Name: ______________________ Group Members: ____

Exploration 7-2a: Differential Equation for Compound Interest

Date: ____________

Objective: Write and solve a differential equation for the amount of money in a savings account as a function of time.

When money is left in a savings account, it earns interest equal to a certain percent of what is there. The more money you have there, the faster it grows. If the interest is *compounded continuously,* the interest is added to the account the instant it is earned.

1. For continuously compounded interest, the instantaneous rate of change of money is directly proportional to the amount of money. Define variables for time and money, and write a **differential equation** expressing this fact.

2. **Separate the variables** in the differential equation in Problem 1, then integrate both sides with respect to t. Transform the integrated equation so that the amount of money is expressed explicitly in terms of time.

3. The integrated equation from Problem 2 will contain e raised to a power containing *two* terms. Write this power as a product of two different powers of e, one that contains the time variable and one that contains no variable.

4. You should have the expression e^C in your answer to Problem 3. Explain why e^C is always positive.

5. Replace e^C with a new constant, C_1. If C_1 is allowed to be positive or negative, explain why you no longer need the ± sign that appeared when you removed the absolute value in Problem 2.

6. Suppose that the amount of money is \$1000 when time equals zero. Use this **initial condition** to evaluate C_1.

7. If the interest rate is 5% per year, then d(money)/d(time) = 0.05(money) in dollars per year. What, then, does the proportionality constant in Problem 1 equal?

8. How much money will be in the account after 1 year? 5 years? 10 years? 50 years? 100 years? Do the computations in the most time-efficient manner.

9. How long would it take for the amount of money to double its initial value?

10. What did you learn as a result of doing this Exploration that you did not know before?

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Name: ______________ Group Members: ______________________________

Exploration 7-3a: Differential Equation for Memory Retention

Date: __________

Objective: Write and solve a differential equation for the number of names remembered as a function of time.

Ira Member is a freshman at a large university. One evening he attends a reception at which there are many members of his class whom he has not met. He wants to predict how many new names he will remember at the end of the reception.

1. Ira assumes that he meets people at a constant rate of R people per hour. Unfortunately, he forgets names at a rate proportional to y, the number he remembers. The more he remembers, the faster he forgets! Let t be the number of hours he has been at the reception. What does dy/dt equal? (Use the letter k for the proportionality constant.)

2. The equation in Problem 1 is a **differential equation** because it has differentials in it. By algebra, separate the variables so that all terms containing y appear on one side of the equation and all terms containing t appear on the other side.

3. Integrate both sides of the equation in Problem 2. You should be able to make the integral of the reciprocal function appear on the side containing y.

4. Show that the solution in Problem 3 can be transformed into the form

   $$ky = R - Ce^{-kt}$$

   where C is a constant related to the constant of integration. Explain what happens to the absolute value sign that you got from integrating the reciprocal function.

5. Use the initial condition $y = 0$ when $t = 0$ to evaluate the constant C.

6. Suppose that Ira meets 100 people per hour, and that he forgets at a rate of 4 names per hour when $y = 10$ names. Write the particular equation expressing y in terms of t.

7. How many names will Ira have remembered at the end of the reception, $t = 3$ h?

8. What did you learn as a result of doing this Exploration that you did not know before?

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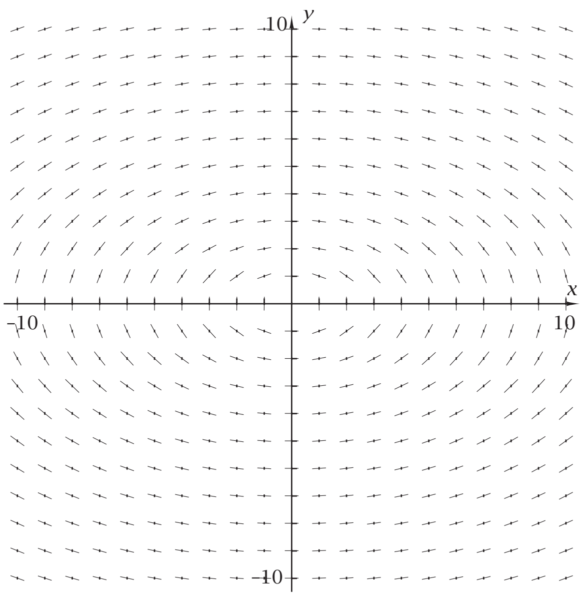
Name: ____________________ Group Members: ____

Exploration 7-4a: Introduction to Slope Fields

Date: ____________

Objective: Find graphically a particular solution of a given differential equation, and confirm it algebraically.



The figure above shows the **slope field** for the differential equation

$$\frac{dy}{dx} = -\frac{0.36x}{y}$$

1. From the differential equation, find the slope at the points (−5, −2) and (−8, 9). Mark these points on the figure. Tell why the slopes are reasonable.

2. Start at the point (0, 6). Draw a graph representing the particular solution of the differential equation which contains that point. The graph should be "parallel" to the slope lines and be some sort of average of the slopes if it goes between lines. Go both to the right and to the left. Where does the graph seem to go after it touches the *x*-axis? What geometric figure does the graph seem to be? Why should you not continue below the *x*-axis?

3. Start at the point (−5, −2) from Problem 1 and draw another particular solution of the differential equation. How is this solution related to the one in Problem 2?

4. Solve the differential equation algebraically. Find the particular solution that contains (0, 6). Verify that the graph really *is* the figure you named in Problem 2.

5. What did you learn as a result of doing this Exploration that you did not know before?

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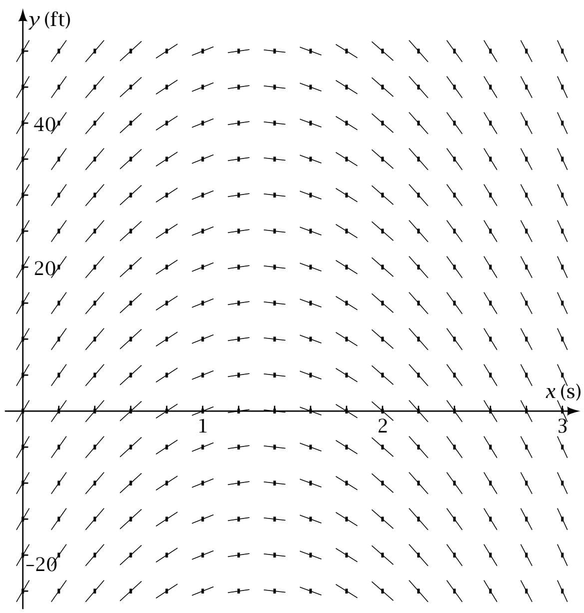
Name: ______________________ Group Members: ____

Exploration 7-4b: Slope Field Practice

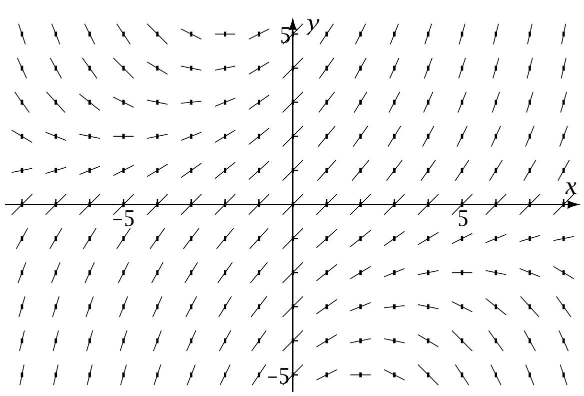
Date: ____________

Objective: Solve a differential equation graphically, using its slope field, and make interpretations about various particular solutions.

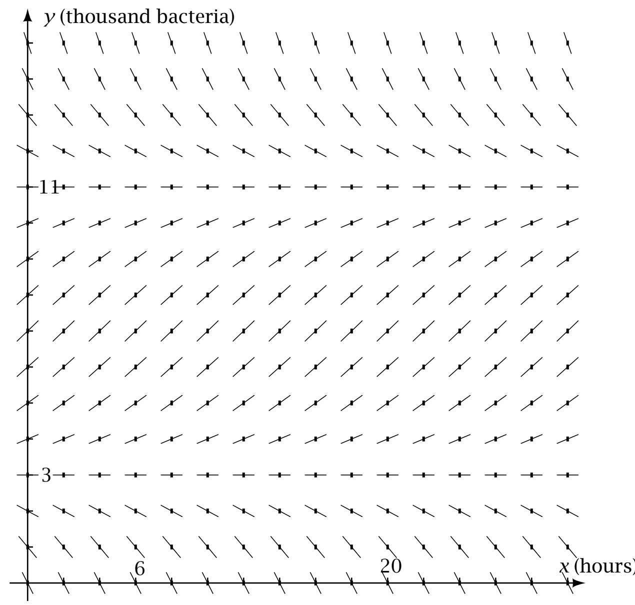
1. The figure shows the slope field for the vertical displacement, y, in feet, as a function of x, in seconds, since a ball was thrown upward with a particular initial velocity. Sketch y as a function of x if the ball starts at $y = 5$ ft when $x = 0$. Approximately when will the ball be at its highest? Approximately when will it hit the ground? If the ball is thrown starting at $y = -20$ when $x = 0$, at approximately what two times will it be at $y = 0$?



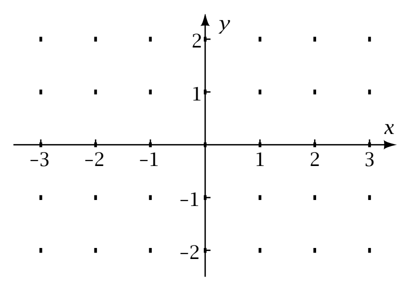
2. Sketch the particular solutions for these initial conditions: $(-7, 2)$, $(-5, -1)$, and $(8, -4)$. Describe the difference in the patterns.



3. The figure shows the slope field for bacteria count, y, in thousands, as a function of time, x, in hours. At $x = 0$, $y = 15$. At $x = 6$, a treatment reduces y to 4. At $x = 20$, another treatment reduces y to 2. Sketch the three branches of the particular solution. Tell what eventually happens to the number of bacteria and what would have happened without the treatments.



4. Plot the slope field for $\frac{dy}{dx} = \frac{x}{y}$ at the grid points. On the slope field, plot the particular solutions for the initial conditions $(2, 1)$, $(0, -1)$, and $(-2, -1)$.



5. What did you learn as a result of doing this Exploration that you did not know before?

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Name: ______________ Group Members: ______________________________

Exploration 7-5a: Introduction to Euler's Method

Date: __________

Objective: Given a differential equation, find an approximation to a particular solution by a numerical method.

For the differential equation

$$\frac{dy}{dx} = 0.3y$$

1. Write an equation for the differential dy.

2. Find dy at the point (1, 2) if $dx = 0.5$.

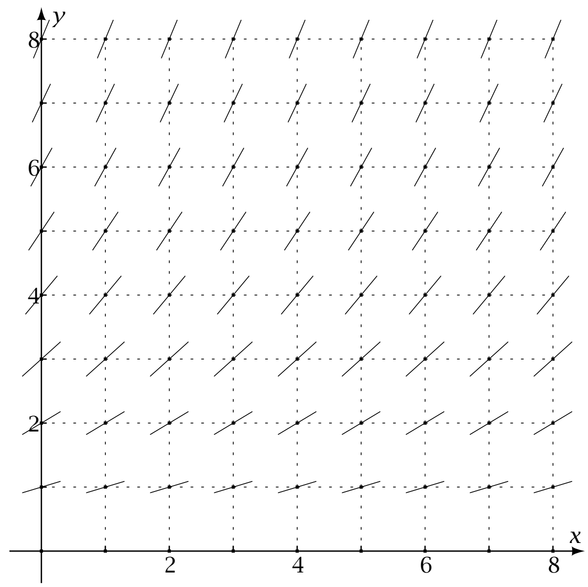
3. Add this value of dy to 2 to estimate the value of y at $x = 1.5$. Plot (1, 2) and (1.5, y) on the figure in Problem 6. Show the values of dx and dy.

4. Estimate the value of y at $x = 2$ by finding dy at the point (1.5, y) from Problem 3 and adding it to that (approximate) y-value.

5. Repeat the calculations in Problem 4 for each 0.5 unit of x up to $x = 6$. Record the y-values in this table. This technique is called **Euler's method** for solving differential equations numerically.

x	y	dy
1	2	0.3
1.5	2.3	0.345
2	2.645	
2.5		
3		
3.5		
4		
4.5		
5		
5.5		
6		

6. The figure shows the slope field for this differential equation. Plot the y-values from Problem 5 on the slope field. Connect the points with line segments. Show dx and dy for $x = 5.5$ to $x = 6$.



7. Solve the differential equation in Problem 1 algebraically. Find the equation of the particular solution that contains (1, 2). If $x = 5$, how well does the value of y by Euler's method agree with the actual value?

8. What did you learn as a result of doing this Exploration that you did not know before?

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Name: ______________ Group Members: ______________

Exploration 7-6a: Logistic Function— The Roadrunner Problem

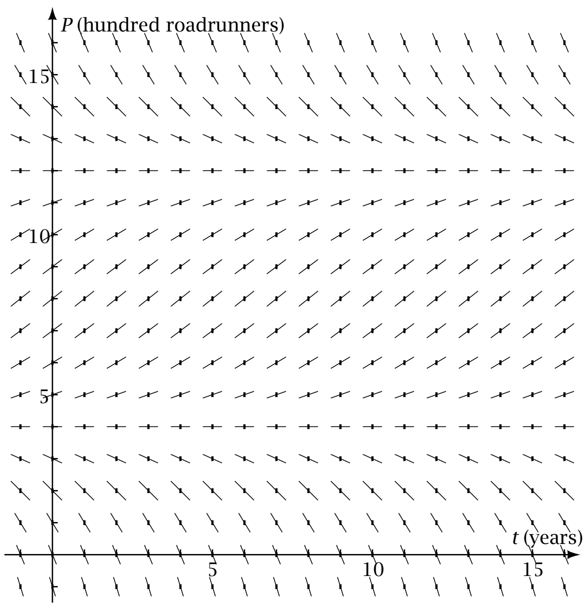
Date: __________

Objective: Analyze a logistic population growth problem graphically, numerically, and algebraically.

The figure shows the slope field for the differential equation

$$\frac{dP}{dt} = 0.05(P - 4)(12 - P)$$

where P is the population of roadrunners (in hundreds) in a particular region at time t, in years.



1. Show algebraically that P is
   - Increasing, if $P = 5$.
   - Decreasing, if $P = 3$ or $P = 15$.
   - Not changing ("stable") if $P = 4$ or $P = 12$.

   Does the slope field agree with the calculations?

2. Assume that $P = 5$ when $t = 0$. Using $dt = 0.1$, show the steps in the Euler's method estimation of P when $t = 0.1$ and $t = 0.2$.

3. Use Euler's method with $dt = 0.1$ to estimate P for these values of t, using the initial condition (0, 5). Round to one decimal place.

t	P
2	
4	
6	
8	
10	
12	
14	
16	

4. Plot the values from Problem 3 on the given figure and connect with a smooth curve. Does the curve follow the slope lines?

(Over)

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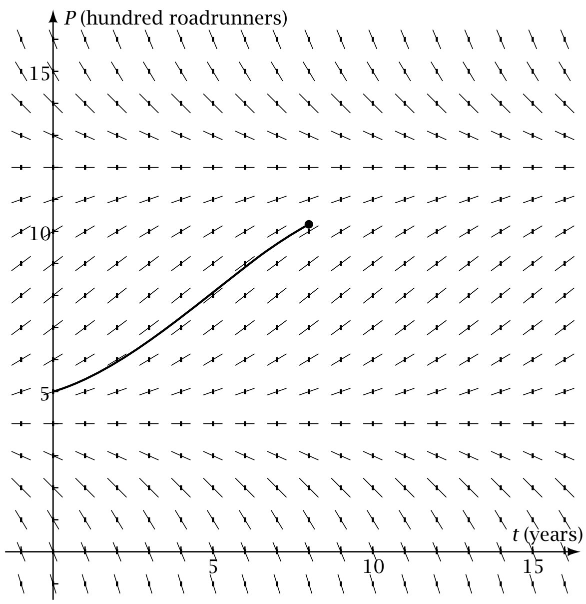
5. This copy of the given slope field shows the particular solution in Problem 4 up to time $t = 8$. Suppose that at this time the Game and Wildlife Commission brings 600 more roadrunners into the region to help increase the population. Use $P(8)$ from Problem 3 to get a new initial condition. Sketch the graph on this slope field. What actually happens to the roadrunner population?



6. Suppose that P had been 3.99 (399 roadrunners) at $t = 0$. Use Euler's method with $dt = 1$ to estimate the year in which the roadrunners become extinct.

7. Estimate the extinction year again, using $dt = 0.1$. Sketch this solution on the slope field. Based on what you know about Euler's method, why is the predicted extinction earlier than in Problem 6?

8. Suppose that 300 roadrunners had been introduced at $t = 8$, using the initial condition in Problems 6 and 7. Sketch and describe the particular solution.

9. Separating the variables and integrating gives

$$\frac{1}{8}\ln\left|\frac{P-4}{12-P}\right| = 0.05t + C$$

as you will learn when you study partial fractions. Use the initial condition (0, 5), as in Problem 2, to get the algebraic solution for P as a function of t. How close does the Euler's solution at $t = 8$ in Problem 3 come to the algebraic solution for $P(8)$?

10. What did you learn as a result of doing this Exploration that you did not know before?

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Name: ______________ Group Members: ______________

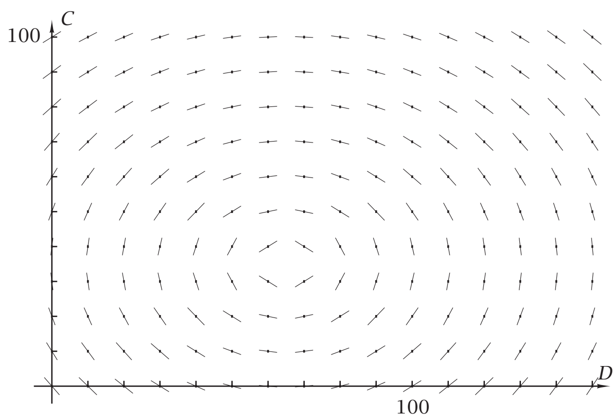
Exploration 7-6b: A Predator-Prey Problem

Date: ________

Objective: Analyze the slope field for a differential equation modeling the populations of predatory coyotes and their deer prey.

The figure shows the slope field for a differential equation that represents the relative populations of deer, *D*, and coyotes, *C*, that prey on the deer. The slope at any point represents

$$\frac{dC}{dD} = \frac{dC/dt}{dD/dt}$$



1. At the point (80, 20), there are relatively few coyotes. Would you expect the deer population to be increasing or decreasing at this point? Which way, then, would the graph of the solution start out under this condition?

2. Sketch the graph of the solution to the differential equation subject to the initial condition (80, 20). Describe what you expect to happen to the two populations under this condition.

3. Suppose that the initial number of deer is increased to 120, with the same 20 coyotes. Sketch the graph. What unfortunate result do you predict to happen under this condition?

4. Suppose that the initial deer population had been 140, with the same 20 coyotes. Sketch the graph. What other unfortunate thing would you expect to happen under this condition?

5. At the point (100, 35), what seems to be happening to the two populations? What seems to be happening at the point (65, 80)?

6. Does there seem to be an "equilibrium" condition for which neither population changes? Explain.

7. What did you learn as a result of doing this Exploration that you did not know before?

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Name: ____________ Group Members: ____________

Exploration 7-6c: U.S. Population Project

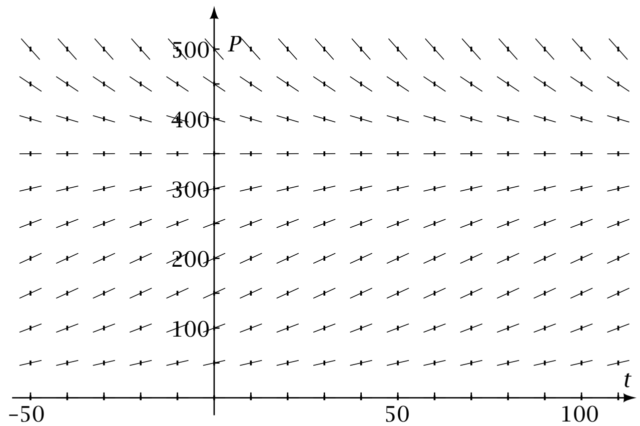
Date: ________

Objective: Use actual census data to make a mathematical model of U.S. population as a function of time.

Year	t	Millions
1940	0	131.7
1950	10	151.4
1960	20	179.3
1970	30	203.2
1980	40	226.5
1990	50	248.7

1. The table shows the U.S. population in millions. For 1950, 1960, 1970, and 1980, find symmetric difference quotients, $\Delta P/\Delta t$, and record them in the table. Then calculate $(\Delta P/\Delta t)/P$, the fractional increase in population, for these years. Store P in L1 and $(\Delta P/\Delta t)/P$ in L2.

2. By regression, find the best-fitting linear function for $(\Delta P/\Delta t)/P$ as a function of P. Assuming that the instantaneous rate of change of population, $(dP/dt)/P$, follows the same linear function, write a differential equation for dP/dt. What special name is given to a differential equation of this form?



3. The figure shows the slope field for the differential equation you should have gotten in Problem 2. Accounting for the unequal scales on the two axes, are the slope lines for $P = 200$ reasonable? Plot this slope field on your grapher. Does it agree?

4. Use Euler's method with $dt = 1$ to calculate the population for each 10 years from $t = -50$ through $t = 100$. Use 1940 as the initial condition. Plot the points on the graph and connect them with a smooth curve.

t	P	t	P
−50		30	
−40		40	
−30		50	
−20		60	
−10		70	
0	131.7*	80	
10		90	
20		100	

5. How well do the Euler's method numbers agree with the given data points?

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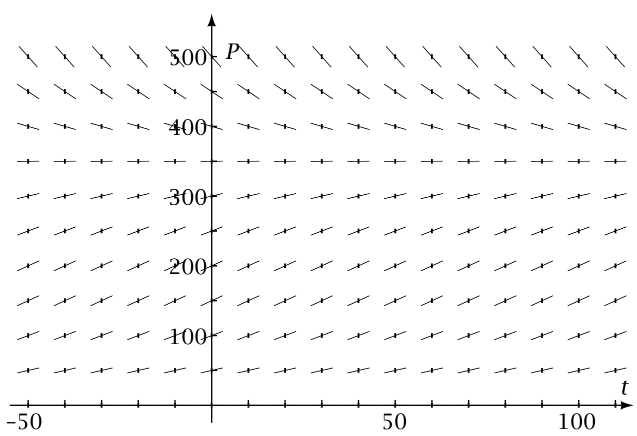
Exploration 7-6c: U.S. Population Project *continued*

Date: __________

6. According to this mathematical model, what is the maximum sustainable population of the United States? Show how you can calculate this from the differential equation.

7. Suppose that in the year 2010, immigration brings 200 million more people to the United States. Use Euler's method to predict the populations each 10 years for the next 40 years, using $dt = 1$ y. What seems to be happening?

8. Plot the results of Problem 7 on this copy of the slope field and connect the points with a smooth curve. Does the curve follow the slope lines?



9. On the Internet, find U.S. Census data for 1900 through 1930. Tell the URL of the Web site you used. How well do the predicted figures agree with the actual census? Think of some reasons for any discrepancies.

10. What did you learn as a result of doing this Exploration that you did not know before?

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