6. Counting and Probability

6. Counting and Probability

The study of arrangements of objects in an important part of discrete mathematics. We must count many different types of problems. For example, counting is use to determine whether there are enough telephone numbers to meet demand. Furthermore, counting techniques are used extensively when probabilities of events are computed.

6.1 Sum Rule, Product Rule, Inclusion-Exclusion Principle

Counting problems arise throughout mathematics and computer science. There are two basic counting principles, namely, the *sum rule* and the *product rule*.

6.1.1 The Sum Rule

If a first task can be done in n_1 ways and a second task in n_2 ways, and if these tasks cannot be done at the same time, then there are $n_1 + n_2$ ways to do either task.

Example 6.1-1

A number is chosen from the set of integers from 1 to 20 inclusive. If it is either an odd number or a multiple of 4, find the number of ways to choose this number.

Solution

The first task, choosing an odd number, can be done in 10 ways. The second task, choosing a multiple of 4, can be done in 5 ways. Since these two tasks cannot be done at the same time, from the sum rule there are 10 + 5 = 15 possible ways to choose this number.

We can extend the sum rule to more than two tasks. Suppose that the tasks T_1, T_2, ……, T_m can be done in n_1, n_2, ……, n_m ways, respectively, and no two of these tasks can be done at the same time. Then the number of ways to do one of these tasks is $n_1 + n_2 + \ldots\ldots + n_m$.

Example 6.1-2

A customer can choose a mobile phone number from one of four lists. The four lists contain 24, 15, 32 and 10 possible numbers, respectively. How many possible numbers are there to choose from?

Solution

The customer can choose a mobile phone number from the first list in 24 ways, from the second list in 15 ways, from the third list in 32 ways and from the fourth list in 10 ways. Hence, there are 24 + 15 + 32 + 10 = 81 numbers to choose from.

If A and B are disjoint sets, then the number of elements in the union of these two sets is the sum of the numbers of elements in them. If no two tasks can be done at the same time, the number of ways to choose an element from one of the sets is equal to the number of elements in the union. Therefore the sum rule can be phrased in terms of sets as follows:

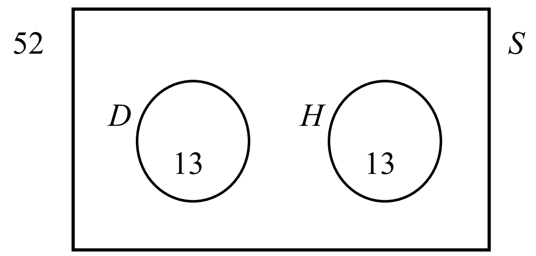
$$|A \cup B| = |A| + |B|$$

Example 6.1-3

A card is drawn at random from an ordinary pack of 52 playing cards. Find the number of ways that the card is a diamond or a heart.

Solution

Let S be the set of the pack of 52 cards, D be the set of diamond cards and H be the set of heart cards.



Since a card cannot be both a diamond and a heart, then

$$|D \cup H| = |D| + |H| = 13 + 13 = 26$$

i.e. the number of ways to draw a card of diamond or heart is 26.

In general, if $A_1, A_2, \ldots\ldots, A_m$ are disjoint sets, then the number of ways to choose an element from one of the set is equal to the number of elements in the union.

i.e. $$|A_1 \cup A_2 \cup \ldots\ldots \cup A_m| = |A_1| + |A_2| + \ldots\ldots + |A_m|$$

6.1.2 The Product Rule

Suppose that a procedure can be broken down into two tasks. If there are n_1 ways to do the first task and n_2 ways to do the second task after the first task has been done, then there are $n_1 n_2$ ways to do the procedure.

Example 6.1-4

The seats in a mass lecture hall are to be labeled with a letter and a positive integer not greater than 10. How many seats can be labeled differently?

Solution

The procedure of labeling a seat consists of two tasks. The first task is assigning one of the 26 letters and then the second task is assigning one of the 10 positive integers. By the product rule, there are 26 x 10 = 260 ways to label a seat. Therefore, 260 seats can be labeled differently.

In general, we can extend the product rule to any number of tasks. Suppose that a procedure is carried out by performing the tasks T_1, T_2, ……, T_m. If task T_i can be done in n_i ways after the tasks T_1, T_2, ……, and T_{i-1} have been done, then there are $n_1 n_2 \dots n_m$ ways to carry out the procedure.

Example 6.1-5

A password on a computer system consists of six characters. Each of these characters must be a digit. How many passwords are there?

Solution

Each of the six characters can be chosen in 10 ways, since each digit is from 0 to 9. Therefore, by the product rule, there are 10^6 = 1,000,000 different passwords.

If A_1, A_2, ……, A_m are finite sets, then the number of ways to choose an element in the Cartesian product of these sets is equal to the product of the number of elements in each set. Therefore the product rule can be phrased in terms of sets as follows:

$$|A_1 \times A_2 \times \dots \times A_m| = |A_1| \cdot |A_2| \cdots |A_m|$$

Notice that many counting problems cannot be solved using just the sum rule or the product rule but both of these rules.

Example 6.1-6
Suppose the name of a variable in a computer language can be either a single letter or a letter followed by a digit. Find the number of possible names.

Solution
Let N be the set of different variable names. We can partition the set N into the disjoint subsets consisting of the set of single letter names L and the set of names of single letter followed by a digit D.

Since $N = L \cup D$, then $|N| = |L| + |D|$ by the sum rule. Note that $|L| = 26$, and by the product rule $|D| = 26 \times 10 = 260$. Therefore $|N| = 26 + 260 = 286$.

6.1.3 The Inclusion-Exclusion Principle

When two tasks can be done at the same time, the number of ways to do one of the two tasks is add the number of ways to do each of the two tasks and then subtract the number of ways to do both tasks. In set notation,

$$|A \cup B| = |A| + |B| - |A \cap B|$$

where A and B are not disjoint.

Example 6.1-7
Count the number of bit strings of length ten which begin with a 1010 or end with a 00.

Solution
Let A be the set of strings of ten which begin with 1010 and B be the set of strings of ten which end with 00. Note that A and B are not disjoint.
For A, the first four bits can be chosen in only one way and each of the following six bits can be chosen in two ways, then $|A| = 2^6 = 64$.
Similar for B, the last two bits can be chosen in only one way and each of the first eight bits can be chosen in two ways, then $|B| = 2^8 = 256$.
Both sets contain the bit strings begin with 1010 and end with 00, for which, the first four bits and the last two bits can be chosen in only one way and the other four bits can be chosen in two ways. Then $|A \cap B| = 2^4 = 16$.
Therefore $|A \cup B| = 64 + 256 – 16 = 304$, i.e. there is 304 number of bit strings of length ten begin with a 1010 or end with a 00.

Example 6.1-8

In a class of 40 students, 25 students take computer science and 30 students take additional mathematics. All the 40 students must take at least one of these subjects. How many students take both subjects?

Solution

Let C be the set of students take computer science and A be the set of students take additional mathematics. Then $|C| = 25$, $|A| = 30$, and $|C \cup A| = 40$.

By the inclusion-exclusion principle,

$$|C \cup A| = |C| + |A| - |C \cap A|$$
$$40 = 25 + 30 - |C \cap A|$$
$$|C \cap A| = 15$$

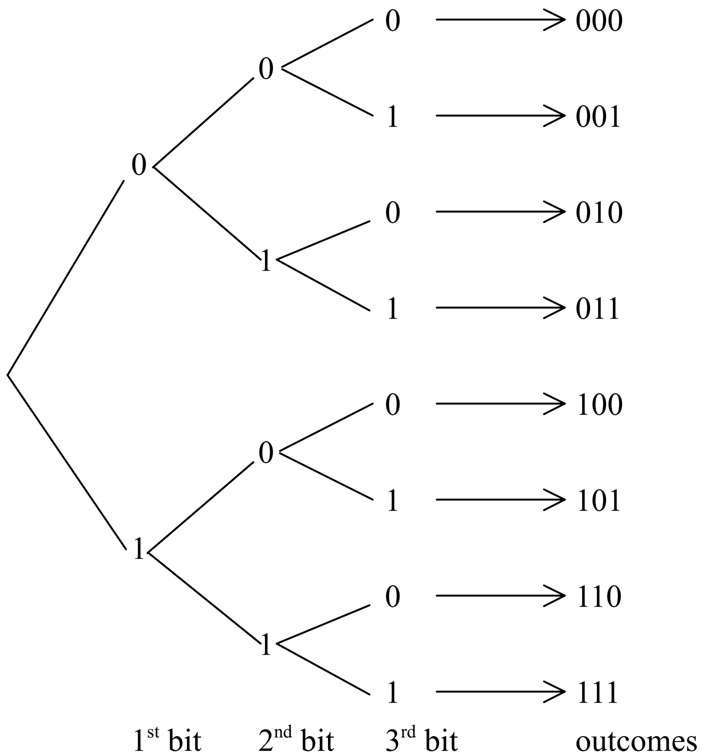
i.e. 15 students take both subjects.

A useful way of tackling counting problems is to draw a **tree diagram**. The method is illustrated in the following example.

Example 6.1-9

List all the bit strings of length three by means of a tree diagram. Hence find the number of bit strings of length three which do not have two consecutive 0s.

Solution



From the tree diagram, there are five bit strings of length three without two consecutive 0s.

6.2 Permutations and Combinations

In this section, techniques will be introduced for counting the unordered selections of distinct objects and the ordered arrangements of objects of a finite set.

6.2.1 Arrangements

The number of ways of arranging n unlike objects in a line is $n!$.

Note: $n! = n\,(n - 1)\,(n - 2) \cdot \cdot \cdot 3 \times 2 \times 1$

Example 6.2-1

Find the number of ways of arranging the letters A, B and C.

Solution

The first letter can be chosen in three ways, either A or B or C. Then, the second letter can be chosen in two ways and the third letter can be chosen in only one way.
Therefore, by the product rule, the number of ways of arranging the three letters is
$3 \times 2 \times 1 = 3! = 6$.
The arrangements are ABC, ACB, BCA, BAC, CAB and CBA.

Example 6.2-2

It is known that the password on a computer system contain the three letters A, B and C followed by the six digits 1, 2, 3, 4, 5, 6. Find the number of possible passwords.

Solution

There are 3! ways of arranging the letters A, B and C, and 6! ways of arranging the digits 1, 2, 3, 4, 5, 6.
Therefore the total number of possible passwords is $3! \times 6! = 4320$.
i.e. 4320 different passwords can be formed.

The number of ways of arranging in a line n objects, of which p are alike, is $\frac{n!}{p!}$.

Example 6.2-3

Find the number of ways of arranging the letters A, B and B.

Solution

In stead of the letters A, B and C we have A, B and B, then the 6 arrangements listed in example 6.2-1 reduced to ABB, BBA and BAB.

Therefore the number of arranging the 3 letters, of which 2 are alike, is $\frac{3!}{2!} = 3$.

The result can be extended as follows:
The number of ways of arranging in a line n objects of which p of one type are alike, q of a second type are alike, r of a third type are alike, and so on, is $\dfrac{n!}{p!q!r!\cdots}$.

Example 6.2-4
Find the number of ways that the letters of the word STATISTICS can be arranged.

Solution
The word STATISTICS contains 10 letters, in which S occurs 3 times, T occurs 3 times and I occurs twice.
Therefore the number of ways is $\dfrac{10!}{3!3!2!}$ =50400.
That is, there are 50400 ways of arranging the letter in the word STATISTICS.

Example 6.2-5
A six-digit number is formed from the digits 1, 1, 2, 2, 2, 5 and repetitions are not allowed. How many these six-digit numbers are divisible by 5?

Solution
If the number is divisible by 5 then it must end with the digit 5. Therefore the number of these six-digit numbers which are divisible by 5 is equal to the number of ways of arranging the digits 1, 1, 2, 2, 2.
Then, the required number is $\dfrac{5!}{2!3!}$ = 10.
That is, there are 10 of these six-digit numbers are divisible by 5.

6.2.2 Permutations

A *permutation* of a set of distinct objects is an ordered arrangement of these objects. An ordered arrangement of r elements of a set is called an *r-permutation*.

The number of r-permutations of a set with n distinct elements, i.e. the number of permutations of r objects taken from n unlike objects is

$$P(n, r) = \frac{n!}{(n-r)!} = n(n-1)(n-2)\cdot\cdots\cdot(n-r+1)$$

Note: 0! is defined to 1, so $P(n, n) = \dfrac{n!}{(n-n)!} = \dfrac{n!}{0!} = n!$

Example 6.2-6

Find the number of ways of placing 3 of the letters A, B, C, D, E in 3 empty spaces.

Solution

The first space can be filled in 5 ways. The second space can be filled in 4 ways. The third space can be filled in 3 ways. Therefore there are 5 x 4 x 3 ways of arranging 3 letters taken from 5 letters.

This is the number of permutations of 3 objects taken from 5 and it is written as $P(5, 3)$, so $P(5, 3) = 5 \times 4 \times 3 = 60$.

On the other hand, 5 x 4 x 3 could be written as $\frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1} = \frac{5!}{2!} = \frac{5!}{(5-3)!} = P(5, 3)$.

Notice that the order in which the letters are arranged is important --- ABC is a different permutation from ACB.

Example 6.2-7

How many different ways are there to select one chairman and one vice chairman from a class of 20 students.

Solution

The answer is given by the number of 2-permutations of a set with 20 elements. This is $P(20, 2) = 20 \times 19 = 380$.

Example 6.2-8

There are ten runners in a race and suppose that they have equal chance to win the race. The champion receives a gold medal, the runners-up receives a silver medal, and the second runners-up receives a bronze medal. How many different ways are there to award these medals?

Solution

The number of different ways to award the medals is the number of 3-permutations of a set with 10 elements. Hence, there are $P(10, 3) = 10 \times 9 \times 8 = 720$ possible ways to award the medals.

6.2.3 Combinations

An *r-combination* of elements of a set is an unordered selection of *r* elements from the set. Thus, an *r*-combination is simply a subset of the set with *r* elements.

The number of *r*-combinations of a set with *n* elements, where *n* is a positive integer and *r* is an integer with $0 \leqq r \leqq n$, i.e. the number of combinations of *r* objects from *n* unlike objects is

$$C(n, r) = \frac{n!}{r!\,(n-r)!}\,.$$

There is another common notation for the number of *r*-combinations from a set with *n* elements, namely, $\binom{n}{r}$.

Example 6.2-9

Find the number of combinations of choosing three letters from the five letters A, B, C, D, E.

Solution

Consider example 6.2-6, the one combination ABC gives rise to 3! permutations ABC, ACB, BAC, BCA, CAB and CBA.

So if the number of combinations of 3 letters from the 5 letters A, B, C, D, E is denoted by $C(5, 3)$, then $C(5, 3) \times 3! = P(5, 3)$.

Therefore $C(5, 3) = \dfrac{P(5, 3)}{3!} = \dfrac{5!}{3!\,2!} = \dfrac{5!}{3!\,(5-3)!} = 10$.

Example 6.2-10

How many different ways are there to select two class representatives from a class of 20 students?

Solution

The answer is given by the number of 2-combinations of a set with 20 elements.

The number of such combinations is $C(20, 2) = \dfrac{20!}{2!\,18!} = 190$.

Example 6.2-11

In how many ways can a hand of 5 cards be dealt from an ordinary pack of 52 playing cards?

Solution

We need to consider combinations, as order in which the cards are dealt is not important. Therefore the number of ways is $C(52, 5) = 2{,}598{,}960$.

Example 6.2-12

A committee of 5 members is chosen at random from 6 faculty members of the mathematics department and 8 faculty members of the computer science department. In how many ways can the committee be chosen if (a) there are no restrictions; (b) there must be more faculty members of the computer science department than the faculty members of the mathematics department.

Solution

(a) There are 14 members, from whom 5 are chosen. The order in which they are chosen is not important. So the number of ways of choosing the committee is $C(14, 5) = 2002$.

(b) If there are to be more faculty members of the computer science department than the faculty members of the mathematics department, then the following conditions must be fulfilled.

(i) 5 faculty members of the computer science department
The number of ways of choosing is $C(8, 5) = 56$.

(ii) 4 faculty members of the computer science department and 1 faculty member of the mathematics department
The number of ways of choosing is $C(8, 4) \times C(6, 1) = 70 \times 6 = 420$.

(iii) 3 faculty members of the computer science department and 2 faculty members of the mathematics department
The number of ways of choosing is $C(8, 3) \times C(6, 2) = 56 \times 15 = 840$.

Therefore the total number of ways of choosing the committee is $56 + 420 + 840 = 1316$.

6.3 Events and Probability

What is the probability that a person will win a lottery where 6 numbers are chosen from the first 46 positive integers? The theory of probability is arisen in the study of gambling games. The French mathematician *Blaise Pascal* first developed it in seventeenth century. In eighteenth century, another French mathematician *Laplace* gave a definition of the probability of an event. They both studied gambling.

6.3.1 Finite Probability

An *experiment* is a procedure that yields one set of possible outcomes. The *sample space* is the set of possible outcomes. An *event* is a subset of the sample space. Each possible outcome is called a *sample point* and the set of all possible outcomes is the *possibility space S*.

If the possibility space is finite, then the number of sample points in S is denoted by $n(S)$. Also $n(E)$ denote the number of sample points in an event E, clearly $n(E) \leqq n(S)$.

Consider an example for one throw of a fair die the possibility space

$$S = \{1, 2, 3, 4, 5, 6\} \text{ and } n(S) = 6.$$

Let E_1 be the event that the number is even, then

$$E_1 = \{2, 4, 6\} \text{ and } n(E_1) = 3.$$

Let E_2 be the event that the number is greater than 2, then

$$E_2 = \{3, 4, 5, 6\} \text{ and } n(E_2) = 4.$$

Laplace's definition of probability: If the possibility space S consists of a finite number of equally likely outcomes, then the probability of an event E, written $P(E)$ is defined as

$$P(E) = \frac{n(E)}{n(S)}$$

Refer to the pervious example,

$$P(E_1) = \frac{n(E_1)}{n(S)} = \frac{3}{6} = \frac{1}{2}$$

and $$P(E_2) = \frac{n(E_2)}{n(S)} = \frac{4}{6} = \frac{2}{3}$$

6.3.2 Certain Events and Impossible Events

Suppose there are n sample points in the possibility space and r sample points in an event E, so that $n(S) = n$ and $n(E) = r$.

The probability of the event E occurs is $P(E) = \dfrac{n(E)}{n(S)} = \dfrac{r}{n}$.

Since $0 \leqq r \leqq n$, then $0 \leqq \dfrac{r}{n} \leqq 1$, hence $0 \leqq P(E) \leqq 1$.

That is, the probability of an event E is between 0 and 1 inclusive.

If $P(E) = 0$ then the event cannot happen, i.e. *impossible event*.
If $P(E) = 1$ then the event is certain to happen, i.e. *certain event*.

For example, if a coin with both heads is tossed, the following probabilities will be obtained.

$$P(\text{a tail is obtained}) = 0$$
$$P(\text{a head is obtained}) = 1$$

6.3.3 Complementary Events

Let E be an event in a sample space S. The probability of the event $\overline{E}$, the *complementary event E*, is given by $P(\overline{E}) = 1 - P(E)$.

Proof: Since $n(\overline{E}) = n(S) - n(E)$, then $P(\overline{E}) = \dfrac{n(S) - n(E)}{n(S)} = 1 - \dfrac{n(E)}{n(S)} = 1 - P(E)$.

Example 6.3-1

A sequence of 8 bits is randomly generated. What is the probability that at least one of these bits is 1?

Solution

The possibility space S is the set of all bit strings of length 8, so that $n(S) = 2^8$.

Let E be the event that at least one of the 8 bits is 1, then $\overline{E}$ is the event that all the bits are 0s and $n(\overline{E}) = 1$. It follows that

$$P(E) = 1 - P(\overline{E}) = 1 - \frac{n(\overline{E})}{n(S)} = 1 - \frac{1}{2^8} = 1 - \frac{1}{256} = \frac{255}{256}.$$

Hence, the probability that the bit string will contain at least one 1 bit is $\dfrac{255}{256}$. It is difficult to obtain the probability directly.

6.3.4 The Probability of Union

Let E_1 and E_2 be the events in the sample space S such that $P(E_1)\neq 0$ and $P(E_2)\neq 0$.
Then

$$P(\ E_1 \text{ or } E_2\ ) = P(\ E_1\ ) + P(\ E_2\ ) - P(\ E_1 \text{ and } E_2\ ).$$

Note that 'E_1 or E_2' means 'E_1 occurs, or E_2 occurs, or both E_1 and E_2 occur'.

In set notation $\quad P(\ E_1 \cup E_2\ ) = P(\ E_1\ ) + P(\ E_2\ ) - P(\ E_1 \cap E_2\ ).$

Example 6.3-2

In a class of 40 students, 6 out of 15 boys and 13 out of 25 girls wear glass. What is the probability that a student chosen at random from the class is a boy or someone who wears glasses?

Solution

Let B be the event that the student chosen is a boy and let W be the event that the student chosen wears glasses. Then

$$P(\ B \cup W\ ) = P(\ B\ ) + P(\ W\ ) - P(\ B \cap W\ ) = \frac{15}{40} + \frac{19}{40} - \frac{6}{40} = \frac{28}{40} = \frac{7}{10}.$$

Therefore, the probability that a student chosen at random from the class is a boy or someone who wears glasses is 0.7.

Example 6.3-3

What is the probability that a positive integer selected at random from the set of positive integers from 1 to 60 inclusive is divisible by either 3 or 4?

Solution

Let A be the event that the integer selected is divisible by 3 and let B be the event that the integer selected is divisible by 4. Then $A \cap B$ is the event that it is divisible by both 3 and 4, that is divisible by 12. While $A \cup B$ is the event that it is divisible by both 3 or 4.

Now, $n(\ A\ ) = \frac{60}{3} = 20$, $n(\ B\ ) = \frac{60}{4} = 15$, and $n(\ A \cap B\ ) = \frac{60}{12} = 5$.

Then, $P(\ A \cup B\ ) = P(\ A\ ) + P(\ B\ ) - P(\ A \cap B\ ) = \frac{20}{60} + \frac{15}{60} - \frac{5}{60} = \frac{30}{60} = \frac{1}{2}$.

6.3.5 Mutually Exclusive Events

Let E_1 and E_2 be the events in the sample space S such that E_1 can occur or E_2 can occur but not both E_1 and E_2 can occur, then the two events E_1 and E_2 are said to be *mutually exclusive*. In this case $n(E_1 \cap E_2) = 0$ and $E_1 \cap E_2 = \phi$.

In set notation, when E_1 and E_2 are mutually exclusive events

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) \qquad \text{and} \qquad P(E_1 \cap E_2) = 0$$

This is known as the *addition law* for mutually exclusive events.

Example 6.3-4

Suppose that there are eight runners in a race including John, David and Albert. The probability that John wins the race is $\frac{1}{2}$. David wins the race is $\frac{1}{4}$ and Albert wins the race is $\frac{1}{8}$. Assume there are no dead heats, find the probability that (a) John or David or Albert wins, (b) neither John nor David wins.

Solution

Since we assume that only one runner can win, the events above are mutually exclusive. Let J be the event that John wins the race, D be the event that David wins the race and A be the event that Albert wins the race.

(a) Probability that John or David or Albert wins the race is $P(J \text{ or } D \text{ or } A)$

$$= P(J) + P(D) + P(A) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}.$$

(b) Probability that neither John nor David wins the race is $1 - P(J \text{ or } D)$

$$= 1 - \left(\frac{1}{2} + \frac{1}{4} \right) = \frac{1}{4}.$$

Example 6.3-5

A card is drawn at random from an ordinary pack of 52 playing cards. Find the probability that the card is (a) a heart or a club, (b) a red card or a club, (c) a red card or a king.

Solution

The possibility space S is the pack of 52 playing cards. Let H be the event that a heart is drawn, C be the event that a club is drawn, R be the event that a red card is drawn and K be the event that a king is drawn.
In this case $n(S) = 52$, $n(H) = 13$, $n(C) = 13$, $n(R) = 26$ and $n(K) = 4$. Note that the events H and C, R and C are mutually exclusive events but R and K are not.

(a) Probability that the card is a heart or a club is $P(H \text{ or } C)$
$= P(H) + P(C) = \dfrac{n(H)}{n(S)} + \dfrac{n(C)}{n(S)} = \dfrac{13}{52} + \dfrac{13}{52} = \dfrac{1}{2}$.

(b) Probability that the card is a red card or a club is $P(R \text{ or } C)$
$= P(R) + P(C) = \dfrac{n(R)}{n(S)} + \dfrac{n(C)}{n(S)} = \dfrac{26}{52} + \dfrac{13}{52} = \dfrac{3}{4}$.

(c) Probability that the card is a king or a red card is $P(R \text{ or } K)$
$= P(R) + P(K) - P(R \text{ and } K) = \dfrac{n(R)}{n(S)} + \dfrac{n(K)}{n(S)} - \dfrac{n(R \cap K)}{n(S)} = \dfrac{26}{52} + \dfrac{4}{52} - \dfrac{2}{52} = \dfrac{7}{13}$.

6.3.6 Exhaustive Events

Let E_1 and E_2 be the events in the sample space S such that $E_1 \cup E_2 = S$ then $P(E_1 \cup E_2) = 1$. The events E_1 and E_2 are said to be *exhaustive*.

Example 6.3-6
Two fair coins are tossed. A is the event that at least one tail is obtained.
(a) Describe an event B such that A and B are exhaustive events only.
(b) Describe an event C such that A and C are both mutually exclusive and exhaustive.

Solution
(a) The possibility space $S = \{HH, HT, TH, TT\}$ and the event $A = \{HT, TH, TT\}$.
Let B be the event that at least one head is obtained, then $B = \{HH, HT, TH\}$.
Since $A \cup B = \{HH, HT, TH, TT\} = S$, A and B are exhaustive events.

(b) Let C be the event that no tail is obtained, then $C = \{HH\}$. Since $A \cup C = \{HH, HT, TH, TT\} = S$ and $A \cap C = \phi$, A and C are both mutually exclusive and exhaustive.

Example 6.3-7

A and B are two events such that $P(A) = \frac{2}{3}$, $P(B) = \frac{2}{5}$ and $P(A \cap B) = \frac{1}{15}$. Are A and B exhaustive events?

Solution

Since $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{2}{3} + \frac{2}{5} - \frac{1}{15} = 1$, A and B are exhaustive.

Example 6.3-8

In a class of 40 students all study at least one of the subjects computer science and discrete mathematics. 27 attend the computer science class and 32 attend the discrete mathematics class. Find the probability that a student chosen at random studies both computer science and discrete mathematics.

Solution

Let C be the event that the student chosen is study computer science and let M be the event that the student chosen is study discrete mathematics. Since all students study at least one of the subjects computer science and discrete mathematics, C and M are exhaustive events. Then

$$n(C \cup M) = n(C) + n(M) - n(C \cap M)$$
$$40 = 27 + 32 - n(C \cap M)$$
$$n(C \cap M) = 19$$

Therefore the probability that a student chosen at random studies both computer science and discrete mathematics is $P(C \cap M) = \frac{19}{40}$.

6.4 Conditional Probability

Suppose that an ordinary die is thrown three times, the event E_1, that a 6 appears in the first time, occurs. Given this information, what is the probability of the event E_2, that two 6s appears? This probability is called the *conditional probability* of E_2 given E_1. Does E_1 change the probability of E_2? If not, E_1 and E_2 are called *independent*.

6.4.1 Conditional Probability

Let E and F are two events with $P(E) \neq 0$ and $P(F) \neq 0$. The probability of E, given that F has already occurred, denoted by $P(E \mid F)$, is defined as

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)}$$

This result is often written as

$$P(E \cap F) = P(E \mid F)\, P(F)$$

Note that if E and F are mutually exclusive events then, as $P(E \cap F) = 0$ and $P(F) \neq 0$, it follows that $P(E \mid F) = 0$.

Also $P(E) = P(E \cap F) + P(E \cap \overline{F})$
$= P(E \mid F)\, P(F) + P(E \mid \overline{F})\, P(\overline{F})$

Example 6.4-1
Given that a black card is picked at random from a pack of 52 playing cards, find the probability that it is a spade.

Solution
Let B be the event that a black card is picked, S be the event that a spade is picked.

The required probability is $P(S \mid B) = \dfrac{P(S \cap B)}{P(B)} = \dfrac{13/52}{26/52} = \dfrac{1}{2}$.

Example 6.4-2
When an ordinary die is thrown, find the probability that (a) the number is prime, given that an odd number occurs, (b) the number is prime, given that an even number occurs.

Solution
Let π be the event that a prime number occurs, O be the event that an odd number occurs, E be the event that an even number occurs.

(a) Probability that the number is prime, given that an odd number occurs is $P(\pi \mid O)$

$$= \frac{P(\pi \cap O)}{P(O)} = \frac{2/6}{3/6} = \frac{2}{3}.$$

(b) The required probability is $P(\pi \mid E) = \dfrac{P(\pi \cap E)}{P(E)} = \dfrac{1/6}{3/6} = \dfrac{1}{3}$.

Example 6.4-3

Of a group of computer science major pupils, 70% are boys and 30% are girls. The probability that a boy in this group study discrete mathematics is 0.8 and the probability that a girl in this group study discrete mathematics is 0.6.

(a) Find the probability that a pupil selected at random from this group is (i) a boy study discrete mathematics, (ii) a girl study discrete mathematics, (iii) study discrete mathematics.

(b) Find the probability that a discrete mathematics pupil selected at random from this group is (i) a boy, (ii) a girl.

Solution

Let M be the event that a pupil studying discrete mathematics is selected, B be the event that a boy is selected, G be the event that a girl is selected.

(a) (i) The required probability is $P(M \cap B) = P(M \mid B)\, P(B) = 0.8 \times 0.7 = 0.56$.

(ii) The required probability is $P(M \cap G) = P(M \mid G)\, P(G) = 0.6 \times 0.3 = 0.18$.

(iii) The required probability is $P(M) = P(M \cap B) + P(M \cap G) = 0.74$.

(b) (i) The required probability is $P(B \mid M) = \dfrac{P(M \mid B)P(B)}{P(M)} = \dfrac{0.56}{0.74} = \dfrac{28}{37}$.

(ii) The required probability is $P(G \mid M) = \dfrac{P(M \mid G)P(G)}{P(M)} = \dfrac{0.18}{0.74} = \dfrac{9}{37}$.

Example 6.4-4

The probability that John will be sick tomorrow is 0.6. If John is sick, the probability that he goes to school is 0.3. If he is not sick, the probability that he goes to school is 0.9. Find the probability that John goes to school tomorrow.

Solution

Let S be the event that John is sick tomorrow and let G be the event that John goes to school tomorrow. Then $\overline{S}$ is the event that John is not sick tomorrow.

The required probability is $P(G)$

$= P(G \mid S)\, P(S) + P(G \mid \overline{S})\, P(\overline{S}) = 0.3 \times 0.6 + 0.9 \times 0.4 = 0.54$.

6.4.2 Independent Events

If the occurrence or non-occurrence of an event E does not influence in any way the probability of an event F, then the event F is *independent* of event E and

$$P(\,F \mid E\,) = P(\,F\,)$$

Similarly, if events E and F are independent, then $P(\,E \mid F\,) = P(\,E\,)$.

Since $P(\,E \cap F\,) = P(\,E \mid F\,)\,P(\,F\,)$, then

$$P(\,E \cap F\,) = P(\,E\,)\,P(\,F\,)$$

This is known as the *multiplication law for independent events.*

Note that, if events E and F are independent, then the events E and $\overline{F}$, $\overline{E}$ and F, $\overline{E}$ and $\overline{F}$ are independent.

Example 6.4-5

Two men fire at a target. The probability that John hits the target is 0.8 and the probability that Mark hits the target is 0.9. Find the probability that (a) both John and Mark hit the target, (b) only one hits the target, (c) neither hits the target.

Solution

Let J be the event that John hits the target and let M be the event that Mark hits the target. Obviously, J and M are independent events.

(a) Probability that both John and Mark hit the target is $P(\,J \cap M\,)$
$= P(\,J\,)\,P(\,M\,) = 0.8 \times 0.9 = 0.72$

(b) Probability that only one hits the target is $P(\,J \cap \overline{M}\,) + P(\,\overline{J} \cap M\,)$
$= P(\,J\,)\,P(\,\overline{M}\,) + P(\,\overline{J}\,)\,P(\,M\,) = 0.8 \times 0.1 + 0.2 \times 0.9 = 0.08 + 0.18 = 0.26$

(c) Probability that neither hits the target is $P(\,\overline{J} \cap \overline{M}\,)$
$= P(\,\overline{J}\,)\,P(\,\overline{M}\,) = 0.2 \times 0.1 = 0.02$

6.5 Reference:

1. Discrete Mathematics and its Applications, fourth edition, Kenneth H. Rosen, McGraw-Hill International Editions, Mathematics & Statistics Series.

2. A Concise Course in A-Level Statistics, J. Crawshaw & J. Chambers, ELBS.