Theory of Vibration

Dr. Abhijit Sarkar
Room 210, Machine Design Section
Department of Mechanical Engineering
Indian Institute of Technology, Madras.

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Abstract

Lecture notes for classes conducted between $18^{th} - 25^{th}$ October, 2010.

The objectives for these classes is to derive the equations of motion of following systems through a consistent framework: -
(i) Euler-Bernouli beam.
(ii) Longitudnal vibration of a bar.
(iii) Torsional vibration of a circular rod.

1 Longitudinal Vibration

Purely longitudinal vibration occurs when all particles of the body move in only one direction. This is distinct from the case of quasi-longitudinal vibration wherein the particles have motion in all the tree directions (albeit dominantly in one direction). In this section, we shall study the purely longitudinal vibration. The associated results for the quasi-longitudinal motion will be mentioned only in passing.

1.1 Geometry of deformation

Consider a long bar with cross-sectional dimensions small compared to its length (which is along the x-axis). Consider this bar to be embeded in a hole (of exactly the same size and shape as its cross-section) of larger rigid block. We are interested to find the equations of motion of the particles in the bar (not the block).

Due to the above set-up, we have the following deformation pattern

$$u = u(x,t) \neq 0,\ v = w = 0. \tag{1}$$

In the above equation, u, v and w represents the deformation in x, y and z directions, respectively.

In contrast to the perhaps unrealistic example of a bar being embedded in a block, purely longitudinal deformation pattern also exists in any body which is of large extent in all the three directions (*e.g.* earth's crust).

1.2 Strain

Due to the deformations as given by equation (1), we have the only non-zero strain components to be given by

$$\epsilon_{xx} = \frac{\partial u}{\partial x}. \tag{2}$$

All other strain components are zero.

The above strain conditions imply the the body to be in a state of plane strain.

1.3 Stress

The shear strains being zero, the shear stresses are also zero. Despite, the normal strains $\epsilon_{yy} = \epsilon_{zz} = 0$, the normal stresses σ_{yy} and σ_{zz} are non-zero due to Poisson's effect. These stresses are calculated as follows

$$\epsilon_{xx} = \frac{\sigma_{xx} - \nu\left(\sigma_{yy} + \sigma_{zz}\right)}{E}, \tag{3a}$$

$$\epsilon_{yy} = \frac{\sigma_{yy} - \nu\left(\sigma_{xx} + \sigma_{zz}\right)}{E}, \tag{3b}$$

$$\epsilon_{zz} = \frac{\sigma_{zz} - \nu\left(\sigma_{xx} + \sigma_{yy}\right)}{E}. \tag{3c}$$

Putting $\epsilon_{yy} = \epsilon_{zz} = 0$, in equations (3b) and (3c) above we get

$$\begin{aligned} \sigma_{yy} - \nu\sigma_{zz} &= \nu\sigma_{xx} \\ \sigma_{zz} - \nu\sigma_{yy} &= \nu\sigma_{xx}. \end{aligned}$$

The solution of the above equations yield

$$\sigma_{yy} = \sigma_{zz} = \frac{\nu\left(1+\nu\right)}{1-\nu^2}\sigma_{xx} = \frac{\nu}{1-\nu}\sigma_{xx}.$$

Using this relation in equation (3a), we get

$$E\epsilon_{xx} = \sigma_{xx}\left[1 - \frac{2\nu^2}{1-\nu}\right] = \sigma_{xx}\left[\frac{1-\nu-2\nu^2}{1-\nu}\right] = \sigma_{xx}\left[\frac{(1+\nu)(1-2\nu)}{1-\nu}\right] \Rightarrow \frac{\sigma_{xx}}{\epsilon_{xx}} = B = E\frac{1-\nu}{(1+\nu)(1-2\nu)} \tag{4}$$

Note, from the above relation the ratio of normal stress to normal strain is not the Young's modulus of elasticity (E) but $B = E\frac{1-\nu}{(1+\nu)(1-2\nu)} \approx 1.35E$.

Finally, using the strain displacement relation along with the above relation, we get

$$\sigma_{xx} = B\frac{\partial u}{\partial x}.$$

Thus, the normal force acting on any cross-section is given by

$$F = A\sigma_{xx} = BA\frac{\partial u}{\partial x} \tag{5}$$

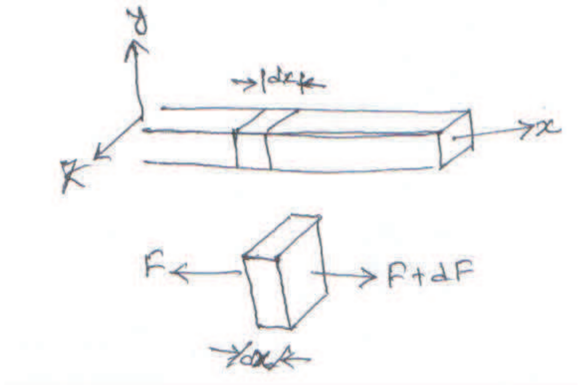

Figure 1: Forces acting on a differential element of the bar

1.4 Linear momentum balance of a differential element

Consider a small length dx of the bar as shown in figure (1). The linear momentum balance of this segment yields

$$dF = \rho A dx \frac{\partial^2 u}{\partial t^2} \Rightarrow \frac{dF}{dx} = \rho A \frac{\partial^2 u}{\partial t^2}.$$

In the above equation, ρ is the density of the material. Using equation (5), we get

$$\rho A \frac{\partial^2 u}{\partial t^2} = \frac{dF}{dx} = BA \frac{\partial^2 u}{\partial x^2}.$$

Thus, the governing equations of motion for longitudinal vibration of a bar is given by

$$\frac{\partial^2 u}{\partial x^2} = \frac{\rho}{B} \frac{\partial^2 u}{\partial t^2}. \tag{6}$$

1.5 Quasi-longitudinal vibration

When a bar is subjected to axial loading only then it undergoes quasi-longitudinal vibration. In this case the lateral surfaces of the bar are traction free and hence the stresses $\sigma_{yy} = \sigma_{zz} = 0$. This implies that the ratio of axial stress to axial strain is $\frac{\sigma_{xx}}{\epsilon_{xx}} = E$ (unlike B in the pure longitudinal case). The remaining part of the derivation proceeds in identically the same fashion giving the governing equation for quasi-longitudinal vibration as

$$\frac{\partial^2 u}{\partial x^2} = \frac{\rho}{E} \frac{\partial^2 u}{\partial t^2}. \tag{7}$$

However, note the following important points of differences between longitudinal and quasi-longitudinal vibration: -

- In longitudinal vibration, the deformation in the transverse directions are zero. In quasi-longitudinal vibration the deformations in the transverse directions, though smaller than the deformation in the axial direction, are non-zero. The geometry of deformation is considerably complicated and is illustrated in figure (2).
- Longitudinal vibration corresponds to plane strain whereas quasi-longitudinal vibration corresponds to the case of plane stress.
- The effect of E (Young's modulus of elasticity) in the equations for quasi-longitudinal vibration is replaced by B for the case of purely longitudinal vibration.

For further reading on this topic the reader is refered to [2, 3].

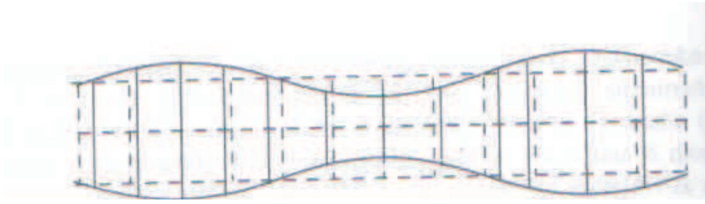

Figure 2: Schematic illustration of quasi-longitudinal deformation. Dotted lines represent the initial configuration of the bar while the solid lines represent the deformed configuration. Figure taken from [2].

2 Torsional vibration of circular rod

2.1 Geometry of Deformation

A long circular rod has its axis along the z-direction. On application of a moment along the axis of such a rod every cross-section of the rod rotates in the xy (or $r\theta$) plane. Thus any radial line on the cross-section is deformed as illustrated in figure (3) (the solid blue line represents the undeformed configuration and the dotted blue line is the same line after deformation).

The rotations for different cross-sections are however different. For example, the cross-section which is fixed does not rotate at all, whereas the cross-section wherein the torque is applied has a maximum rotation. The intermediate cross-section have a linear variation of the rotations between these two extremes. In other words, if a straight line parallel to the axis is drawn on the cylindrical surface of the rod (shown by solid red line in figure (3)), the line remains straight but rotates along with each point of the cross-section. The deformed configuration of this line is shown by dotted green lines.

2.2 Deformation

The problem will be solved in cylindrical coordinate system, with z being along the axis of the rod and the xy plane will be same as the $r\theta$ plane.

Let u_r and u_z be the deformations in the r and z direction. They need to be zero because each cross-section simply rotates about the z axis. Thus, all the points move only along the circumferential direction (θ) resulting in $u_r = u_z = 0$.

The angular rotation or twist of each cross section is defined as the angle made between the deformed and the undeformed configuration of a radial line (see the blue lines in figure(3)). This is denoted by ϕ. As is illustrated in the figure ϕ varies linearly along z, being zero at the face which is fixed and being maximum at the face where the torque is applied. Thus, we have $\phi \propto z$ or $\phi = \alpha z$. The latter relation may be interpreted as follows:- if the distance between any two cross sections is z then the relative rotation between these two cross sections is $\phi = \alpha z$. α is called the twist per unit length. Using the above relation, we find that

$$\alpha = \frac{\partial \phi}{\partial z} \tag{8}$$

Finally, the deformation in the circumferential direction is given by $u_\theta = r\phi = \alpha r z$. Note, u_θ has dimensions of length and not angle. Summing up we have the following relations for the deformation field

$$u_r = u_z = 0, \; u_\theta = \alpha r z. \tag{9}$$

2.3 Strain

Using the deformations in equation (9), we can find the strain field. For this we need to use the strain-displacement relation in cylindrical coordinate system. These relations are elaborately derived in Appendix

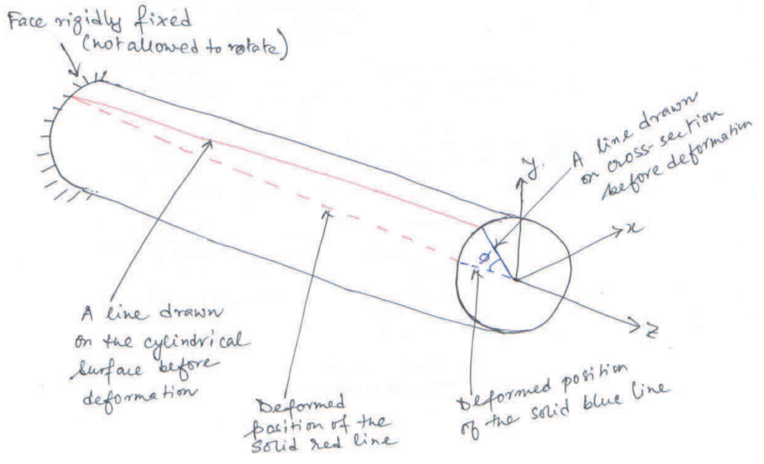

Figure 3: Illustration of the geometry of deformation for torsion

B of reference [1] (students are welcome to approach me in case they find any difficulty in understanding this derivation). The components of the strain tensor ϵ are given as follows (after using equation (9))

$$\begin{aligned}
\epsilon_{rr} = \frac{\partial u_r}{\partial r} &= 0, \\
\epsilon_{zz} = \frac{\partial u_z}{\partial z} &= 0, \\
\epsilon_{\theta\theta} = \frac{1}{R}\frac{\partial u_\theta}{\partial \theta} &= 0, \\
\epsilon_{rz} = \frac{1}{2}\left[\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right] &= 0, \\
\epsilon_{z\theta} = \frac{1}{2}\left[\frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}\right] &= \frac{\alpha r}{2}, \\
\epsilon_{r\theta} = \frac{1}{2}\left[\frac{1}{r}\frac{\partial u_r}{\partial \theta} + r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)\right] &= 0.
\end{aligned}$$

2.4 Stress

Using the strains calculated above, the stresses may be evaluated. As all the normal strain components are zero, all the normal stress components are also zero *viz.* $\sigma_{rr} = \sigma_{\theta\theta} = \sigma_{zz} = 0$. Similarly, the shear stress components $\sigma_{rz} = \sigma_{r\theta} = 0$. The only non-zero stress component is $\sigma_{\theta z} = G\alpha r$. Note, $\epsilon_{\theta z}$ is the engineering strain. The true shear strain is double of the above engineering strain.

2.5 Torque calculation

The non-zero stress $\sigma_{z\theta}$ has the following physical implication : - the plane of cross-section (z plane) has a force acting in the θ direction over each differential element dA over the cross-section. The differential torque about the axis due to this shear stress acting over the differential area dA located at a radial distance r from the axis is

$$dT = r\sigma_{z\theta}dA = G\alpha r^2 dA.$$

Thus, the total torque may be obtained by integrating the above relation over the entire area of the cross section which yields

$$T = \int_A G\alpha r^2 dA = G\alpha \int_A r^2 dA = G\alpha J, \tag{10}$$

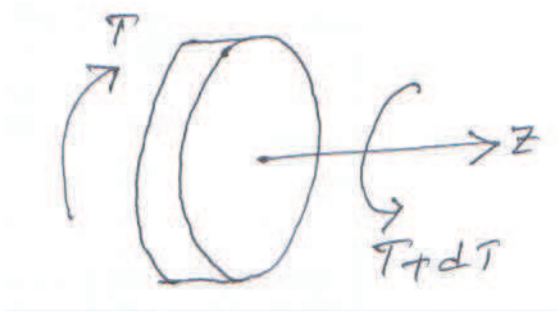

Figure 4: Angular Momentum balance of a differential element of the circular rod.

where J is the polar area moment of inertia.

2.6 Angular momentum balance of a disc of differential length

Consider a small dz portion of the rod as shown in figure (4). This cross-section undergoes a rotational motion. The rotation being a function of position and time is denoted by $\phi(z,t)$. Considering the angular momentum balance of this differential element (in the form of a thin disc) we get

$$dT = I_p \frac{\partial^2 \phi}{\partial t^2},$$

where I_p is the mass moment of inertia of the differential rod. This is given by

$$I_p = \int_M r^2 dm = \int_V r^2 \rho dV = \rho dz \int_A r^2 dA = \rho J dz.$$

ρ is the density of the material.

This gives

$$\frac{dT}{dz} = \rho J \frac{\partial^2 \phi}{\partial t^2}.$$

Using equation (8), we have $\alpha = \frac{\partial \phi}{\partial z}$ and thus

$$\begin{aligned} T = GJ\alpha &= GJ\frac{\partial \phi}{\partial z} \\ \Rightarrow \frac{dT}{dz} &= GJ\frac{\partial^2 \phi}{\partial z^2} \end{aligned}$$

Replacing the above relation in the earlier relation for $\frac{dT}{dz}$, we get the governing equation of motion for torsional vibration of circular rod as

$$\begin{aligned} GJ\frac{\partial^2 \phi}{\partial z^2} &= \rho J \frac{\partial^2 \phi}{\partial t^2}, \\ \Rightarrow \frac{\partial^2 \phi}{\partial z^2} &= \frac{\rho}{G}\frac{\partial^2 \phi}{\partial t^2}. \end{aligned}$$

References

[1] C. S. Jog, *Foundations and Applications of Mechanics, Volume- 1 Continuum Mechanics*, Narosa Publishing House, Second Edition, 2007.

[2] F. J. Fahy and P. Gardonio, *Sound and Structural Vibration*, Academic Press, 2007.

[3] L. Cremmer, M. Heckl, B. A. T. Peterson, *Structure-borne sound*, Springer-Verlag, 2005.