Department of Systems Engineering
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SYST 302: Systems Methodology and Design II #7

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Control Concepts and Techniques

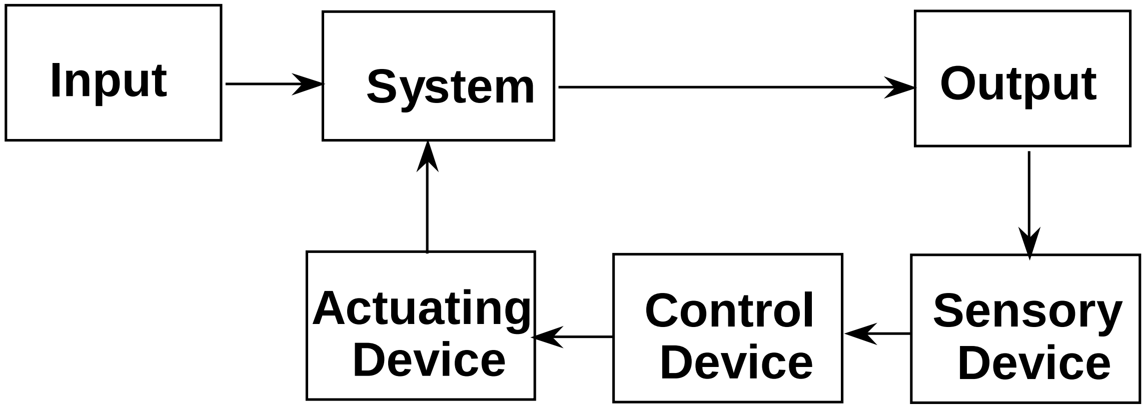
- Concepts and Introduction
- Statistical Process Control
- Optimal Policy Control
- Project Control

Introduction

- **Control System Concepts**
  - Allocation of scarce resources over time
  - Control variables are related to the state variables describing the system conditions to be controlled
  - Maximize performance
  - Open-loop or closed-loop control
- **Control Examples**
  - Missile trajectory determination
  - Air traffic control
  - Quantity or quality regulation such as speed, temperature, etc.

The Control System

- A controlled characteristic
- A sensory device to measure the characteristic
- A control device to compare the measured performance and planned performance
- A activating device to alter the system to the desired condition



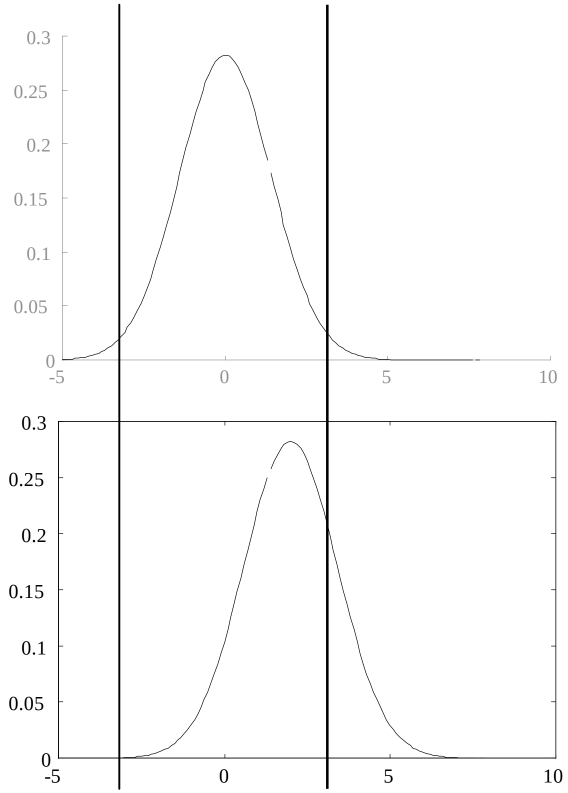
Closed-loop feedback system

Statistical Process Control

– **Pattern Control**
  - **Statistical pattern and variation**
  - **Control limits**

Normal

Abnormal



1. **Control limit**
2. **Type I error**
3. **Type II error**
4. **Minimize the total costs associated with the errors**

Control Charts

– **Detect Parameter Changes**
  – $\overline{X}$ **chart to detect changes in mean of the sample**
  – **R chart to detect the changes in dispersion (range) of the process**

Sample#	Sample Values	Mean	Range
1	$x_{11}, x_{12}, \ldots, x_{1n}$	$\bar{x}_1$	R_1
2	$x_{21}, x_{22}, \ldots, x_{2n}$	$\bar{x}_2$	R_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$
m	$x_{m1}, x_{m2}, \ldots, x_{mn}$	$\bar{x}_m$	R_m

Control Charts
X Chart

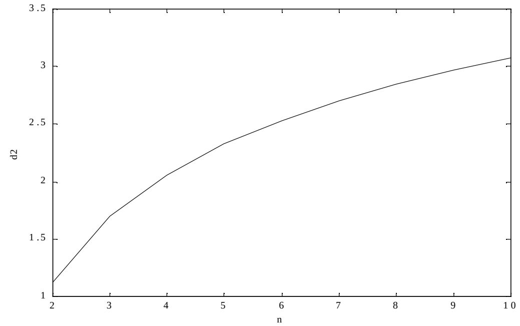
X Chart

- Use Central Limit Theorem
- Approximate normal distribution
- Estimate mean and variance

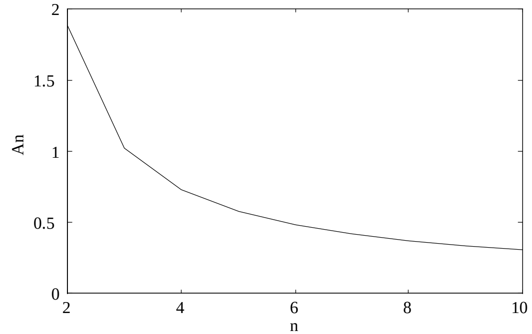
$$\overline{\overline{X}} = \frac{\sum_{i=1}^{m} \overline{X}}{m} = \frac{\sum_{i=1}^{m} \frac{1}{n}\left(\sum_{j=1}^{n} X_{ij}\right)}{m} \Rightarrow \mu = \overline{\overline{X}}$$

$$\overline{R} = \frac{\sum_{i=1}^{m} R_i}{m} \Rightarrow \sigma = \frac{\overline{R}}{d_2} \Rightarrow \sigma_{\overline{X}} = \frac{\sigma}{\sqrt{n}} = \frac{\overline{R}}{d_2\sqrt{n}}$$

$$\overline{X}\ \text{Chart:}\ \mu \pm 3\sigma_{\overline{X}} = \mu \pm \frac{3}{d_2\sqrt{n}}\overline{R} = \mu \pm A_2\overline{R}$$



d2



A2

Factors for X and R Charts

Sample Size	d_2	A_2	D_3	D_4
2	1.128	1.880	0	3.267
3	1.693	1.023	0	2.575
4	2.059	0.729	0	2.282
5	2.326	0.577	0	2.115
6	2.534	0.482	0	2.004
7	2.704	0.419	0.076	1.924
8	2.847	0.373	0.136	1.864
9	2.970	0.337	0.184	1.816
10	3.078	0.308	0.223	1.777

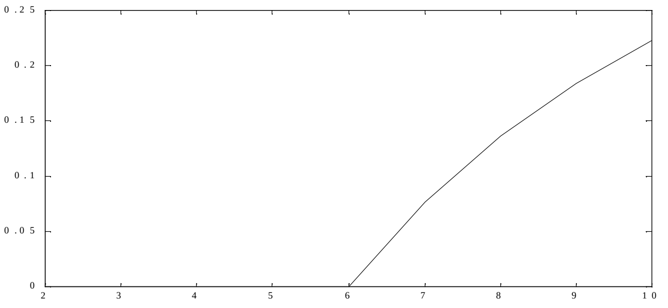
Control Charts
R Chart

R Chart

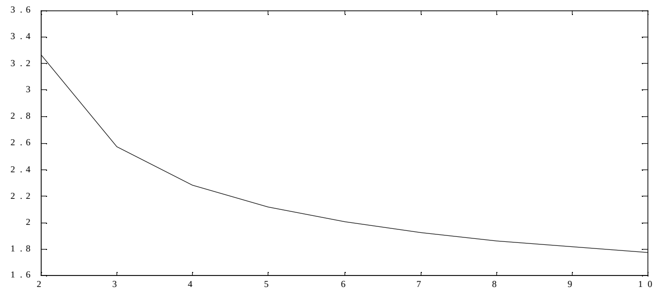
- Similar to X chart
- Define lower and upper control limit

$$LCL_R = D_3\overline{R}$$

$$UCL_R = D_4\overline{R}$$

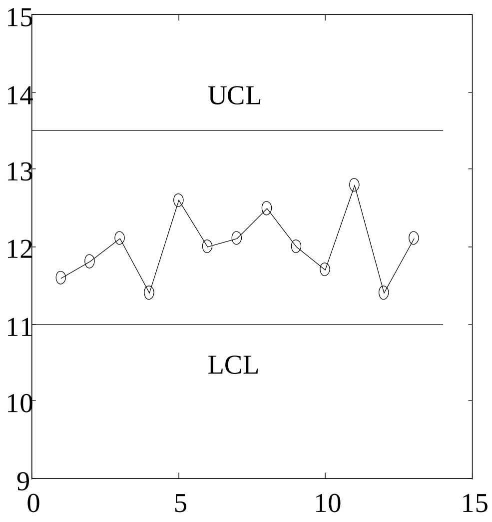


D3

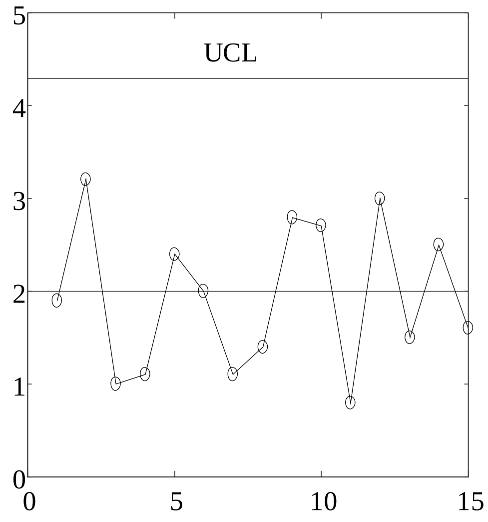


D4

Examples



$\overline{X}$ **chart**



R chart

Ex: Size and weight of the units produced in a assembly line

Issue: revised chart by eliminating outliers and re-calculation

P Charts

P Charts Monitor Binary Process System

- use Binomial distribution
- Mean = np
- Variance = npq=np(1-p)

Convert to proportion:

$$\mu = \frac{np}{n} = p, \quad \sigma^2 = \frac{np(1-p)}{n^2} = \frac{p(1-p)}{n}$$

$$\overline{P} = \frac{\text{total number in the class}}{\text{total number of observations}} \Rightarrow \begin{array}{l} UCL_P = \overline{P} + 2\sigma_P \\ LCL_P = \overline{P} - 2\sigma_P \end{array}$$

$$\sigma_P = \sqrt{\frac{\overline{P}(1-\overline{P})}{n}}$$

P Charts
Example

Use of computer terminal in an office

- **determine proportion of time the terminal is in use**
- **100 observations a day for 21 days**
- **total 546 in-use observations**

$$\overline{P} = \frac{546/21}{100} = 0.26$$

$$\sigma_P = \sqrt{\frac{0.26(1-0.26)}{100}} = 0.044$$

$$LCL_p = \overline{P} - 2\sigma_P = 0.26 - 2(0.044) = 0.172$$

$$UCL_p = \overline{P} + 2\sigma_P = 0.26 + 2(0.044) = 0.348$$

C Charts

C Charts Monitor Multivalued Process System

- use Poisson distribution
- Mean = np
- Variance = np
- Large n, small p

$$\overline{C} = \frac{\sum np}{m} \Rightarrow \begin{array}{l} UCL_P = \overline{C} + 3\sigma_c \\ LCL_P = \overline{C} - 2\sigma_c \end{array}$$

Note that $\sigma_c = \sqrt{\overline{C}}$

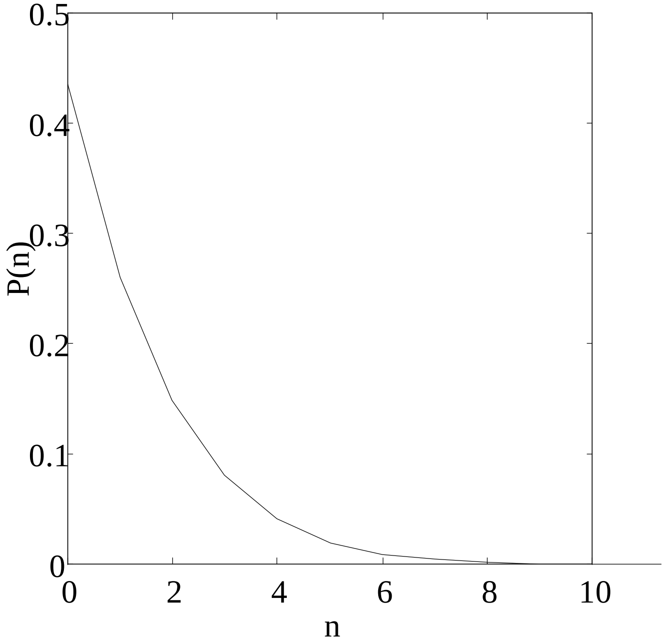
Ex: number of arrivals per hour at a toll gate

Optimal Policy Control

- **Monitor Optimal Decision Policy**
  - **Optimal decision model is valid only when the input parameters remain unchanged**
  - **Use a statistical control chart to detect deviation**
  - **Adapt to the decision environment instability**
- **An Optimal Control Policy Example**
  - **A machine maintenance system**
  - **Single channel (operator) queuing system**
  - **Finite population (N machine in operation)**
  - **The probability of n machine in the (queuing) system is**

$$P_n = \frac{N!}{(N-n)!\,n!}\rho^n P_0 \quad (\text{recall for infinite population} \quad P_n = \rho^n P_0)$$

Control Example



$$\text{With } \rho = 0.03, N = 20$$

$$\sum_{n=0}^{20} P_n = 1 = P_0 \sum_{n=0}^{20} \frac{20!}{(20-n)!\,n!} \rho^n$$

$$\Rightarrow P_0 = 0.435$$

$$\text{with } \alpha = 0.01, \text{since}$$

$$1 - \sum_{n=0}^{6} P_n = 0.0059 \text{ and } 1 - \sum_{n=0}^{5} P_n = 0.0148$$

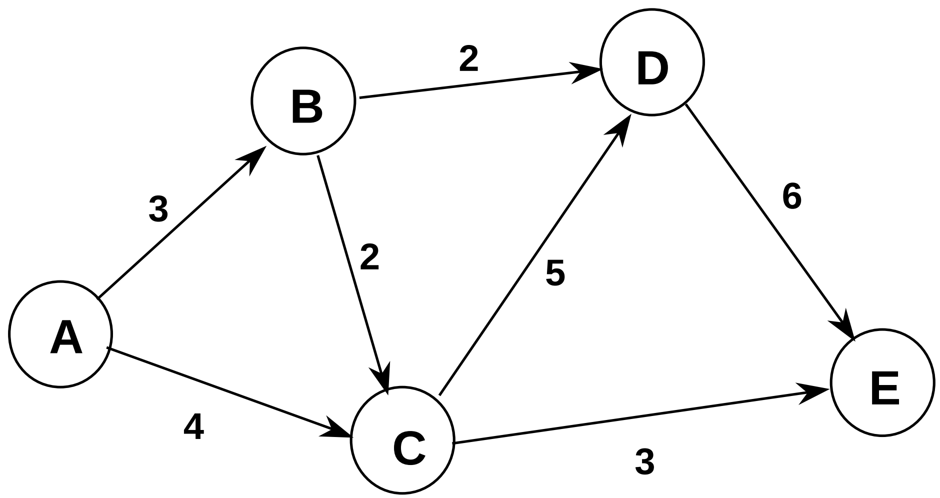
$$\Rightarrow \text{Decision boundary } n = 6$$

Control with CPM and PERT

- **Critical Path Methods (CPM)**
  - **Determine optimal project plan**
  - **Evaluate project progress against the plan**
  - **Planning and control of a large scale system**
  - **Deterministic technique**
- **Program Evaluation and Review Technique (PERT)**
  - **Precise time estimates are not available**
  - **Probabilistic technique**
  - **Compute on-schedule project completion probability**

CPM

Activity-Event network

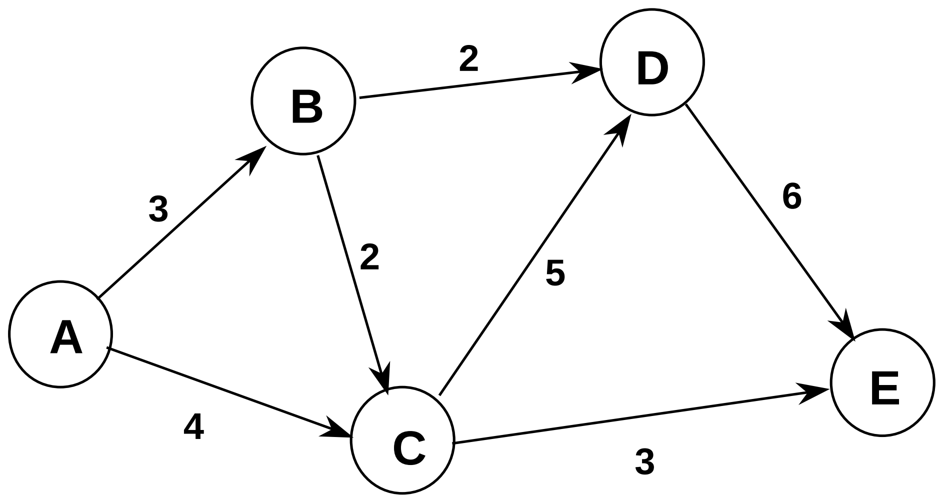


Goal: Finding the critical path

Node: Event - Completion of an Activity
Arc: Activity – Time to Complete

Determine Critical Path

Activity-Event network



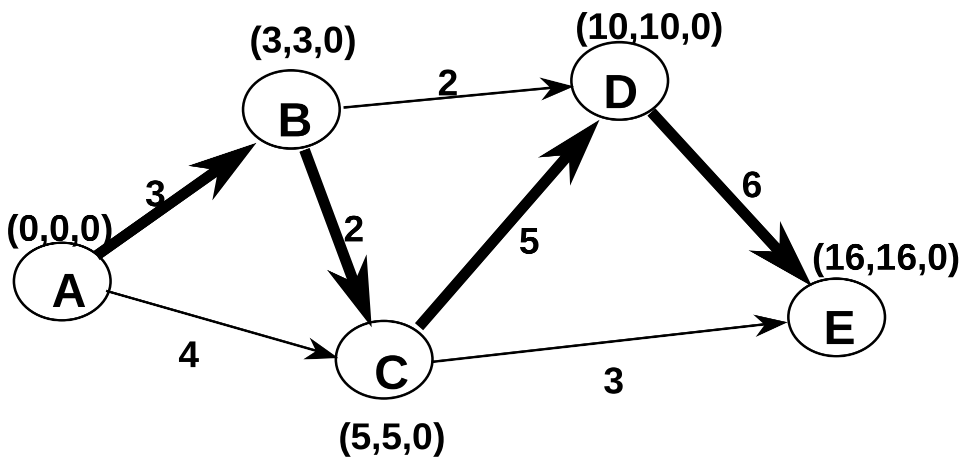
Goal: Finding the critical path

1. For each event, find Et (earliest time) and TL (latest time)
2. On critical path, the slack time TL - Et = 0

Example

Event	Pred	update	Te
A	none	0	0
B	A	3	3
C	A	4	5
C	B	5	
D	B	5	10
D	C	10	
E	C	8	16
E	D	16	

Event	Pred	update	Tl
E	none	16	16
D	E	10	10
C	D	5	5
C	E	13	
B	C	3	3
B	D	8	
A	B	0	0
A	A	1	

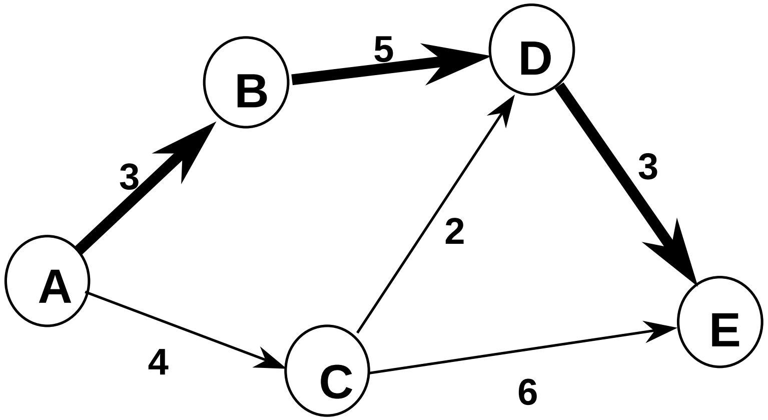


Critical path
A - B - C - D - E

min duration = 16

Economic Aspects of CPM

- Normal schedule: normal time with normal allocation of resources
- Crash schedule: shorten time with additional resources

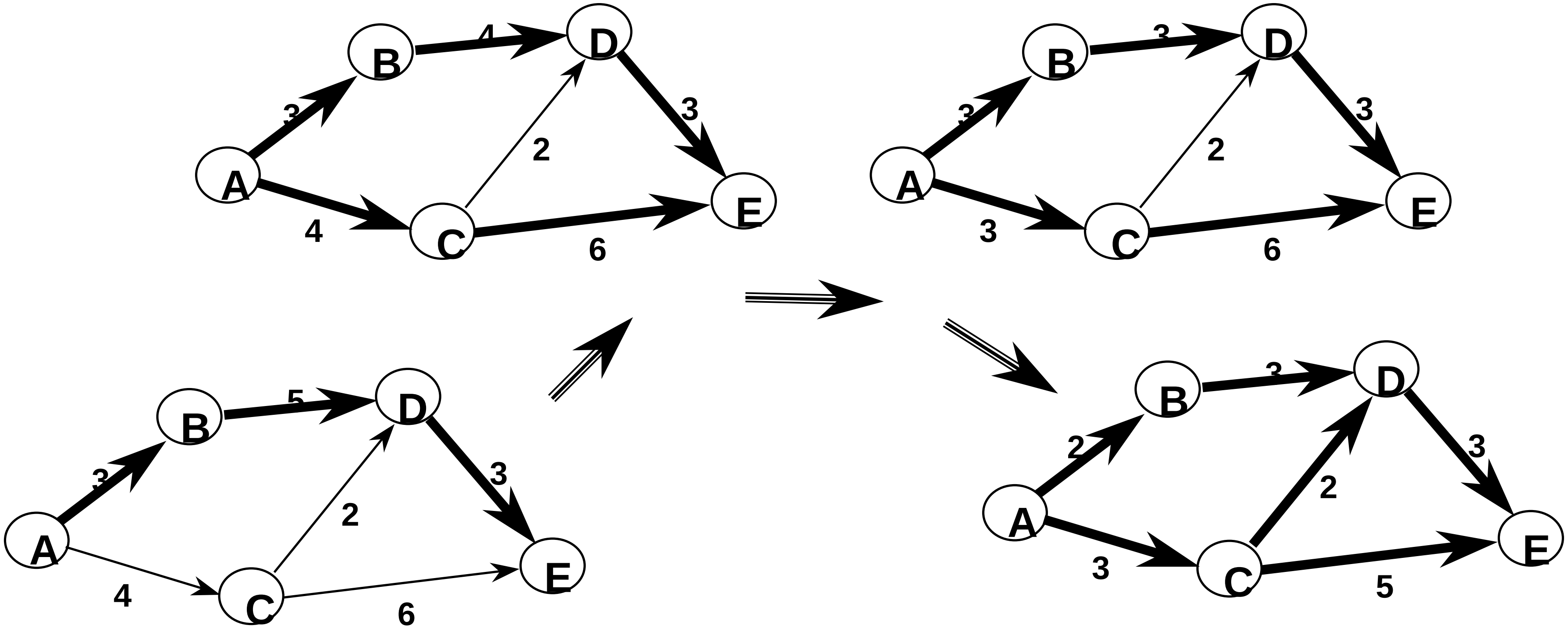


Critical path
A - B - D - E

min duration = 11

Activity	Days Reduced	Crash Cost
AB	1	800
AC	1	750
BD	1	700
	2	900
	3	2,000
CD	0	0
CE	1	950
	2	1,900
DE	0	0

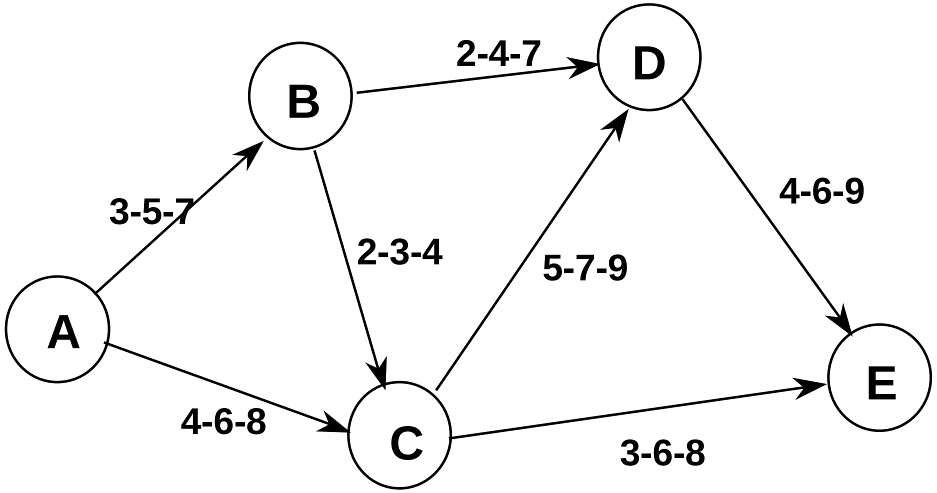
Crash Schedule Savings



Crash Schedule	Cost	Savings	Net Savings
10	700	1,000	300
9	1,650	2,000	350
8	3,400	3,000	-400

PERT

Activity-Event PERT network



Goal: Model time uncertainty and compute probability of project completion

1. **Given optimistic, most likely, and pessimistic time**
2. **Compute expected time and variance**
3. **Find critical path**

PERT Analysis

$$\text{Expected time} = \frac{t_o + 4t_m + t_p}{6}$$

$$\text{variance} = \left(\frac{t_p - t_o}{6}\right)^2$$

$$\text{critical path: } T \propto N(T_C, \sum_{CP} \text{variances}) = N(T_C, \sigma_C^2)$$

Probability of completion by time t_0 is

$$P\{T < t_0) = \int_{-\infty}^{t_0} N(T_C, \sigma_C^2) dT = erf\left(\frac{t_0 - T_C}{\sigma_C}\right)$$