FRIENDSHIP AND RELATIONSHIPS: A SELF-HELP BOOK FOR MATHEMATICIANS

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1 The Invariance of Friendship

Given the finite set of people $P = \{0, 1, \cdots, 6664292109\}$ (population of the Earth as of this writing), we define the Friendship relation F as the mapping $F : P \times P \rightarrow P$ as:

$F(i, j) = j$ if i is friends with j

$F(i, j) = i$ if i is not friends with j

By definition the system $< P, F >$ is invariant/has closure, thus we have a magma. We call this **The Magma of Friendship**.

2 With What Can we Associate Friendship?

We check whether associativity holds in **The Magma of Friendship**.

Let $i, j, k \in P$, $F(F(i, j), k)$, we take this into cases:

i, j, k are not friends:

$F(F(i, j), k) = F(i, k) = i = F(i, j) = F(i, F(j, k))$

i, j, k are all friends:

$$F(F(i,j),k) = F(j,k) = k = F(i,k) = F(i,F(j,k))$$

i, j are friends, but j, k are not friends:

$$F(F(i,j),k) = F(j,k) = j \neq i = F(i,j) = F(i,F(j,k))$$

We see similar situations occur if there is a pair that are friends while another pair are not friends. Thus, **The Magma of Friendship** is not associative.

3 Is There an Alternative to Friendship?

Now, in order to reduce this into a better structure, we determine whether **The Magma of Friendship** is alternative, i.e., that $x(xy) = (xx)y$ and $y(xx) = (yx)x$. For the following given $i, j \in P$ friends we have associativity (as a case we have already checked) and therefore have alternativeness, so WLOG we assume i, j are not friends (this does not reduce into an all not-friends case because WLOG we say x is a friend of himself).

Left Alternativeness:

$$F(F(i,i),j) = F(i,j) = i = F(i,i) = F(i,F(i,j))$$

Right Alternativeness:

$$F(j,F(i,i)) = F(j,i) = j = F(j,i) = F(F(j,i),i)$$

Thus we have alternatives in **The Magma of Friendship**.

4 How Do We IDENTify Friendship?

Friendship does not necessarily have identity. In order to have $i \in P \ni \forall j \in P, F(i,j) = i = F(j,i)$ we would need an i who does not consider himself the friend of anyone else, but everyone else considers themselves to be the friend of

i, or $i = j$. Any such $M \subseteq P$ that has this property (if M is a single element it satisfies this property) we call **sad**.

It is depressing to think that in order for a relationship to have an identity, it needs to be a little bit **sad**.

5 What Have We Learned About Our Relationship?

We have a lot of ways we could go to explore friendship here. We could use Artin's Theorem along with alternativeness. We could possibly use the associator as an equivalence relation, and partition our friends apart. Or, we could attempt to see whether our alternative magma has inverses. We want to determine as much structure, so we don't want to bypass possible other associative structures by Artin's theorem; but we can build up to see whether division is possible, and if it isn't, then we'll take a two-way path partitioning the set into divisible parts and associative ones.

6 So, Can Friends Be Divided?

Left-division:

Given $i, j \in P$, we ask $\exists! k \in P \ni F(i,k) = j$.

Consider $i \neq j$

Let $k = j$, then if i and j are friends, we have a unique divisor. If i and j are not friends, then since $i \neq j$, we have no possible divisors. Thus, we do not necessarily have division. Not only that, but we fortunately see that it is hard to divide friends. However, let us consider collecting such friendships, if we do so, then we create a **Quasigroup of Divided Friends**:

$Q := \{i, j \in P | \exists! k \in P \ni F(i,k) = j\}$

To identify the elements of this set, let us continue this case study:

$i \neq j$, but i and j are friends, if this is so, then $k = j$, furthermore if $i = j$ then trivially $k = i$. That is that Q must collect all people that are co-wise friends with each other.

I should point out that these Q's are not necessarily unique as subsets of the population set. There could be two collections of friends that are all pairwise friends to each other, for example, consider any of the 6 billion subsets consisting of a single person.

7 Let's Be With Friends We Can Associate With!

We collect the associative friendships, let $A = \{i, j, k \in P | F(F(i, j), k) = F(i, F(j, k))\}$ and we call A **The Friendship Semigroup**. What kind of relations do these look like? By our previous analysis, we knew that if there was a pair of non-friends that associativity failed. Thus, like Q, A must collect all pair-wise friends. That is, $\forall A \exists Q \ni Q = A$, and so it seems that friends that associate with one another are divided. Having alternatives in our friendships, it seems, is a much better option.

8 Do We Have Something In Common?

We start with **The Friendship Semigroup**. Now, let us call A by a name, call it e, so that we have something everyone in the group can relate to, an embedded consciousness. At the very least, in the real world, we can call this humanity. Furthermore, this virtual person will act as our identity. That is, humanity is pretty **sad**.

However, by our earlier analysis of associating in friendship, we begin to lose our structure, and the same actually occurs with including e:

$$F(F(i, j), e) = F(j, e) = e \neq i = F(i, j) = F(i, F(j, e))$$

In common dialect, when we create something we can identify with, it turns out everyone can't associate with it as well. We are forced, then, between a dichotomy of having something we can associate with in relationships, or having something we can identify with.

9 What's an Ideal Friendship?

In order to study **The Friendship Semi(Quasi)group**, we look at the ideals inside our earlier defined A.

Let's look at right-ideals first. Given $i \in P$ what does $\{F(i, j), j \in A\} \cup \{i\}$ look like? Well, by definition, it is collecting all the people who i is friends with.

Mutatis Mutandis with left-ideals we collect all the friends of i.

Now, under what conditions does $iA = \{F(i,j), j \in A\} \cup \{i\} = \{F(j,i), i \in A\} \cup \{j\} = jA$? Well, if $F(i,j) = j = F(j,i)$, i.e., if the friendship commutes, they are equal, and it is here we make our biggest jump. If they did not commute, we would have a pairwise non-friendship. That is, in **The Friendship Semi(Quasi)group**, friends are Abelian. Similarly we have $Ai = Aj$, and even $AiA = AjA$. That is, not only are the friendships divided, but we have shown they can be a little two-sided.

10 Let's Cancel This Relationship Altogether.

Now, because we don't have an identity, at least without losing associate with other people and divided ourselves still between others, we can't have a group when more than one person is involved.

Thus, we stop here to cross a different line in our relationships. Let's look at the one-element sets, given that before we were ignoring the trivial case; because, in the non-associativity proof, we implicitly assumed two distinct pairs; it still makes sense to look at this. In fact the surprising relationships we find in ourselves do not contradict the structure in relationships lost to the outside world.

So, we look at $< \{i\}, F >$. i is the identity, it is its own inverse, we have friends with everyone else in the group and thus can both associate and divide amongst ourself. That is, we have a **Lonely Group**, a very **sad** group as well. And in fact, this is it.

11 It All Ends Here

Here we end our tale. With the trivial group as the biggest structure we can find in relationships, we can go no further. Note that if we tried to install field structure onto this group, we could not, as the multiplicative group would be over an empty set, a contradiction.