Department of Systems Engineering
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# SYST 302: Systems Methodology and Design II #5

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# Outline

- Resource Allocation
- Optimization Theory
- Constraint and Unconstraint Optimization
- Numerical Optimization Techniques

# Resource Allocation and Optimization

- **Resource Allocation:** distribute resources to users so that the highest level of a specified objective can be achieved
  - Factory organization for maximum productivity
  - Spacecraft flight plan to minimize fuel consumption
  - Security and safety procedures to minimize risks and hazards
- **Optimization Problems:** to achieve a performance in the extreme within the resource available
  - Given budget resource, optimize performance (maximize reliability, minimize risk)
  - Given specification of performance, minimize the capital required

# Optimization

- **Problem Formulation:** **translate the desired system performance specifications and constraints into a mathematical form**
  - **Set up objective function (revenue, yield, cost, reliability, etc.)**
  - **Identify control variables**
  - **Specify constraints**
- **Optimization Techniques:**
  - **Unconstrained optimization**
  - **Constrained optimization**
  - **Linear/nonlinear programming**
  - **Dynamic programming**

# Unconstrained Optimization Single Variables

Problem: Minimize $f(x)$,

sec*ond - order Taylor series expansion*

$$f(x) = f(x_{\mathbf{0}}) + f_x(x_{\mathbf{0}})(x - x_0) + \tfrac{1}{2} f_{xx}(x_0)(x - x_0)^2 + E(x - x_{\mathbf{0}})$$

where

$$f_x(x_0) \equiv \frac{\partial f(x_0)}{\partial x} \quad \text{is the first order derivative}$$

$$f_{xx}(x_0) = \frac{\partial^2 f(x_0)}{\partial x^2} \quad \text{is the second order derivative}$$

$Ex$: $f(x) = 2x^3 - 3x^2 + 2$

$f_x(x) = 6x^2 - 6x$, $f_{xx}(x) = 12x - 6$

# Unconstrained Optimization Multiple variables

Problem: Minimize $f(\mathbf{x})$, where $\mathbf{x} = [x_1, \ldots, x_m]^T$

sec *ond - order Taylor series expansion*

$$f(\mathbf{x}) = f(\mathbf{x_0}) + f_{\mathbf{x}}(\mathbf{x_0})(\mathbf{x} - \mathbf{x_0}) + \tfrac{1}{2}(\mathbf{x} - \mathbf{x_0})^T f_{\mathbf{xx}}(\mathbf{x_0})(\mathbf{x} - \mathbf{x_0}) + E(\mathbf{x} - \mathbf{x_0})$$

where

$$f_{\mathbf{x}}(\mathbf{x_0}) \equiv \nabla f(\mathbf{x_0}) = [f_{x_1}(\mathbf{x_0}), f_{x_2}(\mathbf{x_0}), \ldots, f_{x_m}(\mathbf{x_0})]$$

$$= \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_m}\right]_{\mathbf{x}=\mathbf{x_0}}$$ is called ***gradient***

$$f_{\mathbf{xx}}(\mathbf{x_0}) = \nabla^2 f(\mathbf{x_0}) = \begin{bmatrix} f_{x_1x_1} & \cdots & f_{x_1x_m} \\ \vdots & \ddots & \vdots \\ f_{x_mx_1} & \cdots & f_{x_mx_m} \end{bmatrix} = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right]_{\mathbf{x}=\mathbf{x_0}}$$ is called ***Hessian***

# Example

$$f(x_1, x_2) = x_1^2 - 6x_1 + 2x_2^2 - 8x_2 + 30$$

$$f_{\mathbf{x}}(\mathbf{x_0}) \equiv \nabla f(\mathbf{x_0}) = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\right]_{\mathbf{x}=\mathbf{x_0}} = \begin{bmatrix} 2x_1 - 6 & 4x_2 - 8 \end{bmatrix}$$

$$f_{\mathbf{xx}}(\mathbf{x_0}) = \nabla^2 f(\mathbf{x_0}) = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} \\ \dfrac{\partial^2 f}{\partial x_1 \partial x_2} & \dfrac{\partial^2 f}{\partial x_2^2} \end{bmatrix}_{\mathbf{x}=\mathbf{x_0}} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$

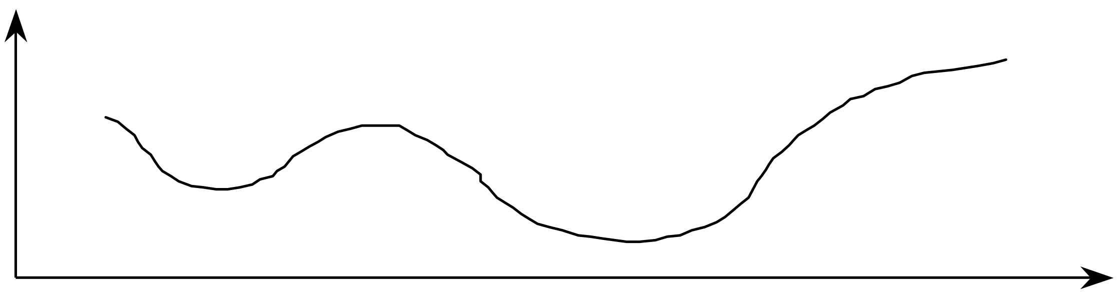
# Local and Global Minimum

**Local Minimu**:

A point $x_0$ is said to be a relative or local minimum if there exists an $\varepsilon > 0$, such that $f(x) \geq f(x_0)$ for all $x$, such that $|x - x_0| < \varepsilon$ and $x \in \Omega$. This definition says that if we can draw an arbitrarily small box around our point $u_0$ and no other values of $f(x)$ within that small box have a lower value than $f(x_0)$, then we have a local minimum.

**Global Minimum**:

A point $x^*$ is said to be a global minimum if $f(x) \geq f(x^*)$ for all $x \in \Omega$.



# Necessary and Sufficient Conditions Single Variable

**Local Minimum**: (Maximun)

Necessary consitions:

first-order $f_x(x_0) \equiv \dfrac{\partial f(x_0)}{\partial x} = 0$

second-order $f_{xx}(x_0) = \dfrac{\partial^2 f(x_0)}{\partial x^2} \geq 0 \quad (\leq 0)$

Sufficient consitions:

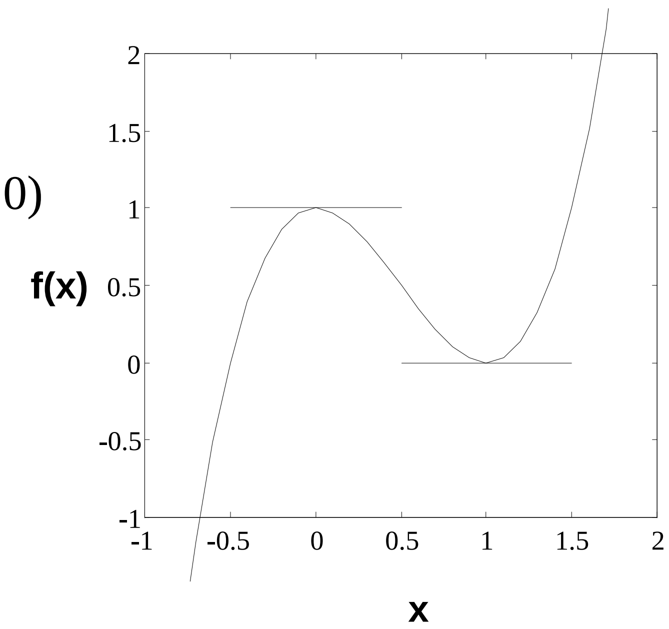
first-order $f_x(x_0) \equiv \dfrac{\partial f(x_0)}{\partial x} = 0$

second-order $f_{xx}(x_0) = \dfrac{\partial^2 f(x_0)}{\partial x^2} > 0 \ (< 0)$

$Ex$: $f(x) = 2x^3 - 3x^2 + 2$

$f_x(x) = 6x^2 - 6x = 0 \Rightarrow x = 1 \ or \ 0$

$f_{xx}(x) = 12x - 6 = 6 \ or \ -6$



# Necessary and Sufficient Conditions Multiple Variables

**Local Minimum**: (Maximun)

Necessary consitions:

first-order $\nabla f(\mathbf{x_0}) = \mathbf{0}$ ($\mathbf{x_0}$: critical point)

second-order $f_{\mathbf{xx}}(\mathbf{x_0}) \equiv Q$ is positive semidefinite ($\mathrm{x}^{\mathrm{T}}\mathrm{Qx} \geq 0, \forall \mathrm{x} \neq 0$)

Sufficient consitions:

first-order $\nabla f(\mathbf{x_0}) = \mathbf{0}$

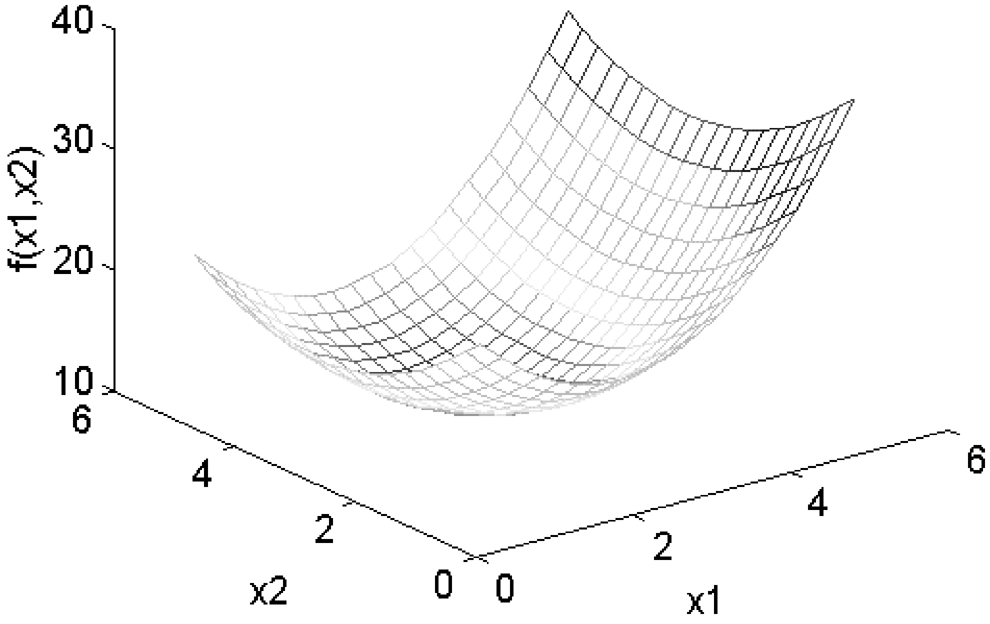
second-order $f_{\mathbf{xx}}(\mathbf{x_0}) \equiv Q$ is positive definite ($\mathrm{x}^{\mathrm{T}}\mathrm{Qx} > 0, \forall \mathrm{x} \neq 0$)

*Ex*: $f(x_1, x_2) = x_1^2 - 6x_1 + 2x_2^2 - 8x_2 + 30$

$f_{\mathbf{x}}(\mathbf{x_0}) \equiv \nabla f(\mathbf{x_0}) = \left[2x_1 - 6 \quad 4x_2 - 8\right] = [0\ 0] \Rightarrow x_1 = 3, x_2 = 2$

$$f_{\mathbf{xx}}(\mathbf{x_0}) = \nabla^2 f(\mathbf{x_0}) = Q = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \Rightarrow Q \text{ is positive definite} \Rightarrow \text{minimum}$$

# Example



# Bridge Design Example

Total cost = cost of superstructure + cost of piers

$$C_T = (AS + B)LC_S + \left(\frac{L}{S} + 1\right)C_p$$

$$\frac{\partial C_T}{\partial S} = ALC_S - \frac{LC_p}{S^2} = 0 \Rightarrow S^* = \sqrt{\frac{C_p}{AC_S}}$$

**Note: Need to choose n=L/S* to a closest integer**

$L$: bridge length (feet)

$W$: superstructure weight (pounds per foot, $W = AS + B$)

$S$: span between piers (foot)

$C_S$: cost of superstructure ($ per pound)

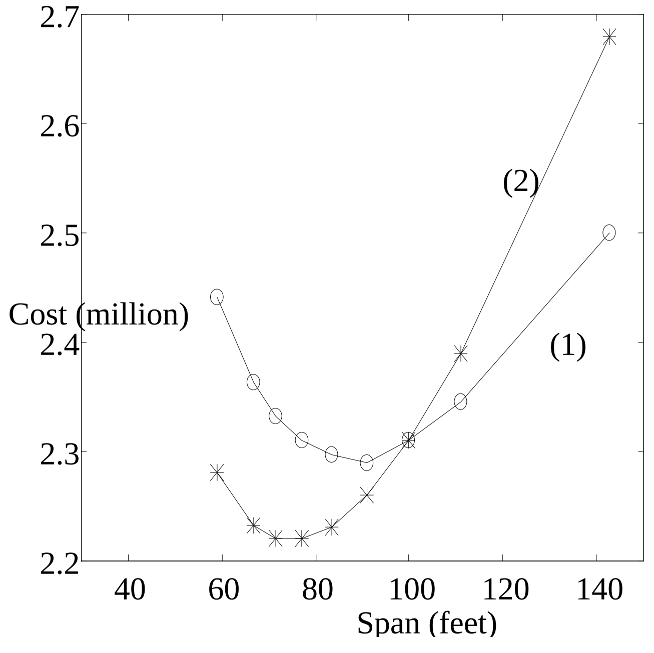
$C_p$: cost of piers ($ per pier)

# Alternative Design

Given $L = 1{,}000 \quad C_s = 0.65, \quad C_p = 80k$

$(1) W_1 = 16S + 600 \quad S^* = 87.7\text{ft}$

$(2) W_2 = 22S \quad S^* = 74.8\text{ft}$



# Conductor Design Example
## cost model

$$\text{Total cost} = \text{Power loss cost} + \text{Investment cost}$$

$$C_T = f(\text{cross-sectional area, electricity cost, unit cost of the conductor,...})$$

$$\text{Issue: cross section} \uparrow, \text{installed cost} \uparrow, \text{power loss cost} \downarrow$$

$$\text{Power loss cost /year} = C_e I^2 R \frac{Hrs}{1000A}$$

$$\textbf{Investment Cost (annual)} = C_T = C_i\left(\frac{A}{P}\right) + (C_m - F_m)W_m\left(\frac{A}{P}\right) + F_m W_m i$$

**where** $W_m = LAD_m/144$, $A$: **area of the cross section (in$^2$)**

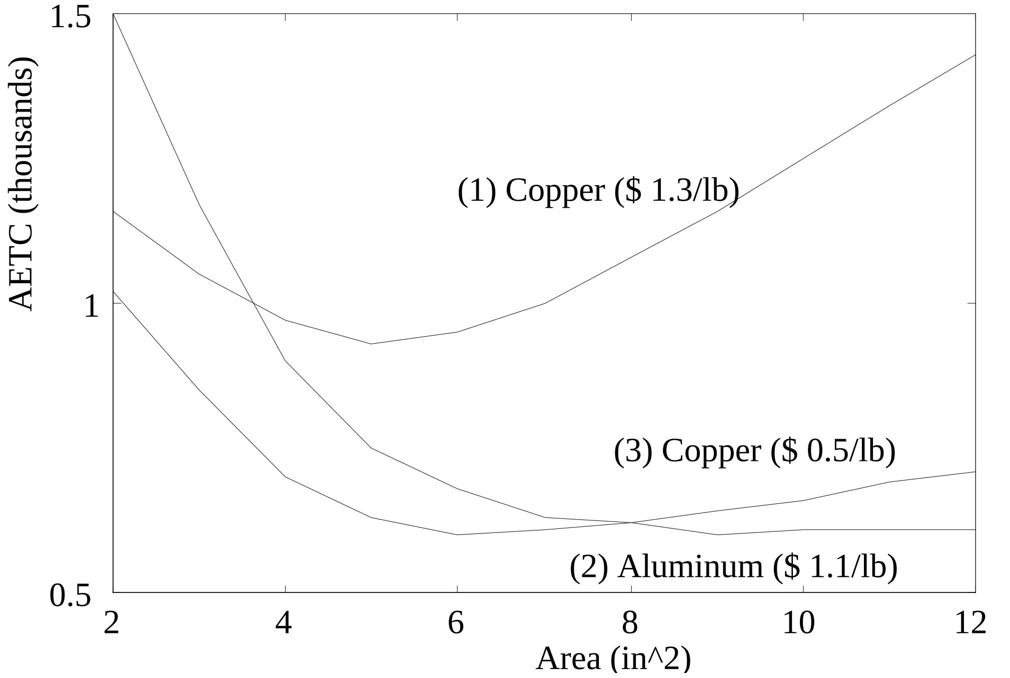
$C_i$ : **installation cost,** $D_m$ : **material density (weight per cubic foot)**

$C_m, F_m$ : **present and future value of the material per unit weight**

# Cost Curves

$$\frac{\partial C_T}{\partial A} = 0 \Rightarrow A^* = \sqrt{\frac{C_e I^2 RH/1000}{(C_m - F_m)\left(\frac{A}{P}\right)L\frac{D_m}{144} + iF_m L\frac{D_m}{144}}}$$

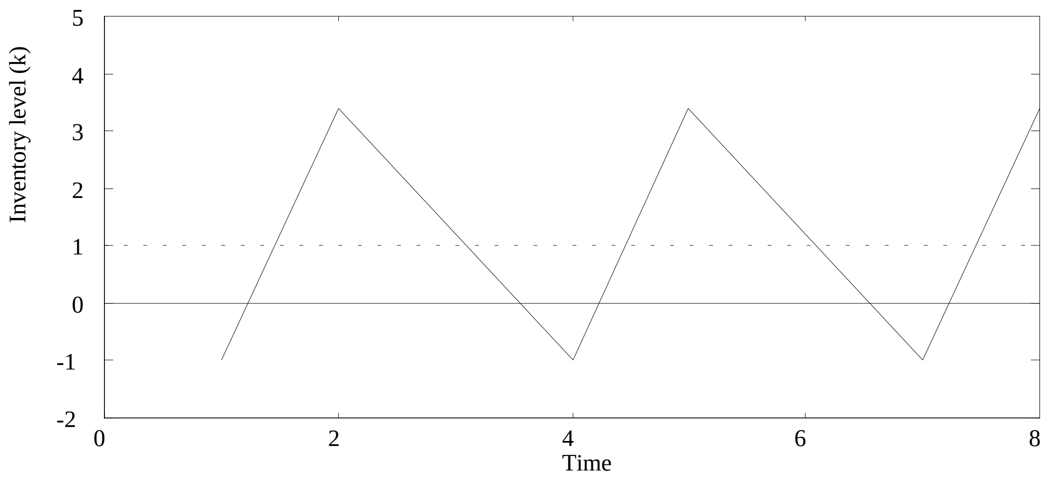
$$\frac{\partial^2 C_T}{\partial A^2} = 2C_e I^2 R\frac{H}{1000A^3} > 0 \Rightarrow \min$$



# Inventory Decision Example

$$\text{Total cost} = \text{item cost} + \text{procurement cost} + \text{holding cost} + \text{shortage cost}$$

$$C_T = f(\text{procurement level, procurement quantity, demand rate, lead time, replenishment rate, ...})$$




Assumptions: replenishment rate is greater than demand rate and that unsatisfied demand is not lost

# Inventory Decision Cost Model

$C_i$ **: item unit cost** $C_p$ **: procurement cost per procurement**

$D$ **: demand rate in units per period**

$L$ : **procurement level** $Q$ : **procurement quantity**

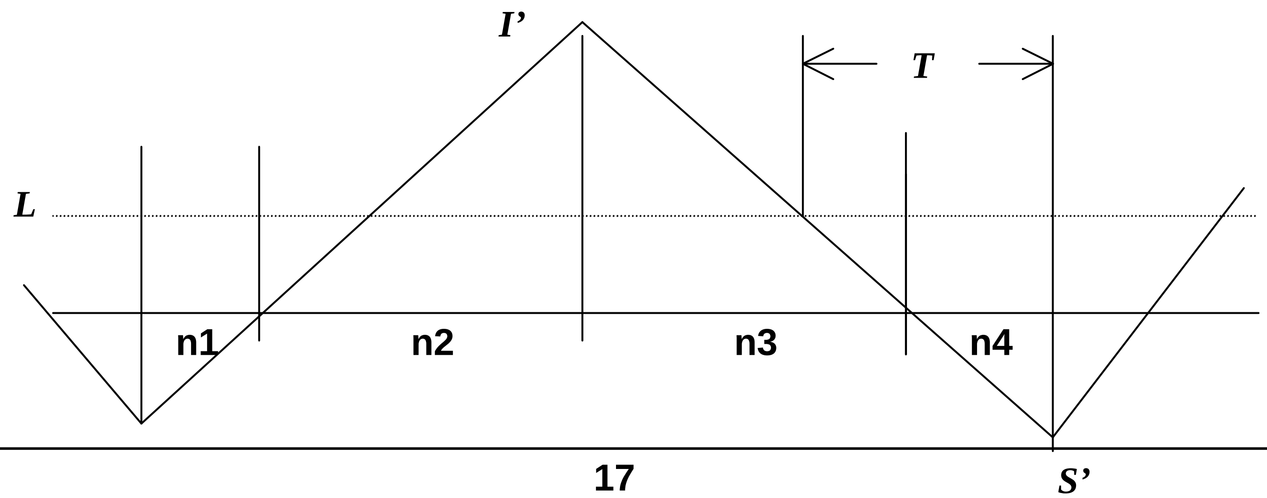
$N = Q/D = n_1 + n_2 + n_3 + n_4$ : **number of periods per cycle**

$I = (n_2 + n_3) I'/2$ : **total inventory per cycle**

$S = (n_4 + n_1) S'/2$ : **total shortage per cycle**

$C_h$ **: unit holding cost per period**

$C_s$ **: unit shortage cost per period**



# Inventory Decision Cost Model

**item cost** $= C_i D$

**procurement cost** $= C_p / N = C_p \frac{D}{Q}$

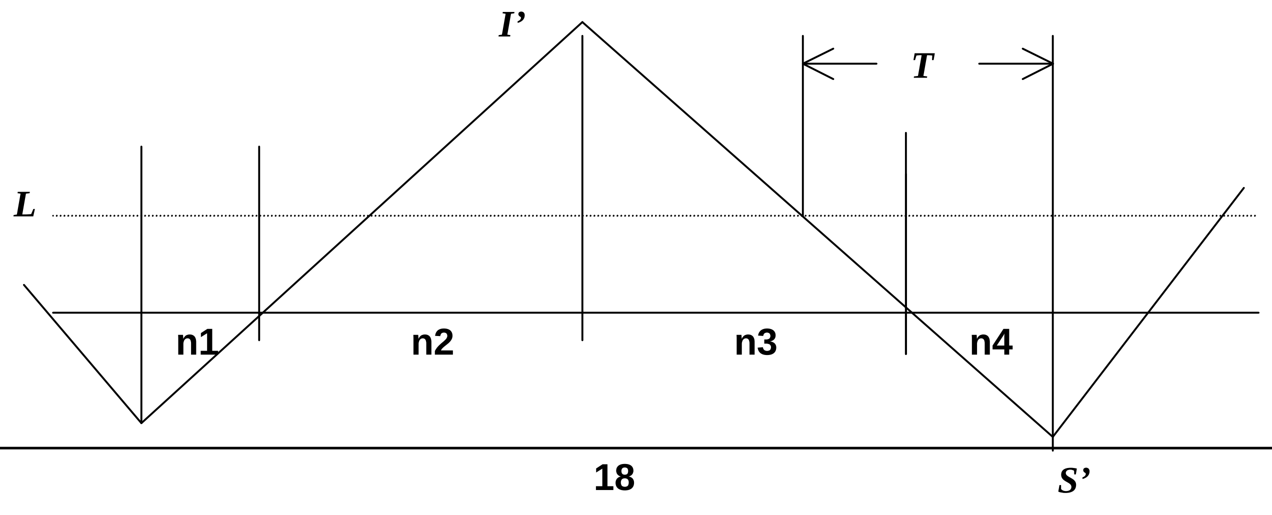
**holding cost** $= C_h I / N = C_h \frac{D}{Q} \frac{(n_2+n_3)I'}{2}$

**shortage cost** $= C_s S / N = C_s \frac{D}{Q} \frac{(n_1+n_4)S'}{2}$

**where** $(n_1+n_2)(R-D) = (n_3+n_4)D;\ n_1+n_2 = Q/R$

$$n_3 + n_4 = \frac{I'-L}{D} + T;\ \ S' = DT - L$$

**Total cost** $= C_T = C_i D + C_p / N + C_h I / N + C_s S / N$



# Optimal Inventory Decision

$$\textit{\textbf{solve}}\ \frac{\partial C_T}{\partial L}=0,\frac{\partial C_T}{\partial Q}=0$$

$$\Rightarrow Q^*=\sqrt{\frac{1}{1-D/R}\left(\frac{2C_pD}{C_h}+\frac{2C_pD}{C_s}\right)}$$

$$L^*=DT-\sqrt{\left(1-\frac{D}{R}\right)\left(\frac{2C_pD}{C_s(1+C_s/C_h}\right)}$$

$$C_T^*=C_iD+\sqrt{\left(1-\frac{D}{R}\right)\left(\frac{2C_pC_hC_sD}{C_s+C_h}\right)}$$

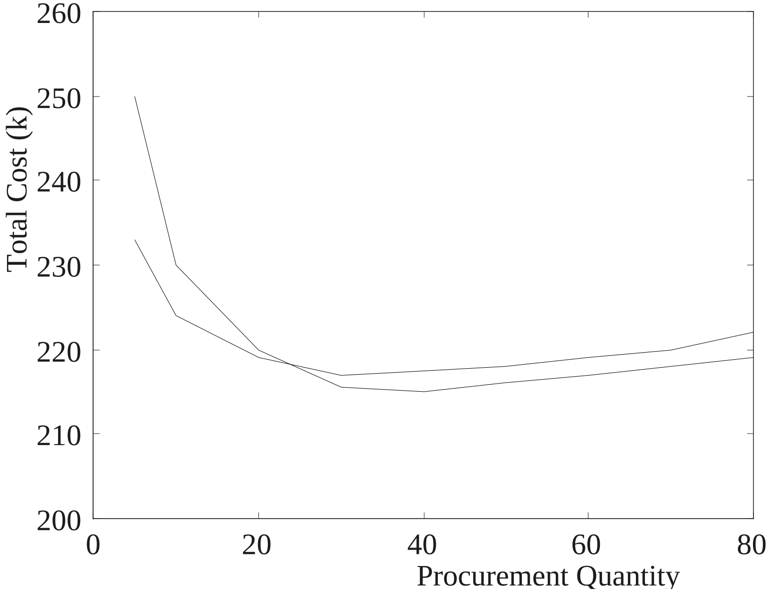
$$\textbf{sufficient condition:}\ \begin{bmatrix}\frac{\partial^2 C_T}{\partial Q^2} & \frac{\partial^2 C_T}{\partial Q\partial L}\\ \frac{\partial^2 C_T}{\partial Q\partial L} & \frac{\partial^2 C_T}{\partial L^2}\end{bmatrix}\ \text{is positive definite}$$

# Space Constrained Operations

Warehouse restriction : max inventory is conatrained

$Ex$ : unconstrained $\Rightarrow Q^* = 43.04, L^* = 20.48$

constrained $(I')$ $\Rightarrow Q^* = 30.57, L^* = 15.6$



# Numerical Optimization Techniques

- **Motivation:** **analytical solution may not be feasible**
  - **Cost function may not be differentiable**
  - **Evaluating the expression for the derivative may be difficult**
- **Algorithm Properties:**
  - ***Iterative* in nature**
  - ***Decent* in sequence**
  - ***Global convergent* is desirable but often not achievable**
- **Numerical Techniques:**
  - **Search methods**
  - **Gradient methods**

# Search Methods

**Constructing shrinking bounding intervals**

**Fibonacci , Golden section**

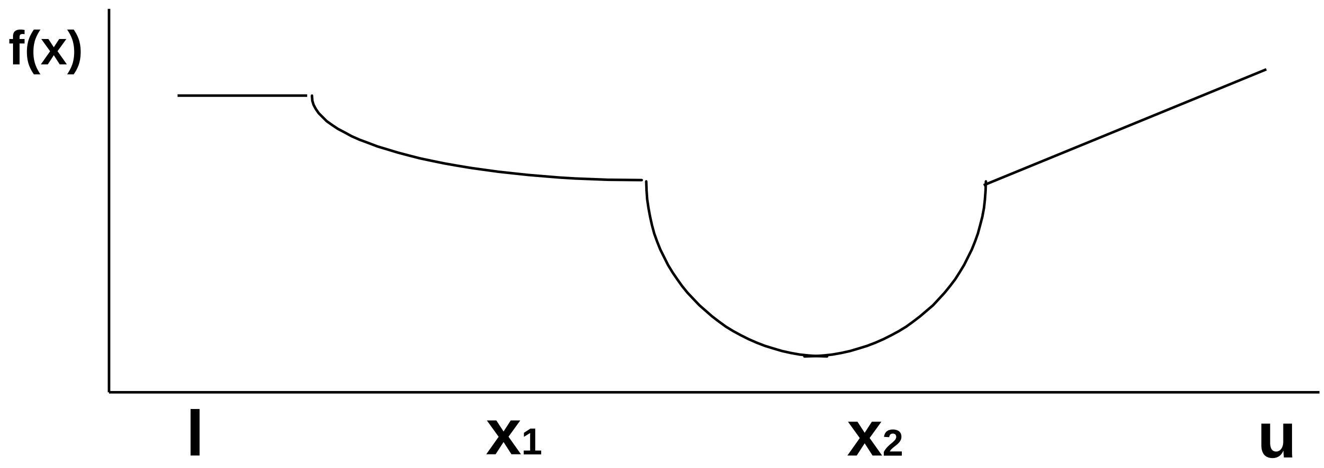
**Often improved by curve - fitting line search**

Basic property of unimodal function :

(1) if $f(x_1) > f(x_2) \Rightarrow x^* \in [x_1, \mathrm{u}]$

(2) if $f(x_1) < f(x_2) \Rightarrow x^* \in [\mathrm{l}, x_2]$

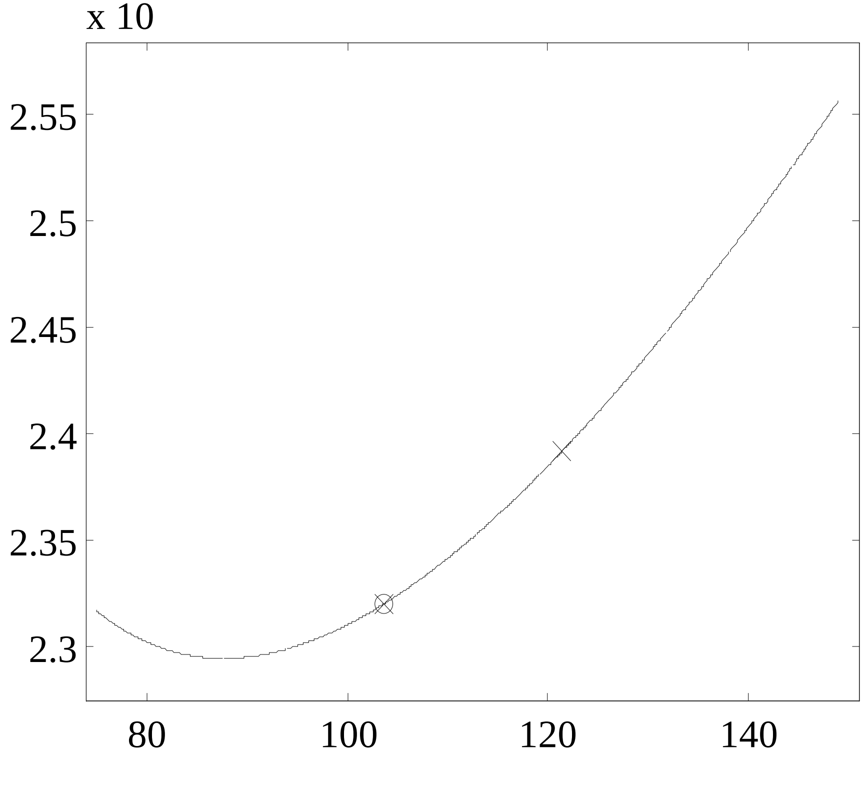
(3) if $f(x_1) = f(x_2) \Rightarrow x^* \in [x_1, x_2]$



# Example - Golden Section Search

**Bridge design example**

$$f(s) = 10400s + 80000000 / s + 470000$$



# Gradient Methods

- **Steepest Decent: oldest and most widely known methods for minimizing multi-variable functions**
  - **Simple and not difficult to analyze**
  - **Many modification on the basic technique exist for better convergence properties**
- **The Method:**

To minimize $f(\mathbf{x})$

$\mathbf{x}_{k+1} = \mathbf{x}_k - \alpha_k \mathbf{g}_k$ where $\mathbf{g}_k = \nabla f(\mathbf{x}_k)$

and $\alpha_k$ is a nonnegative scalar minimizing $f(\mathbf{x}_k - \alpha_k \mathbf{g}_k)$

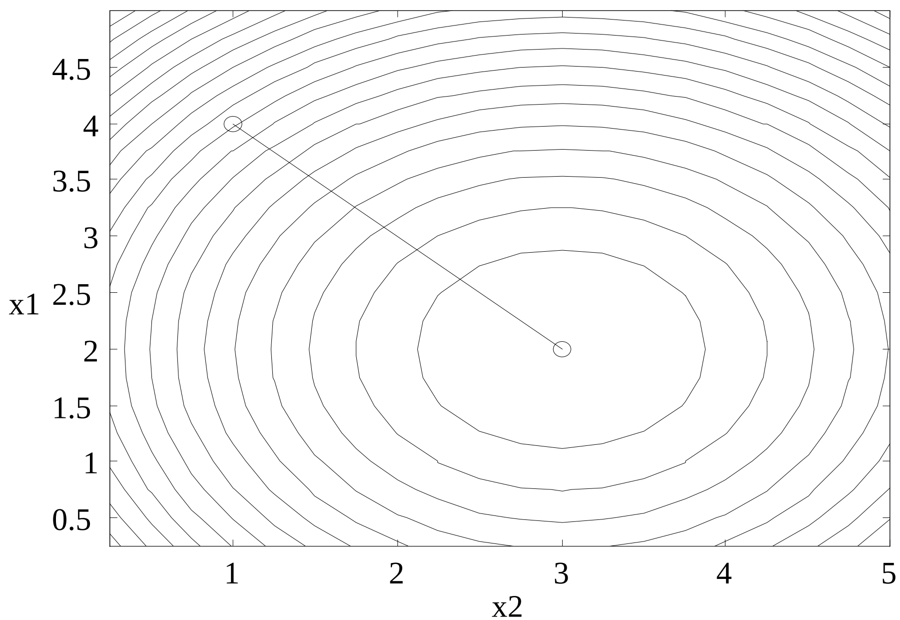
# Example I

$$Ex:\ f(x_1,x_2)=x_1^2-4x_1+x_2^2-6x_2+30;\quad x_1(0)=4, x_2(0)=1$$

$$\nabla f(\mathbf{x(0)})=\begin{bmatrix}2x_1-4 & 2x_2-6\end{bmatrix}_{\mathbf{x(0)}}=\begin{bmatrix}4 & -4\end{bmatrix},\ x_1(1)=x_1(0)-4\alpha, x_2(1)=x_2(0)+4\alpha$$

$$f(\mathbf{x(1)})=(4-4\alpha)^2-4(4-4\alpha)+(1+4\alpha)^2-6(1+4\alpha)+30=32\alpha^2-32\alpha+25\equiv h(\alpha)$$

$$\min h(\alpha)\Rightarrow\alpha=0.5\Rightarrow x_1(1)=2, x_2(1)=3$$



# Example II

$$Ex: f(x_1, x_2) = 2x_1^2 - 6x_1 + x_2^2 - 2x_1x_2; x_1(0) = 0, x_2(0) = 0$$

$$\nabla f(\mathbf{x(0)}) = \left[4x_1 - 6 - 2x_2 \quad 2x_2 - 2x_1\right]_{\mathbf{x(0)}} = \left[-6 \quad 0\right], x_1(1) = x_1(0) + 6\alpha, x_2(1) = x_2(0) + 0\alpha$$

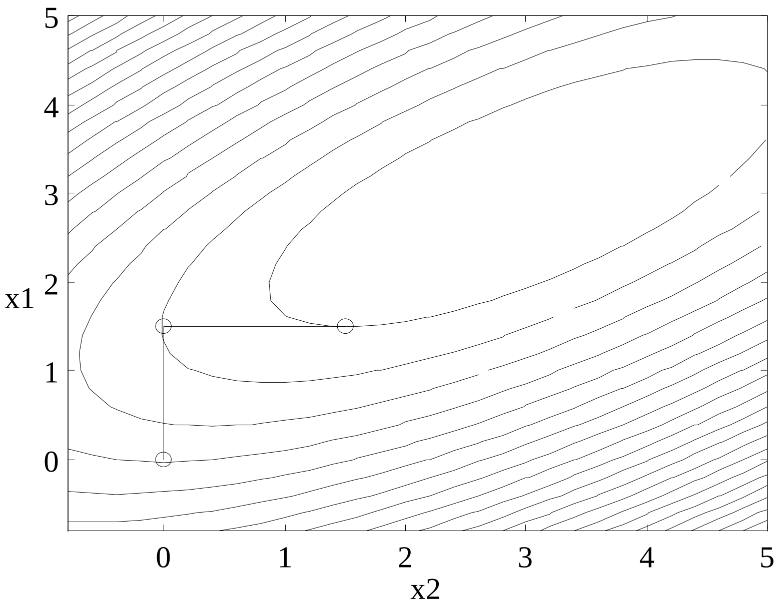
$$f(\mathbf{x(1)}) = 2(6\alpha)^2 - 6(6\alpha) + (0)^2 - 2(6\alpha)(0) = 72\alpha^2 - 36\alpha \equiv h(\alpha)$$

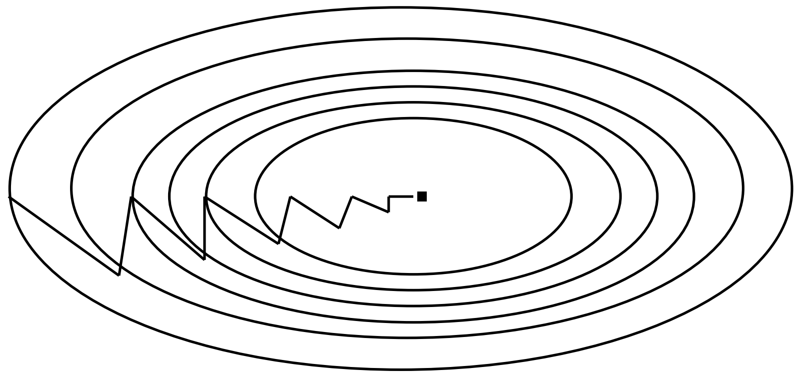
$$\min h(\alpha) \Rightarrow \alpha = 0.25 \Rightarrow x_1(1) = 1.5, x_2(1) = 0$$

$$\nabla f(\mathbf{x(1)}) = \left[4x_1 - 6 - 2x_2 \quad 2x_2 - 2x_1\right]_{\mathbf{x(1)}} = \left[0 \quad -3\right], x_1(2) = x_1(1) + 0\alpha, x_2(2) = x_2(1) + 3\alpha$$

$$f(\mathbf{x(1)}) = 2(1.5)^2 - 6(1.5) + (3\alpha)^2 - 2(1.5)(3\alpha) = 9\alpha^2 - 9\alpha - 4.5 \equiv h(\alpha)$$

$$\min h(\alpha) \Rightarrow \alpha = 0.5 \Rightarrow x_1(1) = 1.5, x_2(1) = 1.5$$



# Newton's Method

Approximate the function locally by a quadratic function

$$f(\mathbf{x}) \cong f(\mathbf{x}_k) + \nabla f(\mathbf{x}_k)(\mathbf{x} - \mathbf{x}_k) + \tfrac{1}{2}(\mathbf{x} - \mathbf{x}_k)'[\nabla f^2(\mathbf{x}_k))](\mathbf{x} - \mathbf{x}_k)$$

the right - hand side is minimized at

$$\mathbf{x}_{k+1} = \mathbf{x}_k - [\nabla f^2(\mathbf{x}_k))]^{-1} \nabla f(\mathbf{x}_k) \Leftarrow \text{Newton's Method}$$

For real quadratic functions $f(\mathbf{x}) = \tfrac{1}{2}\mathbf{x}' Q\mathbf{x} - \mathbf{x}'\mathbf{b}$

$$\Rightarrow \mathbf{x}_{k+1} = \mathbf{x}_k - Q^{-1}(Q\mathbf{x}_k - \mathbf{b}) = Q^{-1}\mathbf{b}$$

reach solution in **one iteration**

# Example

$$Ex: f(x_1, x_2) = 2x_1^2 - 6x_1 + x_2^2 - 2x_1x_2; x_1(0) = 0, x_2(0) = 0$$

$$\nabla f(\mathbf{x(0)}) = \left[4x_1 - 6 - 2x_2 \quad 2x_2 - 2x_1\right]_{\mathbf{x(0)}} = \left[-6 \quad 0\right], \nabla^2 f(\mathbf{x(0)}) = \begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix}$$

$$\left[\nabla^2 f(\mathbf{x(0)})\right]^{-1} \nabla f(\mathbf{x(0)})^T = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 1 \end{bmatrix} \begin{bmatrix} -6 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \end{bmatrix}$$

$$x_1(1) = x_1(0) - (-3) = 3, x_2(1) = x_2(0) - (-3) = 3$$