Supergroups: A Failed Research Programme

Peter Stalin

August 26, 2008

1 Introduction

The idea was to collect the subgroups of a group G as a set A and using the external direct product (e.d.p.) as an operation, make a group with (A, e.d.p), or maybe some subset B of A so that (B,e.d.p.) is a group. Furthermore, if such groups existed, I would've called them Supergroups; but we will show there are none (except the trivial one). I was also thinking that their existence might be able to give me insight into prime groups, since $\mathbf{Z}_p$ has only two subgroups, itself an the identity. However, the ideas started to fall apart when I introduced some new notation.

Either way, the rigorous explanation is as follows.

Def: $sub(G) = \{H | H \leq G\}$

Def: $I = (\{e\}, \oplus)$ (where $\oplus$ is e.d.p. and e is the identity; in other words, I is the trivial group)

2 The Non-Existance of Supergroups

The following theorem shows only trivial supergroups exist.

Thm: If $A \neq I, A \subseteq Sub(G)$ finite then $(A, \oplus)$ is not a magma (and thus not a group).

Pf: Assume $(A, \oplus)$ is a group, then we will prove the intermediary result.

$\forall n, \exists b \in A \ni |b| > n$.

If we show this, then b is an element of infinite order, which contradicts the fact that A is a finite magma.

So, we show this by induction.

Base case:

$\exists b \in A \ni |b| > 1$ because otherwise $|b| = 1$ and this would imply that $b \cong I$, and thus that A is the trivial magma, contradiction.

Step:

Assume $\exists b_1 \in A \ni |b_1| > n$, we will show that $\exists b_2 \in A \ni |b_2| > n + 1$

Since A is a group, $b_1 \oplus b_1 \in A$ and $|b_1 \oplus b_1| = |b_1|^2$, and WLOG $b_1 \ncong I$. Therefore, since $n^2 > n + 1$ for $n > 1$, $|b_1 \oplus b_1| > n + 1$. Thus let $b_2 = b_1 \oplus b_1$ and we are done.

Thus we have an element of infinite order.

Done.

Having an element of infinite order in a finite magma is a contradiction, and thus other than the trivial group, there is no supergroup.
R.A.A.

3 Notation and Alternate Exposition

This, however, brought about a new idea. Consider the following procedure.

Def: $< G >^0 = G$ by convention, $< G > = \cap_{j=1}^{n} \{G_i | \exists k G_i \oplus G_j \cong G_k \in Sub(G)\}$ by convention, and $< G >^l = \cap_{j=1}^{n} \{G_i | \exists k G_i \oplus G_j \cong G_k \in Sub((< G >^{l-1}, \oplus)\}$.

This infrastructure serves as an easier way to understand the previous theorem. To show this point we will prove that:

Thm: $< G >^i = \{I\}$ for some i (again, G finite).

We know that $< G >^i \subseteq < G >^j, i < j$ by definition. Therefore, there are two possibilities, either $< G >^i = < G >^{i+1} \neq I$ or $< G >^i = I$ for some i. Assume the first, then this implies that we have the property of closure, and thus we have a magma, which our previous theorem denies. Therefore, for some $i < G >^i = I$.

Q.E.D.

4 Conclusion

My motivation in using this infrastructure is to see if there is any special properties with this recursion. In other words, perhaps I could save the idea of a supergroup if this iterative process does not immediately go to the trivial supergroup. However, when $j = n, G_n = G$ we have, trivially $\{G_i | \exists k G_i \oplus G \cong G_k \in Sub((< G >^{l-1}, \oplus)\} = I$, and the intersection ensures us that $< G >= I$. And this route is closed as well, but at least it was good exercise.