Error Analysis in Sampling Theory

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Abstract—A basic problem in signal theory is the reconstruction of a band-limited function $f(t)$ from its sampled value $f(nT)$. Because of a number of errors, the computed or physically realized signal is only approximately equal to $f(t)$. The most common sampling errors are: round-off of $f(nT)$, truncation of the series generating $f(t)$, aliasing of frequency components above half the sampling rate $1/T$, jitter in the recording times $nT$, loss of a number of sampled values, and imperfect filtering in the recovery of $f(t)$. In the following we study the effect of these errors on the reconstructed signal and its Fourier transform.

I. INTRODUCTION

Consider a signal $f(t)$ with Fourier transform

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt.$$  \hspace{1cm} (1)

We shall assume that $f(t)$ is band-limited by $\omega_1$, i.e., that

$$F(\omega) = 0 \quad \text{for} \quad |\omega| \geq \omega_1$$  \hspace{1cm} (2)

as in Fig. 1. In this paper, the following interesting generalization of the sampling theorem will be used: If the constant $T$ is such that

$$\omega_2 = \frac{\pi}{T} \geq \omega_1$$  \hspace{1cm} (3)

then $f(t)$ can be recovered from its sampled values $f(nT)$ as

$$f(t) = \sum_{n=-\infty}^{\infty} f(nT) \frac{\sin \omega_0(t - nT)}{\omega_2(t - nT)}$$  \hspace{1cm} (4)

where $\omega_0$ is an arbitrary constant between $\omega_1$ and $2\omega_2 - \omega_1$. 

$$\omega_1 \leq \omega_0 \leq 2\omega_2 - \omega_1$$  \hspace{1cm} (5)

In the familiar form of the theorem [1], [2] the constant $\omega_2$ is taken equal to $\omega_2 = \omega_1$. 

In this case, the kernel $\sin \omega_2(t - nT)/\omega_2(t - nT)$ equals zero at the sampling points $t = mT$, $m \neq n$. In the following we show that (4) is true under the more general condition (5). It will be assumed that $f(t)$ is a deterministic signal with finite energy. The sampling theorem is valid also if $f(t)$ is a stationary stochastic process [3], and most of our results can be extended to such processes.

Since (Fourier inversion formula)

$$f(nT) = \frac{1}{2\pi} \int_{-\omega_1}^{\omega_1} F(\omega)e^{j\omega_0 t} d\omega = \frac{1}{2\omega_2 T} \int_{-\omega_2}^{\omega_2} F(\omega)e^{j\omega_0 t} d\omega$$

we conclude that if $F(\omega)$ is expanded into a Fourier series in the interval $(-\omega_2, \omega_2)$, the coefficients of the resulting expansion will equal $TF(nT)$

$$F(\omega) = \sum_{n=-\infty}^{\infty} T f(nT) e^{-j\omega_0 T} \quad \text{for} \quad |\omega| \leq \frac{\pi}{T}.$$  \hspace{1cm} (7)

Clearly, the above sum is the periodic repetition $F^*(\omega)$ of $F(\omega)$

$$F^*(\omega) = \sum_{n=-\infty}^{\infty} T f(nT) e^{-j\omega_0 T} = \sum_{n=-\infty}^{\infty} F\left(\omega + \frac{2\pi n}{T}\right).$$  \hspace{1cm} (8)

With $p_{\omega_0}(\omega)$ a pulse as in Fig. 1, we thus have

$$F(\omega) = \sum_{n=-\infty}^{\infty} T f(nT) e^{-j\omega_0 T} p_{\omega_0}(\omega)$$  \hspace{1cm} (9)

and (4) follows because

$$\frac{\sin \omega_0(t - nT)}{\pi(t - nT)} \leftrightarrow p_{\omega_0}(\omega) e^{-j\omega_0 T}.$$  \hspace{1cm} (10)

Conversely, given an arbitrary sequence of numbers $a_n$, if we form the sum

$$x(t) = \sum_{n=-\infty}^{\infty} a_n \frac{\sin \omega_0(t - nT)}{\omega_2(t - nT)}$$  \hspace{1cm} (11)

then $x(t)$ is band-limited by $\omega_0$ because its Fourier transform is given by

$$X(\omega) = \sum_{n=-\infty}^{\infty} \frac{\pi a_n}{\omega_2} e^{-j\omega_0 T} p_{\omega_0}(\omega).$$

If $\omega_0 = \omega_2$, then $x(nT) = a_n$; however, if $\omega_0 \neq \omega_2$, then, in general,

$$x(nT) \neq a_n.$$
With a slight modification of the preceding proof one can obtain a further generalization of (4). Consider a function \( k(t) \) with Fourier transforms \( K(\omega) \) such that

\[
K(\omega) = \begin{cases} 
1 & -\omega_1 \leq \omega \leq \omega_1 \\
0 & 2\omega_2 - \omega_1 \leq \omega \leq 2\omega_2 + \omega_1, \ n \neq 0 \\
\text{arbitrary} & \text{elsewhere}
\end{cases}
\]

(Fig. 2). It is easy to see that

\[
F(\omega) = F^*(\omega)K(\omega) = \sum_{n=-\infty}^{\infty} Tf(nT)e^{-jn\omega}K(\omega);
\]

hence

\[
f(t) = \sum_{n=-\infty}^{\infty} Tf(nT)k(t - nT).
\]

This expression is used to relax the filter requirements in reconstructing \( f(t) \) from its sampled values by filtering (Section VII).

\[
\text{Fig. 2. Spectrum of kernel in sampling expansion.}
\]

We note for future use that the integral of \( f(t) \) equals \( T \) times the sum of its sampled values

\[
\int_{-\infty}^{\infty} f(t)dt = \sum_{n=-\infty}^{\infty} Tf(nT) = F(0) \quad T \leq \frac{\pi}{\omega_1}.
\]

This follows from (1) and (9) with \( \omega = 0 \). In particular (see (10) and (13)) for any \( c \)

\[
\sum_{n=-\infty}^{\infty} T\frac{\sin \omega_0(nT - c)}{\pi(nT - c)} = \delta_{\omega_0}(0) = 1 \quad \omega_0 \leq \frac{\pi}{T}
\]

because the function

\[
\frac{\sin \omega_0(t - c)}{\pi(t - c)}
\]

is band-limited by \( \omega_0 \).

Other Forms of the Theorem

The sampling theorem can be written in the form

\[
f(t - a) = \sum_{n=-\infty}^{\infty} f(nT - a)\frac{\sin \omega_0(t - nT)}{\omega_2(t - nT)}
\]

where \( a \) is an arbitrary constant. Indeed, the transform of \( f(t - a) \) equals

\[
e^{-ja\omega}F(\omega);
\]

therefore, \( f(t - a) \) is band-limited with the same bound \( \omega_1 \) as \( f(t) \), and (15) follows from (4). Replacing \( t - a \) by \( t \), we also have

\[
f(t) = \sum_{n=-\infty}^{\infty} f(nT - a)\frac{\sin \omega_0(t + a - nT)}{\omega_2(t + a - nT)}.
\]

As a consequence of (16) we mention the useful identity

\[
\sum_{n=-\infty}^{\infty} \frac{\sin \omega_1(t - nT)}{t_1 - nT} \frac{\sin \omega_2(t - nT)}{\omega_2(t - nT)} = \frac{\sin \omega_1(t_1 - t_2)}{t_1 - t_2}
\]

valid for

\[
\omega_1 \leq \omega_0 \leq \frac{2\pi}{T} - \omega_1
\]

because the function \( \frac{\sin \omega t}{t} \) is band-limited by \( \omega_1 \) and (17) follows from (16) with

\[
f(t) = \frac{\sin \omega t}{t} \quad a = t_1 \quad t + a = t_2.
\]

The sampling theorem can be applied to the function \( f^2(t) \) if the sampling rate is sufficiently high. Since (frequency convolution [2, p. 27])

\[
f^2(t) \leftrightarrow \frac{1}{2\pi} F(\omega) * F(\omega)
\]

the function \( f^2(t) \) is band-limited by \( 2\omega_1 \) where \( \omega_1 \) is the band of \( f(t) \). Therefore,

\[
f^2(t) = \sum_{n=-\infty}^{\infty} f^2(nT)\frac{\sin \omega_0(t - nT)}{\omega_2(t - nT)} \quad \omega_2 = \frac{\pi}{T}
\]

provided \( \omega_2 \geq 2\omega_1 \) and \( \omega_0 \) is such that

\[
2\omega_1 \leq \omega_0 \leq 2\omega_2 - 2\omega_1.
\]

Applying (13) to the function \( f^2(t) \) we obtain

\[
\int_{-\infty}^{\infty} f^2(t)dt = \sum_{n=-\infty}^{\infty} T f^2(nT) \quad T \leq \frac{\pi}{2\omega_1}.
\]

A Useful Bound

A signal \( f(t) \) with finite energy

\[
E = \int_{-\infty}^{\infty} f^2(t)dt
\]

can, in general, take arbitrarily large values. However, if \( f(t) \) is bandlimited by \( \omega_1 \), then

\[
|f(t)| \leq \sqrt{\frac{\omega_1 E}{\pi}}.
\]

To prove the above, we apply Schwarz's inequality [2]

\[
|\int fgdt|^2 \leq \int |f|^2g^2dt \int |g|^2dt
\]

to the inversion formula

\[
f(f) = \frac{1}{2\pi} \int_{-\omega_1}^{\omega_1} F(\omega)e^{i\omega t}d\omega.
\]
We thus have
\[ |f(t)|^2 \leq \frac{1}{4\pi^2} \int_{-\omega_1}^{\omega_1} F(\omega) |\omega|\,d\omega \int_{-\omega_1}^{\omega_1} \left| e^{j\omega t}\right|^2 \,d\omega \]
and since (Parseval’s formula [2])
\[ \frac{1}{2\pi} \int_{-\omega_1}^{\omega_1} F(\omega)^2 \,d\omega = E \]
(22) follows readily.

II. Round-Off Error

In digital recording of the sampled values of a given signal, the stored numbers are \( f(nT) \), differing from \( f(nT) \) by the round-off errors
\[ e_n = f(nT) - \tilde{f}(nT). \] (23)

With
\[ f_r(t) = \sum_{n=-\infty}^{\infty} \tilde{f}(nT) \frac{\sin \omega_2(t - nT)}{\omega_2(t - nT)} \]
the recovered signal from (4), we see that \( f(t) \) differs from \( f_r(t) \) by the total error
\[ e_r(t) = f(t) - f_r(t) = \sum_{n=-\infty}^{\infty} e_n \frac{\sin \omega_2(t - nT)}{\omega_2(t - nT)}. \] (24)

We shall assume that the local errors \( e_n \) are uncorrelated random variables with the same mean \( \eta \) and variance \( \sigma^2 \)
\[ E\{e_n\} = \eta \quad E\{e_ne_m\} = \begin{cases} \sigma^2 + \eta^2, & m = n \\ \eta^2, & m \neq n \end{cases} \] (25)
The error \( e_r(t) \) is thus a stochastic process. We shall prove that it is wide-sense stationary with mean
\[ E\{e_r(t)\} = \eta \] (26)
and autocovariance
\[ C(r) = E[\{e_r(t + r) - \eta\}[e_r(t) - \eta]] = \sigma^2 \frac{\sin \omega_2 T}{\omega_2 T}. \] (27)

Equation (26) follows readily from (25) and (14). Inserting (24) into the middle part of (27) we obtain with (25)
\[ C(r) = \sigma^2 \sum_{n=-\infty}^{\infty} \frac{\sin \omega_2(t + r - nT)}{\omega_2(t + r - nT)} \frac{\sin \omega_2(t - nT)}{\omega_2(t - nT)} \]
and the last part of (27) follows from (17) with \( \omega_0 = \omega_1 = \omega_2 \), \( t + r = t_1 \), \( t = t_2 \). Thus, \( e_r(t) \) is ideal low-pass [3, p. 372] with variance \( C(0) = \sigma^2 \). Hence,
\[ E\{f_r(t)\} = f(t) + \eta \quad \sigma_{f_r}^2 = \sigma^2. \] (28)

Bounds

From (24) we see that the error at the sampling points equals the local error
\[ e_r(nT) = f(nT) - \tilde{f}(nT) = e_n. \]

It is of interest to observe that, even if \( |e_n| \leq \epsilon \) for all \( n \), \( e_r(t) \) may exceed all bounds for some values of \( t \). To see this, we suppose that \( e_0 = 0 \) and
\[ e_n = (-1)^{n+1} \epsilon/ \left| n \right| \quad n \neq 0. \]
We then have from (24)
\[ e_r \left( \frac{t}{2} \right) = \epsilon \sum_{n=0}^{\infty} (-1)^{n+1} \frac{\sin \left( n \frac{\pi}{2} \right)}{\left| n \right| \left( n \frac{\pi}{2} \right)} = \infty. \]

Suppose that the reconstructed signal is determined by
\[ f_r(t) = \sum_{n=-\infty}^{\infty} \tilde{f}(nT) \frac{\sin \omega_1(t - nT)}{\omega_2(t - nT)} \]
[see (4) with \( \omega_0 = \omega_1 \)]. In this case, \( f_r(t) \) and hence the error \( e_r(t) \) are band-limited by \( \omega_1 \). Therefore, if \( \omega_2 \geq 2\omega_1 \), then \( e_r(t) \) is bounded by its total energy \( E_r \) [see (20)]
\[ |e_r(t)| \leq \sqrt{\frac{\omega_1 E_r}{\pi}}. \]

We note that if \( f_r(t) \) is given by (29), then, in general, \( f_r(nT) \neq \tilde{f}(nT) \).

III. Truncation Error

In a numerical evaluation of \( f(t) \), a finite sum must be used
\[ f_N(t) = \sum_{n=-N}^{N} f(nT) \frac{\sin \omega_2(t - nT)}{\omega_2(t - nT)} \]
(30)
differing from \( f(t) \) by the truncation error
\[ e_N(t) = f(t) - f_N(t). \] (31)

In this section we shall study the nature of the approximation of \( f(t) \) by \( f_N(t) \) and shall give bounds on \( e_N(t) \).

Frequency-Domain Analysis

We begin with an investigation of the effects of truncation on the Fourier transform \( F(\omega) \) of \( f(t) \). This analysis is useful if one’s objective is the determination of \( F(\omega) \) from \( f(nT) \) (spectral analysis). From (30) we see that the sampled values of the truncated sum \( f_N(t) \) are given by
\[ f_N(nT) = \begin{cases} f(nT) & |n| \leq N \\ 0 & |n| > N \end{cases} \] (32)
Clearly, \( f_N(t) \) is band-limited by \( \omega_1 \) [see (11)]. Denoting by \( F_N(\omega) \) its Fourier transform, we conclude from (7) that
\[ F_N(\omega) = \sum_{n=-N}^{N} T f(nT) e^{-jnT} \quad |\omega| \leq \frac{\pi}{T}. \] (33)

In other words, \( F_N(\omega) \) equals the truncated sum of the Fourier series expansion (7) of \( F(\omega) \). The problem is, thus, reduced to the study of truncated Fourier series.
This problem is extensively treated. We mention below certain conclusions.

It can be shown that [2, p. 46] \( F_N(\omega) \) is given by the weighted average

\[
F_N(\omega) = \int_{-\omega_2}^{\omega_2} F(y) k_1(\omega - y) dy
\]

of \( F(\omega) \) with the Fourier kernel

\[
k_1(\omega) = \frac{T \sin (N + \frac{1}{2}) T \omega}{2 \sin \frac{T \omega}{2}}
\]
as weight. As \( N \) tends to infinity, \( k_1(\omega) \) tends to an impulse \( \delta(\omega) \), and \( F_N(\omega) \) to \( F(\omega) \). For a given \( N \), the error \( F(\omega_0) - F_N(\omega_0) \) is small if the curvature of \( F(\omega) \) at \( \omega = \omega_0 \) is small. If \( F(\omega) \) is discontinuous at \( \omega_0 \) then the error cannot be eliminated no matter how large \( N \) is (Gibbs' phenomenon). If \( F(\omega) \) drops rapidly to zero at \( \omega = \omega_0 \), then the sampling rate \( 1/T \) should be made equal to \( \omega_0/\pi \) so as to eliminate the end-point discontinuity. An increase in the sampling rate will make the Gibbs' phenomenon more pronounced (Fig. 3).

**Fejér Approximation**

Although, for a given \( N \), \( F_N(\omega) \) gives the best mean-square approximation of \( F(\omega) \), the maximum error near discontinuity points of \( F(\omega) \) can be reduced by favoring linearly the low-frequency components of the Fourier expansion of \( F(\omega) \) (Fejér sum [2, p. 46]). This fact can be utilized in sampling theory. With

\[
\phi(nT) = f(nT) \left( 1 - \frac{|n|}{N} \right)
\]

we use as an estimate of \( f(t) \) the function

\[
\Phi_N(t) = \sum_{n=-N}^{N} \phi(nT) \sin \omega_2(t - nT) \quad \frac{\omega_2(t - nT)}{}.
\]

The transform \( \Phi_N(\omega) \) of \( \Phi_N(t) \) is now given by the weighted average

\[
\Phi_N(\omega) = \int_{-\omega_2}^{\omega_2} F(y) k_2(\omega - y) dy
\]
of \( F(\omega) \) with the Fejér kernel

\[
k_2(\omega) = \frac{T \sin^2 \frac{NT \omega}{2}}{2 \pi \sin \frac{T \omega}{2}}
\]
as weight. Since \( k_2(\omega) \geq 0 \), the Gibbs' phenomenon is eliminated. Again, as \( N \) tends to infinity, \( k_2(\omega) \) tends to \( \delta(\omega) \), and \( \Phi_N(\omega) \) to \( F(\omega) \).

The estimate \( \Phi_N(t) \) is preferable also for another reason. Suppose that the function \( f(t) \) is positive semi-definite, i.e., that its Fourier transform is non-negative \( F(\omega) \geq 0 \).

This is, for example, the case if \( f(t) \) is the autocorrelation of a stationary stochastic process \( x(t) \) and \( F(\omega) \) its power spectrum [3, p. 349]. Since \( k_1(\omega) \) takes negative values, the transform \( F_N(\omega) \) of \( f_N(t) \) is not necessarily non-negative [see (34)], i.e., \( f_N(t) \) may not be positive semi-definite. However,

\[
k_2(\omega) \geq 0 \quad \text{and, hence,} \quad \Phi_N(\omega) \geq 0
\]
as we see from (38). The estimate \( \Phi_N(t) \) provides thus a positive semi-definite approximation of \( f(t) \).

**Time-Domain Analysis**

We shall now develop bounds for the error

\[
e_N(t) = f(t) - f_N(t) = \sum_{|n|>N} f(nT) \frac{\sin \omega_2(t - nT)}{\omega_2(t - nT)}.
\]

Since \( \omega_2 T = \pi \), this error can be written in the form

\[
e_N(t) = \frac{\sin \omega_2 f}{\omega_2} \left[ \sum_{n=-\infty}^{-N+1} (-1)^n f(nT) \frac{-t - nT}{t - nT} \right. \]

\[
+ \sum_{n=N+1}^{\infty} (-1)^n f(nT) \frac{t - nT}{t - nT} \right].
\]

From the above it follows that

\[
|e_N(t)| \leq \frac{|\sin \omega_2 f|}{\omega_2} \left| \sum_{|n|>N} \frac{|f(nT)|}{|t - nT|} \right|.
\]

We next show that for \( |t| < NT \)

\[
|e_N(t)| \leq \frac{|\sin \omega_2 f|}{\sqrt{\pi \omega_2}} \left\{ \left[ \frac{1}{NT} \sum_{n=-\infty}^{-N+1} f^2(nT) \right]^{1/2} \right. \]

\[
+ \left[ \frac{1}{NT} \sum_{n=N+1}^{\infty} f^2(nT) \right]^{1/2} \right\}.
\]

To prove (43) we apply the discrete form

\[
\left( \sum \alpha_n \beta_n \right) \leq \sum \alpha_n^2 \sum \beta_n^2
\]
of Schwarz's inequality to each sum in (41)
But for \( |t| \leq NT \)
\[
\sum_{n=N+1}^{\infty} \frac{T}{(l-nT)^2} \leq \int_{t=NT}^{\infty} \frac{d\tau}{(l-\tau)^2} = \frac{1}{NT-l} ;
\]
hence
\[
\left| \sum_{n=N+1}^{\infty} \frac{(-1)^n f(nT)}{t-nT} \right| \leq \frac{1}{T(NT-l)} \sum_{n=N+1}^{\infty} f^2(nT) \left[ \sum_{n=N+1}^{\infty} \frac{1}{(l-nT)^2} \right]^{1/2} .
\]
Reasoning similarly for the first sum in (41) we readily obtain the desired estimate (43). This estimate was communicated to the author by Jagerman [4].

IV. ALIASING OF HIGH FREQUENCIES

Suppose that the band limit \( \omega_1 \) of \( F(\omega) \) is larger than \( \omega_2 \)
\[
\omega_1 > \omega_2 = \frac{\pi}{T} \]
[Fig. 4(a)]. We form the sum
\[
f_a(t) = \sum_{n=-\infty}^{\infty} f(nT) \frac{\sin \omega_1(t-nT)}{\omega_1(t-nT)} .
\]
This sum is no longer equal to \( f(t) \). Its Fourier transform
\[
F_a(\omega) = F^*(\omega) \rho_\omega(\omega)
\]
[see (8)] is shown in Fig. 4(b). We denote by
\[
e_a(t) = f_a(t) - f(t)
\]
the “aliasing error” and by \( E_a(\omega) \) its transform [Fig. 4(c)]. With \( U(\omega) \) a unit-step function, \( E_a(\omega) U(\omega + \omega_2) \) is the portion of \( E_a(\omega) \) in \( \omega > 0 \). Shifting this quantity to the left by \( \omega_s \), we obtain the “quadrature component”
\[
Q(\omega) = E_a(\omega + \omega_2) U(\omega + \omega_2)
\]
shown in Fig. 4(d). This function is such that \( Q(-\omega) = -Q^*(\omega) \); hence, [2], its inverse Fourier transform is purely imaginary
\[
j q(t) \leftrightarrow Q(\omega).
\]
From the low-pass band-pass transformation theorem it follows that [2, p. 132]
\[
e_a(t) = 2q(t) \sin \omega_2 t .
\]
A simple bound on \( e_a(t) \) can be derived in terms of the area of its spectrum
\[
B = \int_{-\infty}^{\infty} |E_a(\omega)| d\omega.
\]
From the Fourier inversion formula and (46) we conclude that
\[
|q(t)| = \frac{1}{2\pi} \int_{-\infty}^{\infty} Q(\omega) e^{i\omega t} d\omega \leq \frac{B}{4\pi} .
\]

V. JITTER

In the physical applications of the sampling theory, one attempts to sense the signal \( f(t) \) at \( t=nT \) and to regenerate \( f(t) \) from its samples \( f(nT) \). In real problems, the sampled numbers are \( f(nT-\mu_n) \) where \( \mu_n \) are the deviations of the sampling times from \( nT \), and the problem is to determine \( f(t) \) in terms of \( f(nT-\mu_n) \). Timing errors in the recovery mechanism may also cause jitter in the delays of the kernel \( (\sin \omega_2 t)/\omega_2 t \); however, their effect can be treated similarly.

Nonlinear Time Transformation

In the following we assume that the delays \( \mu_n \) are known numbers and develop a method for determining \( f(t) \) from \( f(nT-\mu_n) \). To this end, we form the signal
\[
\theta(\tau) = \sum_{n=-\infty}^{\infty} \frac{\mu_n}{\omega_2(\tau-nT)} .
\]
This signal is band-limited by \( \omega_2 \), and its sample values equal \( \mu_n \) [see (11)]
\[
\theta(nT) = \mu_n .
\]
We next let \( t = \tau - \theta(\tau) \) and we assume that this function has a single-valued inverse \( \tau = \gamma(t) \)
\[
t = \tau - \theta(\tau) \quad \tau = \gamma(t) .
\]
This assumption is not too restrictive (Fig. 5.). From (49) it follows that
\[
\text{if} \quad \tau = nT, \quad \text{then} \quad t = nT - \theta(nT) = nT - \mu_n .
\]
Thus, the nonlinear transformation (50) transforms the points \( nT-\mu_n \) of the \( \tau \)-axis into the points \( nT \) of the \( t \)-axis. This transformation is useful also in other applications [5].

With
\[
g(\tau) = f(\tau - \theta(\tau)) = f(t)
\]
we see from (51) that
\[ g(nT) = f(nT - \mu_n). \] (53)

But the numbers \( f(nT - \mu_n) \) are given; hence, \( g(\tau) \) is known at a sequence of equidistant points \( \tau = nT \). If, therefore, its transform
\[ G(\omega) = \int_{-\infty}^{\infty} e^{-j\omega \tau} g(\tau) d\tau \]
is band-limited by \( \omega_0 \), then
\[ g(\tau) = \sum_{n=-\infty}^{\infty} g(nT) \frac{\sin \omega_0 (\tau - nT)}{\omega_0 (\tau - nT)} \] (54)

and, hence,
\[ f(t) = g(\gamma(t)) = \sum_{n=-\infty}^{\infty} f(nT - \mu_n) \frac{\sin \omega_0 [\gamma(t) - nT]}{\omega_0 [\gamma(t) - nT]} . \] (55)

To determine the validity of the last two equations, we shall find \( G(\omega) \). It will be assumed that \( \theta(\tau) \) is sufficiently small that we may neglect higher-order terms in the Taylor expansion of \( f[\tau - \theta(\tau)] \)
\[ g(\tau) = f[\tau - \theta(\tau)] \approx f(\tau) - \theta(\tau) f'(\tau) . \] (56)

With \( \Theta(\omega) \) and \( j\omega F(\omega) \) the transforms of \( \theta(\tau) \) and \( f'(\tau) \), respectively, we conclude from (56) and the frequency-convolution theorem [2] that
\[ G(\omega) \approx F(\omega) - \frac{1}{2\pi} \Theta(\omega) * [j\omega F(\omega)]. \] (57)

The function \( \Theta(\omega) \) equals zero for \( |\omega| \geq \omega_0 \) and the function \( F(\omega) \) equals zero for \( |\omega| \geq \omega_0 \). Hence, \( G(\omega) \) is band-limited by \( \omega_0 + \omega_0 \). Equation (54) is, therefore, only approximately true. The two sides differ by an aliasing error whose magnitude can be determined as in Section IV. If the time-jitter \( \mu_n \) is small, then this error is negligible.

**Equivalence Between Jitter Error and Round-off Error**

In many applications, the delays \( \mu_n \) are not known numbers but random variables, and the unknown signal \( f(t) \) is estimated by the sum
\[ f_j(t) = \sum_{n=-\infty}^{\infty} f(nT - \mu_n) \frac{\sin \omega_0 (t - nT)}{\omega_0 (t - nT)} . \] (58)

This sum differs from \( f(t) \) by the "jitter error"
\[ e_j(t) = f(t) - f_j(t) . \] (59)

With
\[ \delta_n = f(nT) - f(nT - \mu_n) \] (60)
the difference between the unknown \( f(nT) \) and the recorded samples \( f(nT - \mu_n) \), the jitter error, can be written in the form
\[ e_j(t) = \sum_{n=-\infty}^{\infty} \delta_n \frac{\sin \omega_0 (t - nT)}{\omega_0 (t - nT)} . \] (61)

From the above we see that \( e_j(t) \) is equivalent to the round-off error \( e_\&(t) \) in (24); hence, it can be analyzed with the techniques of Section II. However, the following problem arises: the increments \( \delta_n \) are related to the random variables \( \mu_n \) by the nonlinear transformation (60) involving the unknown function. Therefore, even if we assume that the statistical properties of the delays \( \mu_n \) are known, we cannot determine the properties of the increments \( \delta_n \). But, as we show presently, it is possible to derive some useful conclusions about \( e_j(t) \) by establishing bounds for \( \delta_n \). This can be accomplished by bounding the variation \( f(t_1) - f(t_2) \) of \( f(t) \).

With
\[ M_1 = \frac{1}{\pi} \int_{0}^{\omega_1} |F(\omega)| d\omega \]
the absolute first moment of \( |F(\omega)| \), it can be shown that [2, p. 34]
\[ |f(t_1) - f(t_2)| \leq M_1 |t_1 - t_2| \] (62)
for any \( t_1 \) and \( t_2 \). Applying the above to (60) we conclude that
\[ |\delta_n| \leq M_1 |\mu_n| . \]

We shall assume that the random variables \( \mu_n \) are independent with zero mean and equal variance
\[ E\{\mu_n\} = 0 \quad E\{\mu_n^2\} = \sigma^2 . \]

Since, for a given \( n \), \( \delta_n \) depends only on \( \mu_n \), the random variables \( \delta_n \) are also independent and
\[ E\{\delta_n^2\} \leq M_1 \sigma^2 . \] (63)

If, in an interval of order \( \sigma M_1 \), the function \( f(t) \) is sufficiently smooth that it can be approximated by a straight line, then the mean of \( \delta_n \) is also zero. It then follows from (61), (63), and (17) with \( \omega_1 = \omega_0 = \omega_2 \), \( t_1 = t_2 = t \) that
\[ E\{e^2_j(t)\} = \sum_{n=-\infty}^{\infty} E\{\delta_n^2\} \frac{\sin^2 \omega_0 (t - nT)}{\omega_0^2 (t - nT)^2} \leq M_1 \sigma^2 . \] (64)

If it is known that \( f(t) \) is positive definite, then its variation (62) and, hence, the second moment of \( \delta_n \) can be estimated with closer bounds [6].
VI. INFORMATION LOSS

In certain applications, the sampled values \( f(nT) \) of the signal \( f(t) \) are not known for every \( n \). This is not uncommon if \( f(nT) \) is stored on magnetic tape. The missing values of \( f(nT) \) might be estimated in terms of the known samples; however, we shall not consider this question. We shall assume that, in the sampling expansion (4), the missing data are replaced by zero. With this assumption, the regenerated signal can be written in the form

\[
f(t) = \sum_{n=-\infty}^{\infty} f(nT) \frac{\sin \omega_2 (t - nT)}{\omega_2 (t - nT)} - \sum_{n=-\infty}^{\infty} \beta_n f(nT) \frac{\sin \omega_2 (t - nT)}{\omega_2 (t - nT)} \tag{65}\]

where \( \beta_n = 1 \) for the values of \( n \) for which \( f(nT) \) is missing, and \( \beta_n = 0 \) otherwise. Thus the missing samples cause an error

\[
e(t) = f(t) - f(t) = \sum_{n=-\infty}^{\infty} \beta_n f(nT) \frac{\sin \omega_2 (t - nT)}{\omega_2 (t - nT)} \tag{66}\]

Sometimes not the entire number \( f(nT) \), but only part of it is lost. This is the case, for example, in digital recording when a digit channel is defective. The resulting error is then given by

\[
e_d(t) = \sum_{n=-\infty}^{\infty} \alpha_n \frac{\sin \omega_2 (t - nT)}{\omega_2 (t - nT)} \tag{67}\]

where \( \alpha_n \) takes the values 0 or \( a \), if \( a \) is the magnitude of the digit of the defective channel.

We shall assume that the numbers \( \alpha_n \) form a sequence of independent variables taking the values 0 and \( a \) with probabilities \( p \) and \( 1 - p \), respectively. We thus have

\[
E[\alpha_n] = aP[\alpha_n = a] = ap \quad E[\alpha_n^2] = a^2 p. \tag{68}\]

It can be shown that if the misses are rare, i.e., if

\[
p \ll 1 \quad \tag{69}\]

then the time instants \( t_n = nT \) such that \( \alpha_n = a \) are Poisson distributed random points with parameter \( p/T \). Thus, the probability that in a time-interval \( (t_0, t_1) \) of length \( t_1 - t_0 = t \) there are \( k \) misses, equals

\[
e^{-pt/T} \frac{(pt/T)^k}{k!}. \tag{70}\]

It is true of course that the time instants \( t_n \) are synchronized whereas this is not the case for truly random points; however, (70) is only an approximation valid for \( t \gg T \) and this fact can be ignored.

From the above it follows that, if (69) is true, then the error \( e_d(t) \) can be considered as uniform shot noise [3] generated by the function

\[
a \frac{\sin \omega_2 t}{\omega_2 t} \]

and its statistical properties can be derived from the well-known theory of shot noise [3]. In particular, one can show that [3, p. 358] the mean of \( e_d(t) \) is given by

\[
E[e_d(t)] = \frac{p}{T} \int_{-\infty}^{\infty} \sin \omega_2 t \frac{dt}{\omega_2 t} = \frac{ap}{\omega_2 T} = ap \tag{71}\]

and its autocovariance by

\[
C(r) = \frac{p}{T} \int_{-\infty}^{\infty} a^2 \sin \omega_2 (t + r) \sin \omega_2 t \frac{dt}{\omega_2 (t + r)} = a^2 p \frac{\sin \omega_2 r}{\omega_2 r}. \tag{72}\]

Comparing (67) with (24) we see that the error \( e_d(t) \) is of the same form as the round-off error \( e_r(t) \). In fact, (71) is a special case of (26). The variance of \( \alpha_n \), however, equals \( a^2 p - a^2 p^2 \). It would seem, therefore, that (72) is not in agreement with (27). But (72) was valid only for \( p < 1 \) and, if this is true, then \( p - p^2 \approx p \); hence, (72) agrees with (27). The error \( e_l(t) \) can be analyzed similarly.

VII. PHYSICAL REGENERATION OF A BAND-LIMITED SIGNAL

We shall now investigate the possibility of regenerating a band-limited signal \( f(t) \) from its sampled values \( f(nT) \) by a physical device. This can be done, in principle, by creating a sequence of equidistant impulses (Fig. 6)

\[
f^*(t) = \sum_{n=-\infty}^{\infty} T f(nT) \delta(t - nT) \tag{73}\]

of area \( T f(nT) \) and passing \( f^*(t) \) through an ideal filter. However, impulses cannot be created and ideal filters do not exist, even if no limitation is placed on their complexity. Thus, the above scheme can be only approximately realized, and the purpose of this section is to discuss means for improving the approximation.

![Fig. 6. Ideal low-pass filter for regeneration of f(t) when sampling rate is minimum.](image)

FILTER REQUIREMENTS AND CHOICE OF SAMPLING RATE

Since the transform of \( \delta(t - nT) \) equals \( e^{-jnT\omega} \), the transform of \( f^*(t) \) is given by the sum

\[
F^*(\omega) = \sum_{n=-\infty}^{\infty} T f(nT) e^{-jnT\omega}. \tag{74}\]
Comparing (74) with (8), we conclude that $F^*(\omega)$ is the periodic repetition of $F(\omega)$

$$F^*(\omega) = \sum_{n=-\infty}^{\infty} F\left(\omega + \frac{2\pi n}{T}\right). \quad (75)$$

Suppose that $f^*(t)$ is the input to a system with impulse response $h(t)$ and system function $H(\omega)$. For the output to equal $f(t)$, $H(\omega)$ must be such that

$$F^*(\omega)H(\omega) = F(\omega). \quad (76)$$

We shall assume that the cutoff frequency $\omega_1$ of $F(\omega)$ is specified. The requirements on storage capacity demand that $T$ be taken as large as possible. If we assign to it the maximum permissible value

$$T = \frac{\pi}{\omega_1}$$

then, in order to satisfy (76), we must have

$$H(\omega) \approx p_{\omega_1}(\omega)$$

(Fig. 6). Thus, the filter must be ideal low-pass with cutoff frequency $\omega_1$. Such a filter is, of course, impossible to realize and difficult to approximate.

The demands on the filter can be considerably relaxed if the sampling rate $1/T$ is increased. It is easy to see from (76) that if

$$\omega_2 = \frac{\pi}{T} > \omega_1$$

then the output

$$\sum_{n=-\infty}^{\infty} f(nT)h(t-nT)$$

of the filter equals $f(t)$ provided $H(\omega)$ is such that [see also (12)]

$$H(\omega) = 1 \quad \text{for} \quad |\omega| < \omega_1 \quad (77a)$$

$$H(\omega) = 0 \quad \text{for} \quad |\omega| > 2\omega_2 - \omega_1 \quad (77b)$$

$$H(\omega) \text{arbitrary for} \quad \omega_1 \leq |\omega| \leq 2\omega_2 - \omega_1 \quad (77c)$$

(Fig. 7). By increasing $\omega_2$ we have thus removed the requirement of a sharp cutoff. The existence of a free attenuation interval $(\omega_1, 2\omega_2-\omega_1)$ facilitates the approximate realization of the above conditions by a real filter (with a delay to be discussed shortly). Such a filter cannot, of course, satisfy (77b) exactly (see Paley-Wiener condition [2, p. 215]). The high-frequency components of $F^*(\omega)$ cannot be entirely eliminated, with the result that the output of the filter is only approximately equal to $f(t)$. The error depends on the choice of the filter and decreases with decreasing $T$.

From the foregoing we see that the size of $T$ is dictated by two conflicting factors: storage capacity and filter requirements. The final choice depends on the specific problem.

**Real Pulses**

In the preceding discussion, it was assumed that the input to the filter $H(\omega)$ is a sequence of impulses. Such a sequence is, of course, not possible to realize. However, as we show presently, it is not necessary even to attempt to approximate it. The unknown $f(t)$ can be recovered using as input to a suitable filter $H_1(\omega)$ the signal

$$f_0(t) = \sum_{n=-\infty}^{\infty} f(nT)h_0(t-nT) \quad (78)$$

(Fig. 8) where $h_0(t)$ is an arbitrary function. The choice of $h_0(t)$ is dictated by the ease of realizing it.

The requirements on the filter $H_1(\omega)$ can be easily determined from the preceding discussion. Clearly, $f_0(t)$ can be considered as the output of a hypothetical filter with input an impulse train $f^*(t)$ as in (73), and impulse response $h_0(t)$. With $H_0(\omega)$ the Fourier transform of $h_0(t)$

$$h_0(t) \leftrightarrow H_0(\omega)$$

we easily conclude that if the product $H(\omega) = H_1(\omega)H_0(\omega)$ satisfies (76) then the output of the two filters $H_0(\omega)$ and $H_1(\omega)$ in cascade, with input $f^*(t)$, will equal $f(t)$. Therefore, the output of $H_1(\omega)$ alone with input $f_0(t)$ will also equal $f(t)$. Thus,

$$H_1(\omega) = \frac{H(\omega)}{H_0(\omega)}$$

with $H(\omega)$ any function satisfying (76).

**Example**

Suppose that $h_0(t)$ is a pulse

$$h_0(t) = \begin{cases} 1 & |t| \leq \delta \\ \frac{1}{2\delta} & |t| > \delta \end{cases}$$

as in Fig. 9. Since its transform

$$H_0(\omega) = \frac{\sin \delta \omega}{\delta \omega}$$

is small for large $\omega$, the presence of $h_0(t)$ causes a reduction in the high-frequency content of the input to $H_1(\omega)$, and so its attenuation requirements can be relaxed. However, since $H_0(\omega)$ is not constant for $|\omega| < \omega_1$, the spectrum of $F(\omega)$ is distorted, and this distortion must be compensated by $H_1(\omega)$.
Fig. 9. Spectrum of impulse train, and output filter.

Thus, to satisfy (77a), we must have

$$H_1(\omega) = \frac{\delta \omega}{\sin \delta \omega} \quad \text{for} \quad |\omega| < \omega_1.$$  \hspace{1cm} (79)

A simpler expression can be obtained if we assume that $\delta \omega_1 \ll 1$. Since

$$\frac{X}{\sin X} \approx \frac{X}{X - X^3/3!} \approx 1 + \frac{X^2}{6} \quad \text{for} \quad |X| \ll 1$$

(79) yields

$$H_1(\omega) \approx 1 + \frac{X^2}{6} \quad \text{for} \quad |\omega| \leq \omega_1$$

which is not particularly difficult to realize.

We close with a final remark. It is clear from the sampling expansion (4) that, for a specific $t$, the function $f(t)$ depends in general on all its samples, past and future. But the output of a physical filter, with input the sequence of impulses $f^*(t)$, equals

$$\sum_{n=-\infty}^{[t/T]} T f(nT) h(t-nT)$$

because $h(t) = 0$ for $t < 0$. And since this sum contains only past values of $f(nT)$, it cannot equal $f(t)$, no matter how $h(t)$ is chosen. This apparent contradiction to our previous conclusion is easily explained from the fact that (77b) can be approximately satisfied only if sufficient delay is allowed. In other words, (76) must read

$$F^*(\omega) H(\omega) = F(\omega) e^{-j\omega t_0}.$$  \hspace{1cm} (80)

The output of the filter is no longer $f(t)$ but it equals $f(t - t_0)$. Thus, to evaluate $f(t)$ for $t = t_1$, we may use the input up to time $t_1 + t_0$. To satisfy (80) exactly, one must make $t_0$ infinite.

REFERENCES


Cylindrical Waveguide as a Power Transmission Medium—Limitations Due to Mode Conversion

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Abstract—A cylindrical waveguide operating in the $\text{TE}_{01}$ mode can transmit powers of the order of thousands of megawatts in frequency ranges below $10$ GHz with acceptable copper losses over distances of hundreds of miles, provided it can be made with sufficient accuracy to avoid excessive power transfer into other unwanted high-loss modes. The tolerances on the guide straightness appear to be the most severe limitation on the use of this type of guide as a high power system component. These tolerances are evaluated here and found to be of the order of $10$ to $100$ mils in distances of a few hundred feet for typical values of the other parameters such as $1.0$ GHz in a 2-meter-diameter guide providing 1 dB power loss in a distance of 200 miles. Tolerances on diameter variations and cross-section ellipticity are also evaluated and found to be of the same order of magnitude as the deviations from straightness, but should be easier to control by the proper manufacturing process. It is also shown that low loss bends (about 0.01 dB/mile) with bending radii of the order of $0.5$ mile are possible if the length of the bend is accurately controlled and the guide has a dielectric lining of the proper thickness.

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