHE) and is represented as follows$^{b}$

$$c(p)\phi(p) = \left[ G(p, q) - \frac{\partial G(p, q)}{\partial n_{q}} \right] dS_{q} + \phi_{inc}(p),$$

(21)

where

$$c(p) = 1, \text{ if } p \text{ is in } \Omega;$$

$$c(p) = \frac{1}{2}, \text{ if } p \text{ is on } \Gamma;$$

$$c(p) = 0, \text{ if } p \text{ is not in } \Omega \text{ or } \Gamma;$$

$\phi_{inc}(p)$ is the incident field at the field point $p$ and $G$ is the Green’s function in Eq. (20).

In other words, if we know the distribution of source points on a closed surface, then the acoustic potential at any field point, $p$, lying outside this closed surface may be found using Eq. (21). The symbol $G(p, q)$ emphasises that the Green’s function is to be computed for the field point, $p$, with respect to the source point, $q$, lying on the infinitesimal area $dS_{q}$. All the $dS_{q}$ values together make up $\Gamma$.

By meshing the boundary, $\Gamma$ is decomposed into small values of $\Delta \Gamma_{j}$ in such a way that

$$\Gamma \approx \Gamma = \sum_{j}^{N} \Delta \Gamma_{j}, \quad (j = 1, 2...N).$$

(22)

We assume a piecewise constant approximation over each $\Delta \Gamma_{j}$. The value of $\phi$ and $\partial \phi/\partial n_{q}$ in each $\Delta \Gamma_{j}$ are given by $\phi(q_{j})$ and $\partial \phi(q_{j})/\partial n_{q}$, which are their respective values at the centroid $q_{j}$ of $\Delta \Gamma_{j}$. This approximation becomes increasingly accurate as the discretisation becomes finer. Thus Eq. (21) is rewritten as

$$c(p)\phi(p) = \sum_{j} \int_{\Delta \Gamma_{j}} \phi(q_{j}) \left[ G(p, q_{j}) - \frac{\partial G(p, q_{j})}{\partial n_{q}} \right] dS_{q_{j}}$$

$$- \sum_{j} \int_{\Delta \Gamma_{j}} G(p, q_{j}) \frac{\partial \phi(q_{j})}{\partial n_{q}} dS_{q_{j}} + \phi_{inc}(p),$$

(23)

With the piecewise constant approximation, the quantities $\phi(q_{j})$ and $\partial \phi(q_{j})/\partial n_{q}$ are treated as constants (though unknown as yet) over each $\Delta \Gamma_{j}$. Thus, the above equation then becomes

$$c(p)\phi(p) = \sum_{j} Q_{j}(p)\phi_{j} - \sum_{j} L_{j}(p)\frac{\partial \phi_{j}}{\partial n_{q}} + \phi_{inc}(p),$$

(24)

where

$$Q_{j}(p) = \int_{\Delta \Gamma_{j}} G(p, q_{j}) dS_{q_{j}};$$

(25)

$$L_{j}(p) = \int_{\Delta \Gamma_{j}} \frac{\partial G(p, q_{j})}{\partial n_{q}} dS_{q_{j}}.$$}

Now, we take the field points $p$ at the centroid $p_{i}$ of each $\Delta \Gamma_{i} (i = 1, 2...N)$. This leads to a set of $N$ simultaneous equations. Replacing $p_{i}$ with just $i$, the equations can be written compactly as

$$c_{i}\phi_{i} = \sum_{j} Q_{ij}\phi_{j} - \sum_{j} L_{ij}\frac{\partial \phi_{j}}{\partial n_{q_{j}}} + \phi_{inc_{i}}, \quad (i, j = 1, 2...N),$$

(27)

where the running index $i$ is over the field points. In the notation to follow, ‘‘ brackets denote a vector and a boldfaced capital letter is a matrix. Thus, the matrix form of the equation above becomes

$$[\mathbf{Q} - \frac{1}{2}] [\phi] = \mathbf{L} \left[ \frac{\partial \phi}{\partial n} \right] + [\phi_{inc}],$$

(28)

where $\mathbf{Q}$ and $\mathbf{L}$ are $N \times N$ matrices, $\mathbf{I}$ is the $N \times N$ identity matrix, and $[\phi_{inc}]$ is the known vector of the incident acoustic field.

It is to be noted that in an uncoupled radiation/scattering problem, either $\partial \phi/\partial n_{q}$ or $\phi$ is known so that there are $N$ unknowns to be solved for. However, in a structural-acoustic coupled problem such as dealt with here, neither is known. This set of equations is augmented by another set obtained from the structural response formulation leading finally to as many equations as unknowns. This will be shown later.

As reported by Shenck,$^{3}$ Eq. (28) has non-unique solutions at certain frequencies. This is true of even the Differen-
tiated Surface Helmholtz Equation (DSHE). However, the frequencies for this non-uniqueness are different for the two equations. Burton and Miller$^{4}$ have shown that a linear combination of the SHE (Eq. (21)) and DSHE circumvents this problem. In the DSHE, the differentiation is taken in the direction of the normal through the field point $p$. The DSHE equation is given by

$$\frac{\partial}{\partial n_{p}} \left[ \left[ \phi(q) - G(q) \right] dS_{q} = c(p) \frac{\partial \phi(p)}{\partial n_{q}},$$

(29)

where $c(p)$ are as in Eq. (21). Following the same steps which were used in discretising the Surface Helmholtz Equation, we obtain a set of simultaneous equations for DSHE:

$$\sum_{j} R_{ij}\phi_{j} = \sum_{j} \frac{\delta_{q_{j}}}{2} + Q_{ij}\phi_{j}, \quad (i = 1, 2...N).$$

(30)
Assembled into a matrix form, the equations are given by

$$[R](\phi) = \left[ \frac{1}{2} + Q^* \right] \left[ \frac{\partial \phi}{\partial n} \right],$$  \hfill (31)

where the \((i, j)\)-th element of the matrices \(R\) and \(Q^*\) are defined as follows: (again we replace \(p\) by index \(i\) and \(q\) by index \(j\))

$$R_{ij} = \frac{\partial}{\partial n_j} \int_{\partial V_i} \frac{\partial G}{\partial n_j} dS_{ij},$$ \hfill (32)

$$Q^*_{ij} = \int_{\partial V_i} \frac{\partial G}{\partial n_j} dS_{ij}. \hfill (33)$$

After linearly combining the SHE and DSHE the equation is given by

$$[Q - \frac{1}{2} + aR]\{\phi\} = [L + a\left(\frac{1}{2} + Q^*\right)]\left\{ \frac{\partial \phi}{\partial n} \right\} + \{\phi^{inc}\}, \hfill (34)$$

where \(a\) is complex, when the wavenumber \(k\) is purely real or imaginary and real when \(k\) is complex.\(^{21}\)

Using Eq. (18), the above equation for the acoustic potential may be expressed in terms of the acoustic pressures and velocities as follows

$$[A]\{\hat{P}\} = [B]\{V\} + \{P^{inc}\}, \hfill (35)$$

where \(\hat{P}\) and \(V\) are column vectors whose \(j\)-th components are the pressure and the normal velocity on the \(j\)-th element. \(P^{inc}\) is a vector of incident acoustic pressure at the field points. \(A\) and \(B\) are given by

$$[A] = \left[ Q - \frac{1}{2} + aR \right]^{-1},$$ \hfill (36)

and

$$[B] = [L + a\left(\frac{1}{2} + Q^*\right)]. \hfill (37)$$

### Finite Element Method

A brief description of the Finite Element Method for structural dynamics is presented here, starting from the principle of virtual work. The principle of virtual work states that the virtual work done by external forces is absorbed by the work of the internal and inertial forces for any small kinematically admissible displacements. The statement in a mathematical form is given by\(^{3}\)

$$\int_V \delta\{u\}^T \{F\} dV + \int_S \delta\{u\}^T \{\Theta\} dS + \sum \delta\{u\}^T \{P_{conc}\}$$

$$= \int_V \delta\{\varepsilon\}^T \{\sigma\} + \delta\{u\}^T \rho\{\ddot{u}\} dV. \hfill (38)$$

In the above equation, \(V\) represents the volume of interest and \(S\) is the boundary of the domain. \(\{F\} , \{\Theta\}, \{P_{conc}\}\) are the body forces, traction forces, and concentrated forces. The sum on the left side represents the virtual work done by these external forces in moving through a virtual displacement \(\delta u\). The product of \(\delta\{\varepsilon\}\), the virtual strain and \(\{\sigma\}\), the stress, denotes the potential energy stored in the volume \(V\), and \(\rho\{\ddot{u}\}\) is the work done by the inertial forces, where \(\rho\) is the density and \(\{\ddot{u}\}\) is the acceleration. It should be mentioned that all boldface variables are vectors.

In FEM the volume \(V\) is discretised into smaller values of \(\Delta V_i\) in such a way that \(V = \sum \Delta V_i\) and \(S = \sum \Delta S_i\). In this case each integral in Eq. (38) may be simplified as follows

$$\int_V \{\varepsilon\} = \sum_i \int_{\Delta V_i} \{\varepsilon\}; \hfill (39)$$

$$\int_S \{\sigma\} = \sum_i \int_{\Delta S_i} \{\sigma\}. \hfill (40)$$

Thus each integrand in Eq. (38) represents the corresponding quantities within the smaller elements \(\Delta V_i\), rather than the entire domain. This permits us to approximate the displacement \(u\) and acceleration \(\ddot{u}\) within the element from their nodal values, using suitable interpolation functions \(N_r\)

$$u = [N]\{d\}; \quad \ddot{u} = [N]\{\ddot{d}\},$$

where \(\{d\}\) is a vector of element nodal displacements and \(N\) is composed of \(N_r\).\(^{2}\) If \(B\) is the strain displacement matrix and \(E\) the matrix of material constants, then

$$\{\varepsilon\} = [B]\{d\}; \quad \{\sigma\} = [E]\{\varepsilon\}.$$

Using these simplifications, the virtual displacements and virtual strains may be related to the virtual displacements of the nodes \(\delta\{d\}\) as follows

$$\delta\{u\} = [N]\delta\{d\}; \quad \delta\{\varepsilon\} = [B]\delta\{d\}. \hfill (41)$$

Thus, the virtual work equation for each element may be written as

$$\int_{\Delta V_i} [B]^T E [B] dV \{d\} + \int_{\Delta V_i} \rho \delta\{d\}^T [N] [N] dV \{\ddot{d}\}$$

$$- \int_{\Delta S_i} \delta\{d\}^T [N]^T \{F\} dS - \int_{\Delta S_i} \delta\{d\}^T [N]^T \{\Theta\} dS$$

$$- \delta\{d\}^T \{P_{conc}\} = 0, \hfill (41)$$

where all the concentrated forces act at nodal points.

In Eq. (41) the virtual displacements are arbitrary; hence it can be rewritten in a matrix form given by

$$[M] \{\ddot{d}\} + [K] \{d\} = \{F^{ext}\}, \hfill (42)$$

where

$$[M] = \int_{\Delta V_i} \rho [N]^T [N] dV; \hfill (43)$$

$$[K] = \int_{\Delta V_i} [B]^T E [B] dV; \hfill (44)$$

and

$$\{F^{ext}\} = \int_{\Delta V_i} \{F\} dV + \int_{\Delta S_i} \{N\}^T \{\Theta\} dS + \{P_{conc}\}. \hfill (45)$$

The above element matrices are appropriately assembled into global matrices by accounting for the nodal connectivities.
We have implemented linear solid tetrahedral elements to arrive at the structural matrices $M$ and $K$. The exact form of these matrices may be found in any standard FEM textbook. As part of validation, the parent and the inclusion material properties were taken to be identical and the code was run. The results are presented in Fig. A1. Figure A1(a) is for a low wavenumber $k = 1 \text{ m}^{-1}$, and Fig. A1(b) is for $k = 20 \text{ m}^{-1}$ which has been used throughout in this study. Comparing Fig. A1(a) with Fig. 9 and examining Fig. A1(b) one sees the regular scattering response of the plate.

**Figure A1.** The results with parent and inclusion materials taken as the same. (a) $k = 1 \text{ m}^{-1}$; (b) $k = 20 \text{ m}^{-1}$.