Department of Systems Engineering

George Mason University

SYST 302: Systems Methodology and Design II #3

Kuo-Chu Chang
Fairfax, Virginia

Probability Concepts

- Concepts and Theory
- Probability Distribution and Random Variables
- Probability Inference

Probability Concept

- **Motivation**
  - **Decision making under uncertainty (risk)**
  - **Need probability model to quantitatively evaluate different criteria**
  - **Rely on statistical regularity to predict future behavior**
- **Concept**
  - **Probability is usually associated with an** *event*
  - **An** *event* **is a subset of a** *sample space*
  - **A** *sample space* **is a set of all possible** *outcomes*
  - **An** *outcome* **is the result of a** *random experiment*
  - **A** *random experiment* **is a process by which a observation is obtained (ex: throwing a die)**

Fundamentals

- **Definitions**
  - **Given a sample space *S*, for each event *A*, there is a number *P(A)* called probability of *A*, such that the following** *axioms of probability* **are satisfied**
  - $0 \le P(A) \le 1$

    $P(S) = 1$

    $P(A \cup B) = P(A) + P(B) \;\; if \;\; A \cap B = \varnothing$

Thm: $P(A^C) = 1 - P(A) \;\; and \;\; P(S) = P(A) + P(A^C) = 1$

where $A^C = S \setminus A$

Thm: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Conditional Probability

- **Conditional Probability P(B|A)**
  - **The probability of an event B under the condition that an event A occurs**

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} \quad or \quad P(A \cap B) = P(B \mid A)P(A)$$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \quad or \quad P(A \cap B) = P(A \mid B)P(B)$$

$$Bayes Rule: P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A \mid B)P(B)}{P(A)}$$

- **Independent Events**

$$if \;\; P(A \cap B) = P(A)P(B) \;\; then \;\; A \;\; and \;\; B \;\; are \;\; independent$$

$$\Rightarrow P(A|B) = P(A) \;\; or \;\; P(B|A) = P(B)$$

Random Variables

- **Random Variable**
  - **A variable x, about which we are uncertain, and to which we can assign a real number**
  - **Defined on the sample space**
- **Discrete Random Variable**
  - **Can assume finite number of discrete values**
  - **Probability is well defined for each value**
  - **Described by probability mass function**
- **Continuous Random Variable**
  - **Can assume any real value in a specified range**
  - **Probability is well defined for any interval**
  - **Described by probability density function**

Probability Distribution

- **Probability Mass Function (pmf)**
  - **Define the probability that a random variable x takes on a value**

$$f(x) = \begin{cases} P_j & if \;\; x = x_j \\ 0 & otherwise \end{cases}$$

CDF: $F_x(x) = P(x_j \le x) = \sum_{x_j < x} f(x_j) = \sum_{x_j < x} P_j$

- **Probability Density Function (pdf)**
  - **Define the probability that a random variable x takes on a value in an interval [a b]**

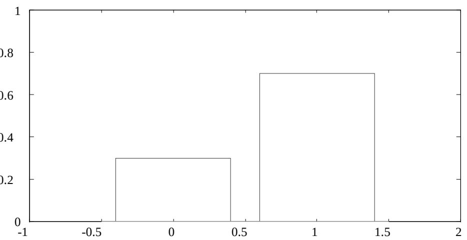
$$P(a \le x \le b) = \int_a^b f_x(x)dx \qquad \text{where} \;\; \int_{-\infty}^{\infty} f_x(x)dx = 1$$

CDF: $F_x(x_0) = P(x \le x_0) = \int_{-\infty}^{x_0} f_x(x)dx$ Note that $P(x > x_0) = 1 - F_x(x_0)$

Discrete Random Variables

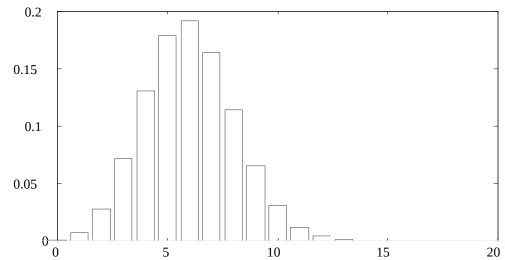
(1) *Bernoulli R.V.* $\quad S_x = \{0,1\}$

$$P_0 = P, \ P_1 = 1 - P$$



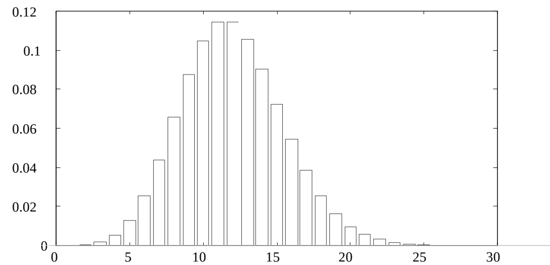
(2) *Binomial R.V.* $S_x = \{0,1,2,\ldots,n\}$

$$P_k = \frac{n!}{(n-k)!k!} P^k (1-P)^{n-k}, \quad k = 0,1,2,\ldots,n$$



(3) *Poisson R.V.* $S_x = \{0,1,2,\ldots\}$

$$P_k = \frac{\alpha^k}{k!} e^{-\alpha}, \quad k = 0,1,2,\ldots \ and \ \ \alpha > 0$$



Examples

(1) *Binomial*

The probability of having at least two defects in 10 samples when the defect probability is 0.01

$$P_{\geq 2} = 1 - P_0 - P_1 = 1 - (0.99)^{10} - \frac{10!}{9!1!}(0.01)^1(0.99)^9 \approx 0.043$$

(2) *Poisson*

If on the average, 3 cars enter a parking lot per minute, what is the probability that during any minute 4 or more cars will enter the lot?

$$P_{\geq 4} = 1 - P_0 - P_1 - P_2 - P_3 = 1 - e^{-3}\left(\tfrac{3^0}{0!} + \tfrac{3^1}{1!} + \tfrac{3^2}{2!} + \tfrac{3^3}{3!}\right) \approx 0.353$$

Continuous Random Variables

$(1)\ Uniform\quad R.V.\quad S_x = [a, b]$

$f(x) = 1/(b-a), a \le x \le b$

$(2) Exponentia\quad l R.V. S_x = [0, \infty)$

$f(x) = \lambda e^{-\lambda x}, x \ge 0 and\ \lambda > 0$

$(3) Gaussian\quad R.V. S_x = (-\infty, \infty)$

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}}, -\infty < x < \infty\ and\ \sigma > 0$$

Examples

(1) *Exponential*

The average life time of a system is 4 hours.

What is the probability that a system will fail within the first two hours of operation?

$$P_{\le 2} = \int_0^2 \frac{1}{4} e^{-\frac{t}{4}} dt \approx 0.39$$

(2) *Gaussian*

If on the average, a product weigh 10 kg with STD = 1kg, what is the probability that the product is lighter than 9 kg?

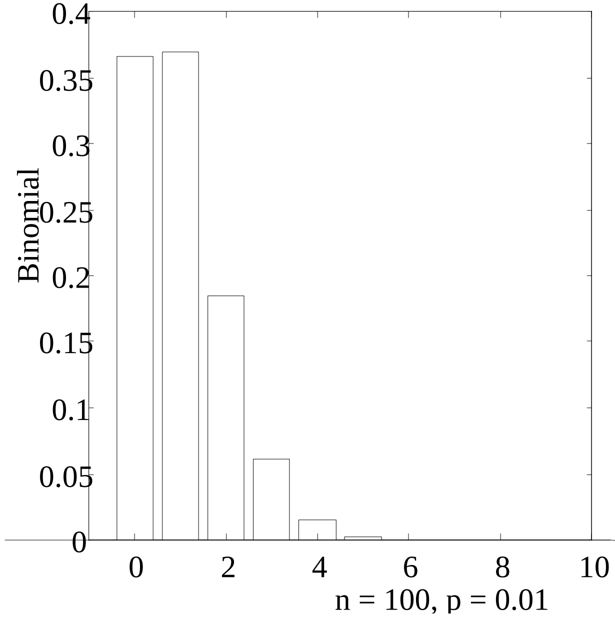
$$P_{\le 9} = \int_{-\infty}^{9} \frac{1}{\sqrt{2\pi}} e^{-\frac{(t-10)^2}{2}} dt \approx 0.159$$

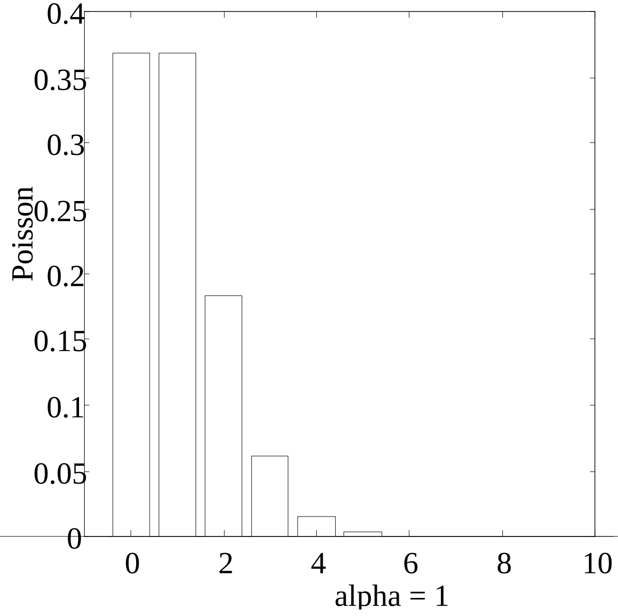
Expectation and Variance

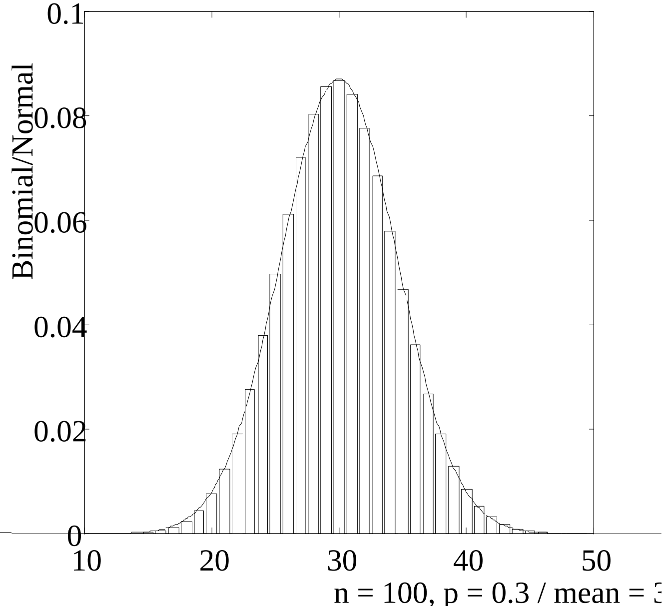
- **Expectation (Mean)** $\overline{x} = E(x) = \int_{-\infty}^{\infty} x f_x(x)dx$
- **Variance** $\sigma_x^2 = E[(x-\overline{x})^2] = \int_{-\infty}^{\infty} (x - E[x])^2 f(x)dx = E[x^2] - E[x]^2$

	pdf	mean	variance
Bernoulli	$P_0 = 1-p,\ P_1 = p$	p	$p(1-p)$
Binomial	$\frac{n!}{(n-k)!k!} P^k q^{n-k}$	np	npq
Poisson	$\frac{\alpha^k}{k!} e^{-\alpha}$	α	α
Uniform	$1/(b-a)$	$(a+b)/2$	$(b-a)^2/12$
Exponential	$\lambda e^{-\lambda x}$	$1/\lambda$	$1/\lambda^2$
Normal	$\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}}$	m	σ^2

Binomial and Poisson/Normal







$$P_k = \frac{n!}{(n-k)!k!} P^k (1-P)^{n-k}$$

$$P_k = \frac{\alpha^k}{k!} e^{-\alpha}$$

$$m = 30, \sigma^2 = 21$$

Sample Mean and Variance

Sample Mean: $\bar{x} = \dfrac{x_1 + x_2 + \cdots + x_n}{n} = \dfrac{\sum_i x_i}{n}$

Sample Variance*: $\sigma^2 = \dfrac{\sum_i (x_i - \bar{x})^2}{n-1} = \dfrac{\sum_i x_i^2 - \left(\sum_i x_i\right)^2 \Big/ n}{n-1}$

*** Unbiased estimate**

Sum of Random Variables and Central Limit Theorem

Let $S_n = x_1 + x_2 + \cdots + x_n$

where $x_1, x_2, \ldots, x_n$ are i.i.d. with mean μ and variance σ^2,

then

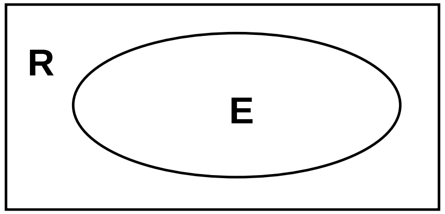
$$\lim_{n\to\infty} f_{S_n}(s) = N(n\mu, n\sigma^2)$$

or

$$\lim_{n\to\infty} f_{Z_n}(z) = N(0,1) \qquad \text{where } Z_n = \frac{S_n - n\mu}{\sqrt{n}\sigma}$$

Monte Carlo Simulation

- **Motivation**
  - **Uncertain decision environment requires probabilistic models**
  - **Formal mathematic solutions may be difficult or impossible**
  - **Monte Carlo analysis provides a powerful tool through simulation**
- **Applications**
  - **System performance evaluation**
  - **System operation characteristics analysis**
- **A Simple Example**
  - **Determine the area of an irregular shape**



(1) Generate 2 - D points in R uniformly

$$\text{(2) Area (E)} \approx \frac{\#\ \text{of points in E}}{\text{total}\ \#\ \text{of points}}\ \text{Area(R)}$$