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SYST 302: Systems Methodology and Design II #9

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Outline

- Concepts and Introduction
- Markov Chain
- State Transition
- Markov Chain and Reliability

Markov Process and Markov Chain

Markov Processes: **if the future of the random process given the present is independent of the past, i.e.,**

with $t_1 < t_2 < \ldots < t_k < t_{k+1}$

Discrete state: $P(a \le x_{k+1} \le b | x_k, x_{k-1}, \ldots, x_1) = P(a \le x_{k+1} \le b | x_k)$

Continuous state: $f_{x_{k+1}}(x_{k+1} | x_k, x_{k-1}, \ldots, x_1) = f_{x_{k+1}}(x_{k+1} | x_k)$

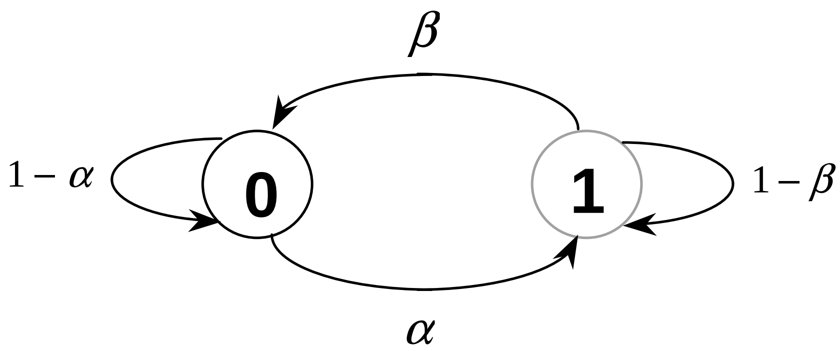
This is known as conditional independence; the values of x_{k+1} is conditionally independent of $x_{k-1}, \ldots, x_1$ given x_k

Markov Chain: **a discrete state Markov process, for time-invariant Markov chains, we have**

Transition Probabilities : $P_{ij} = P(x_{k+1} = j | x_k = i)$ for $1 \le i, j \le N,\ k = 0, 1, 2, \ldots$

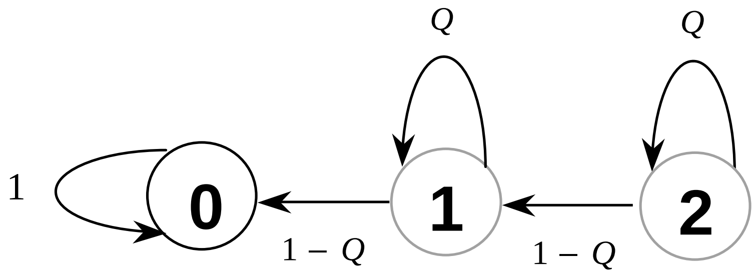
State Transition Diagram

Two-state Markov chain



$$P=\begin{bmatrix}1-\alpha & \alpha \\ \beta & 1-\beta\end{bmatrix}$$

Three-state Markov chain



$$P=\begin{bmatrix}1 & 0 & 0 \\ 1-Q & Q & 0 \\ 0 & 1-Q & Q\end{bmatrix}$$

Transition Probability Matrix

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1N} \\ p_{21} & p_{22} & \cdots & p_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ p_{N1} & p_{N2} & \cdots & p_{NN} \end{bmatrix}, \text{ where } \sum_{j=1}^{N} p_{ij} = 1$$

Note that probability theory requires that the rows of P sum to 1.0; this is called a *stochastic matrix*.

$\Rightarrow$ (1) $\lambda(P)$ are all real, (2) $\lambda_{\max}(P) = 1$

The n-step Transition Probability

2-step transition probability

$$P_{ij}(2) \equiv P(x_{k+2} = j | x_k = i) = \sum_{m=1}^{N} P(x_{k+2} = j, x_{k+1} = m | x_k = i)$$

$$= \sum_{m=1}^{N} P(x_{k+2} = j | x_{k+1} = m, x_k = i) P(x_{k+1} = m | x_k = i)$$

$$= \sum_{m=1}^{N} P(x_{k+2} = j | x_{k+1} = m) P(x_{k+1} = m | x_k = i) \equiv \sum_{m=1}^{N} P_{jm}(1) P_{mi}(1) \Rightarrow P(2) = P(1)P(1) = P^2$$

For n-step transition, icbest

$$P(n) = P^n \Rightarrow \pi(n) = \pi(n-1)P = \pi(n-2)P \cdot P = \ldots = \pi(0)P^n,$$

where $\pi(0) = [\pi_{01}, \pi_{02}, \ldots, \pi_{0N}]$ is the initial state probability (pmf)

Classes of States

We say that **state *j* is accessible from state *i*** if for some $n \geq 0$, $P_{ij}(n) > 0$, that is, if there is a sequence of transitions from ***i*** to ***j*** that has nonzero probability. We say that **state *i* and *j* communicate** if they are accessible to each other.

We say that two states belong to the same **class** if they communicate.

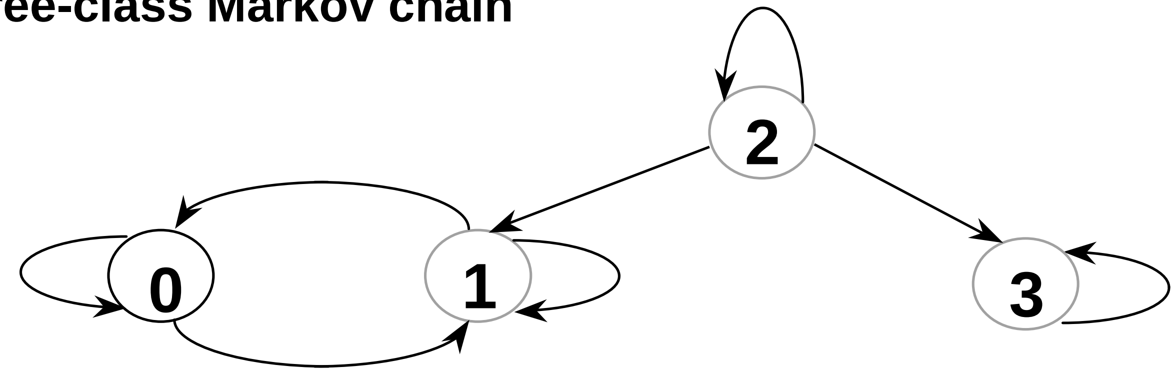
A Markov chain that consists of a single class is called **irreducible.**

A state ***i*** is said to be **recurrent** if the process returns to the state with probability 1, otherwise, it is said to be **transient.**

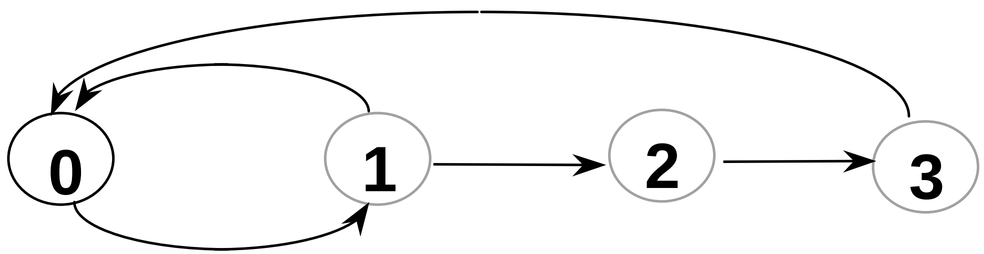
A state ***i*** is said to have **period** d if it can only recur at times that are multiples of d, if $d = 1$, it is called **aperiodic.**

Some Markov Chains

A three-class Markov chain



Periodic Markov chain



Limiting Probabilities

Limiting Theorem: for an irreducible, aperiodic, recurrent Markov chain, the state probability will reach a steady (equilibrium) state pmf π_S obtained as:

$$\lim_{n\to\infty} \pi(n) = \pi(n-1)P_{ij} \Rightarrow \pi_S = \pi_S P$$

$$\Rightarrow P' \pi_S' = \pi_S' \Rightarrow (P' - I)\pi_S' = 0$$

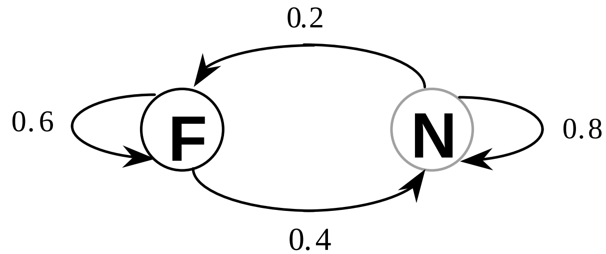
which indicates that $\lambda = 1$ is one of the eigenvalues of P, and the corresponding *normalized* eigenvector is the steady state of the Markov chain

Note that the steady state does not depend on the initial state

Steady State Example

Power station load: **on any given day, the station could be on full load or not, denoted as F and N respectively, and the load varies such that it can be represented as a Markov process with the transition matrix**

$$P = \begin{array}{c|cc} & F & N \\ \hline F & & \\ N & & \end{array} \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix}$$



$$\pi(n) = \pi(n-1)P = \pi(n-2)P \cdot P = \cdots = \pi(0)P^n$$

$$\text{but } P = \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix},\ P^2 = \begin{bmatrix} 0.44 & 0.56 \\ 0.28 & 0.72 \end{bmatrix},\ P^3 = \begin{bmatrix} 0.376 & 0.624 \\ 0.312 & 0.688 \end{bmatrix},\ P^4 = \begin{bmatrix} 0.3504 & 0.6496 \\ 0.3248 & 0.6752 \end{bmatrix}$$

$$P^5 = \begin{bmatrix} 0.3402 & 0.6598 \\ 0.3299 & 0.6701 \end{bmatrix},\ P^6 = \begin{bmatrix} 0.3361 & 0.6639 \\ 0.3320 & 0.6680 \end{bmatrix}, \ldots, P^n = \begin{bmatrix} 0.3333 & 0.6667 \\ 0.3333 & 0.6667 \end{bmatrix},\ \text{for } n > 11$$

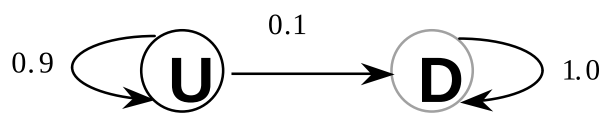
$$\Rightarrow \pi(n) = \pi(0)P^n = [\tfrac{1}{3}, \tfrac{2}{3}]$$

$$\text{In fact, } (P' - I)\mathbf{x} = \begin{bmatrix} -0.4 & 0.2 \\ 0.4 & -0.2 \end{bmatrix}\mathbf{x} = \mathbf{0} \Rightarrow \mathbf{x} = [\tfrac{1}{3}, \tfrac{2}{3}]' \ \text{(eigenvector for } \lambda = 1\text{)}$$

Transient Behavior

System failure without repair: **on any given day, the system could be either up or down, denoted as U and D respectively, and assuming it behaves according to the Markov process with the transition matrix:**

$$P = \begin{array}{c|c} & \begin{array}{cc} U & D \end{array} \\ \hline \begin{array}{c} U \\ D \end{array} & \begin{bmatrix} 0.9 & 0.1 \\ 0.0 & 1.0 \end{bmatrix} \end{array}$$



$$\pi(n) = \pi(n-1)P = \pi(n-2)P \cdot P = \cdots = \pi(0)P^n$$

$$\text{but } P = \begin{bmatrix} 0.9 & 0.1 \\ 0.0 & 1.0 \end{bmatrix},\ P^2 = \begin{bmatrix} 0.81 & 0.19 \\ 0.0 & 1.0 \end{bmatrix}, \ldots, P^n = \begin{bmatrix} (0.9)^n & 1-(0.9)^n \\ 0.0 & 1.0 \end{bmatrix}$$

$$\Rightarrow \pi(n) = \pi(0)P^n = [\pi_{01}(0.9)^n, \pi_{01}(1-(0.9)^n) + \pi_{02}] \ \Rightarrow \lim_{n\to\infty} \pi(n) = [0,1]$$

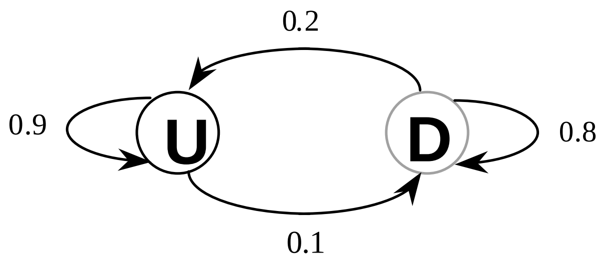
$$\text{In fact, } (P' - I)\mathbf{x} = \begin{bmatrix} \text{-}0.1 & 0.0 \\ 0.1 & 0.0 \end{bmatrix}\mathbf{x} = \mathbf{0} \Rightarrow \mathbf{x} = [0,1]'$$

In general, the steady state may not be unique, i.e., may depend on the initial state

Markov Chain for Reliability Analysis

Markov Chain can be considered as the system availability (steady state probability)

$$P = \begin{array}{c|c} & \begin{array}{cc} U & D \end{array} \\ \hline \begin{array}{c} U \\ D \end{array} & \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} \end{array}$$



$$\pi(n) = \pi(n-1)P = \pi(n-2)P \cdot P = \cdots = \pi(0)P^n$$

$$\Rightarrow \pi(n) = \pi(0)P^n = [0.667, 0.333]$$

$$R = \pi_F(n) = 0.667$$

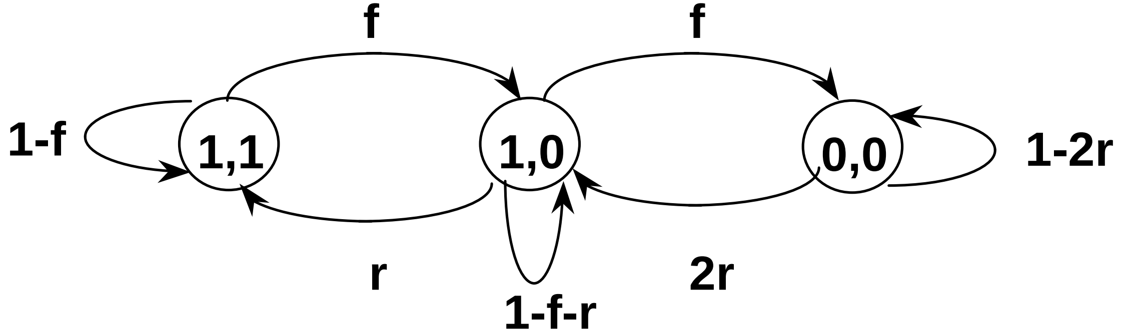
Reliability Analysis Example

State Space: (system with standby component)

(1, 1) - one component working, one in standby
(1, 0) - one component working, one down and being repaired
(0, 0) - both components down and being repaired.

Transition Probabilities:

f: the probability of either of these components will fail in a unit time period while operational
r: the probability that a component that has failed will be repaired in a unit time period, when both components fail, double the crew and consequently double the repair probability to 2r



System Availability

Transition Matrix: $$\mathbf{P} = \begin{bmatrix} 1-f & f & 0 \\ r & 1-f-r & f \\ 0 & 2r & 1-2r \end{bmatrix}$$

If our uncertainty about the state of the system reaches an equilibrium, then

$$\boldsymbol{\pi}_{eq} = [\pi_{11} \quad \pi_{10} \quad \pi_{00}] = [\pi_{11} \quad \pi_{10} \quad \pi_{00}] \begin{bmatrix} 1-f & f & 0 \\ r & 1-f-r & f \\ 0 & 2r & 1-2r \end{bmatrix}$$

$$\pi_{11} = (1-f)\pi_{11} + r\pi_{10} \quad \Rightarrow \quad \pi_{11} = \pi_{10} r / f$$

$$\pi_{10} = f\pi_{11} + (1-f-r)\pi_{10} + 2r\pi_{00}$$

$$\pi_{00} = f\pi_{10} + (1-2r)\pi_{00} \quad \Rightarrow \quad \pi_{00} = \pi_{10} f / 2r$$

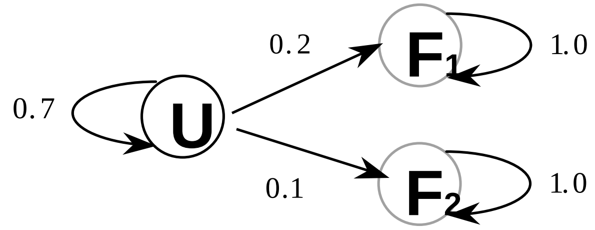
$$\pi_{11} = 2r^2 / (f^2 + 2rf + 2r^2),\ \pi_{10} = 2rf / (f^2 + 2rf + 2r^2), \pi_{00} = f^2 / (f^2 + 2rf + 2r^2)$$

$$\textbf{System availability} = \pi_{11} + \pi_{10} = (2r^2 + 2rf) / (f^2 + 2rf + 2r^2)$$

Limiting Behavior

Consider a system with two failure modes and no repair capability as shown in the figure below with the transition probability matrix:

$$P = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix}$$



$$\pi(n) = \pi(0)P^n$$

$$\text{but } P = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix},\ P^2 = \begin{bmatrix} 0.49 & 0.34 & 0.17 \\ 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix}, \ldots, P^{25} = \begin{bmatrix} 0.0 & 0.667 & 0.333 \\ 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix}$$

$$\Rightarrow \lim_{n\to\infty} \pi(n) = \pi(0)P^n = [0, \tfrac{2}{3}\pi_{01} + \pi_{02}, \tfrac{1}{3}\pi_{01} + \pi_{03}]$$

$$\text{In fact, } (P' - I)\mathbf{x} = \begin{bmatrix} -0.3 & 0.0 & 0.0 \\ 0.2 & 0.0 & 0.0 \\ 0.1 & 0.0 & 0.0 \end{bmatrix}\mathbf{x} = \mathbf{0} \Rightarrow \mathbf{x} = [0,1,0]' \text{ or } \mathbf{x} = [0,0,1]'$$

The steady state can be any point on a line segment depending on the initial state pmf