Introduction:

$A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} * \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ $A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} * \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 13 \end{bmatrix}$ $A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} * \begin{bmatrix} 7 \\ 10 \end{bmatrix} = \begin{bmatrix} 11 \\ 23 \end{bmatrix}$

But notice something different with this:

$A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} * \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$ **Now notice this:** $\begin{bmatrix} 4 \\ -4 \end{bmatrix} = 4 * \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ **The 4 is the Eigenvalue and the vector, which is the same vector in the original equation, is the Eigenvector (or self-vector).**

So, how do we get the Eigenvalue & Eigenvector and how many do we get?

(Note: the Eigenvalue & Eigenvector come as a pair. You can't have one without the other.)**

DEFINITION 26.1

If a is a square matrix, λ is a real scalar and $\vec{X} \in \Re^n$, then $\vec{X}$ is called an *eigenvector* of A with *eigenvalue* λ if and only if $\vec{X} \neq \vec{0}$ and $A\vec{X} = \lambda\vec{X}$.

(Comment: $\vec{X} \neq \vec{0}$, λ can be anything including Zero.)**

THEOREM 26.2

The number λ is an eigenvalue of a square matrix A if and only if $|A-\lambda I| = 0$.

DEFINITION 26.5

Suppose A is an *n x n* matrix and define $C(\lambda) = |A-\lambda I|$. Viewing λ as a variable, $C(\lambda)$ turns out to be a polynomial of degree *n*, called the characteristic polynomial of A. The roots of the equation $C(\lambda)=0$ (called the characteristic equation for A) are the eigenvalue of A.

DEFINITION 26.7

The algebraic multiplicity of the eigenvalue α is the number of times $\lambda - \alpha$ (or $\alpha - \lambda$) appears in the factorization of $C(\lambda)$.

Consider: $A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$

1) **Find the characteristic polynomial.**
2) **Find the eigenvalue.**

Solution: $C(\lambda) = |A - \lambda I|$

1) $$A - \lambda I = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{bmatrix} = (3-\lambda)^2 - 1$$

2) **Solve to find eigenvalue(s)** $C(\lambda) = 0$.

$$(3-\lambda)^2 - 1 = 0 \quad \Rightarrow \quad (3-\lambda)^2 = 1 \quad \Rightarrow \quad 3-\lambda = \pm 1 \quad \therefore$$

$$3-\lambda_1 = 1 \text{ and } 3-\lambda_2 = -1 \quad \Rightarrow \quad \lambda_1 = 2,\ \lambda_2 = 4$$

For any given matrix *n x n*, you always get *n* eigenvalues.
The eigenvalues can repeat so this means that each eigenvalues may not be unique.
The λ can be real or complex.

Example: Determine the algebraic multiplicity of the eigenvalue of :

$$A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \qquad \lambda_1 = 2 \text{ (algebraic multiplicity = 1)}, \quad \lambda_2 = 4 \text{ (algebraic multiplicity = 1)}$$

Example 2: Find the eigenvalue of : $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$|A - \lambda I| = 0 \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} * - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{bmatrix} \Rightarrow (1-\lambda)^2$$

$$(1-\lambda)^2 = 0 \quad \therefore \quad 1-\lambda_1 = 0 \quad \text{and} \quad 1-\lambda_2 = 0 \quad \Rightarrow \quad \lambda_1 = 1,\ \lambda_2 = 1$$

For any given matrix *n x n*, you always get *n* eigenvalues, and since it is a 2 x 2 matrix, there are 2 eigenvalues. $\Rightarrow \quad \lambda_1 = 1,\ \lambda_2 = 1$

Summary: For a given matrix A, we can easily find the eigenvalues by solving the characteristic equation $|A - \lambda I| = 0$.

The next question is how to find the *Associated Eigenvectors*.

DEFINITION 27.1

The dimension of $N(A - \lambda I)$ is called the geometric multiplicity of the eigenvalue λ.

DEFINITION 27.3

Suppose B_i is a basis for $N(A - \lambda_i I)$ for each eigenvalue λ_i of A. We then define a complete set of eigenvectors for A to be the union of these B_i.

DEFINITION 27.6

A complete set of eigenvectors is always linearly independent.

$$A-\lambda I = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{bmatrix} = (3-\lambda)^2 - 1$$

$$(3-\lambda)^2 - 1 = 0 \quad \Rightarrow \quad (3-\lambda)^2 = 1 \quad \Rightarrow \quad 3-\lambda = \pm 1 \quad \therefore$$

$$3-\lambda_1 = 1 \text{ and } 3-\lambda_2 = -1 \quad \Rightarrow \quad \lambda_1 = 2, \ \lambda_2 = 4$$

To find the Eigenvectors:

By the Fund Theorem of Algebra, there exist exactly n roots.

For any matrix *n x n*, there exist *n* Eigenvectors or Eigenvectors $\leq n$.