3. Matrix Arithmetic and Relations

Matrix Arithmetic

3.1 Introduction to Matrix Arithmetic

A *matrix* is a set of real or complex numbers (or *elements*) arranged in *rows* and *columns* to form a *rectangular array*.

A matrix having m rows and n columns is called an m x n (i.e. 'm by n') matrix and is referred to as having *order* m x n.

A matrix is indicated by writing the array within a *large square bracket*.

Example 3.1-1 $\begin{pmatrix} 1 & 1 & 3 \\ 4 & 6 & 9 \end{pmatrix}$ is a 2×3 matrix. Where 1, 1, 3, 4, 6, 9 are the elements of the matrix.

Example 3.1-2 $\begin{pmatrix} 1 & 2 & 2 \\ 4 & 3 & 5 \\ 5 & 2 & 7 \\ 1 & 2 & 6 \end{pmatrix}$ is a 4×3 matrix.

Example 3.1-3 $\begin{pmatrix} 1 & 2 & 2 \\ 6 & 3 & 5 \\ 7 & 2 & 7 \end{pmatrix}$ is a 3×3 matrix.

Short Questions:

So matrix $\begin{pmatrix} 427 & 429 \\ 369 & 371 \end{pmatrix}$ is of order __________.

and matrix $\begin{pmatrix} 427 & 429 \\ 369 & 371 \\ 23 & 12 \\ 24 & 66 \end{pmatrix}$ is of order __________.

3.2 Matrix Notation

Each element in a matrix has its own particular *'address'* or *location* which can be defined by a system of *double suffixes*, the first indicates the row, while the second indicates the column. For example,

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

Short Questions:

$$\begin{pmatrix} 45 & -89 & 17 & 23 \\ 34 & 66 & 88 & 56 \\ 12 & 70 & 87 & 55 \\ 98 & 32 & 71 & 22 \end{pmatrix}$$

The locations of the elements 66, -89 and 71 can be stated as ___________ respectively.
The elements on the locations 31, 42 and 23 are _____________ respectively.

Further, a whole matrix can be denoted by *a single general element* enclosed in square brackets, or by a single letter printed in *bold type*. This is a very neat shorthand and saves much space and writing.

Example 3.2-1 $\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$ can be denoted by $[a_{ij}]$ or by **A**.

Similarly, $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ can be denoted by $[x_i]$ or by **X**.

3.3 Equal Matrices

Two matrices are said to be *equal* if their corresponding elements throughout are equal. Therefore, the two matrices must also be of the *same order*.

That is $[a_{ij}] = [x_{ij}]$ if $a_{ij} = x_{ij}$ for all values of *i and j*.

3.4 Addition and Subtraction of Matrices

To add or subtract, the two matrices must be of the *same order*. The result from the sum or difference is then determined by adding or subtracting the corresponding elements.

Example 3.4-1 $\begin{pmatrix} 7 & 9 & 5 \\ 3 & 6 & 3 \end{pmatrix} + \begin{pmatrix} 427 & 429 & 427 \\ 369 & 371 & 369 \end{pmatrix} = \begin{pmatrix} 7+427 & 9+429 & 5+427 \\ 3+369 & 6+371 & 3+369 \end{pmatrix}$

$$= \begin{pmatrix} 434 & 438 & 432 \\ 372 & 377 & 372 \end{pmatrix}$$

Example 3.4-2 $\begin{pmatrix} 427 & 429 & 427 \\ 369 & 371 & 369 \end{pmatrix} - \begin{pmatrix} 7 & 9 & 5 \\ 3 & 6 & 3 \end{pmatrix} = \begin{pmatrix} 427-7 & 429-9 & 427-5 \\ 369-3 & 371-6 & 369-3 \end{pmatrix}$

$$= \begin{pmatrix} 420 & 420 & 422 \\ 366 & 365 & 366 \end{pmatrix}$$

3.5 Multiplication of Matrices

(a) Scalar Multiplication

Example 3.5-1 $4\begin{pmatrix} 7 & 9 & 5 \\ 3 & 6 & 3 \end{pmatrix} = \begin{pmatrix} 28 & 36 & 20 \\ 12 & 24 & 12 \end{pmatrix}$

i.e. $k[a_{ij}] = [ka_{ij}]$.

This also means that, we can take a *common factor* out of every element in reverse.

Example 3.5-2 $\begin{pmatrix} 14 & 18 & 10 \\ 6 & 12 & 6 \end{pmatrix} = 2\begin{pmatrix} 7 & 9 & 5 \\ 3 & 6 & 3 \end{pmatrix}$

(b) Multiplication of Two Matrices

The two matrices can be multiplied together only when the number of columns in the first is equal to the number of rows in the second.

Example 3.5-3 If $A = [a_{ij}] = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$ and $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

$$\text{then } Ab = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_{11}b_1 + a_{12}b_2 + a_{13}b_3 \\ a_{21}b_1 + a_{22}b + a_{23}b_3 \end{pmatrix}$$

That is each element in the top row of **A** is multiplied by the corresponding element in the first column of **b** and the products added. The second row of the product is similarly done.

Example 3.5-4 If $A = \begin{pmatrix} 1 & 5 \\ 2 & 7 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 8 & 4 & 3 & 1 \\ 2 & 5 & 8 & 6 \end{pmatrix}$,

$$\text{then } AB = \begin{pmatrix} 1 & 5 \\ 2 & 7 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 8 & 4 & 3 & 1 \\ 2 & 5 & 8 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 1\times 8 + 5\times 2 & 1\times 4 + 5\times 5 & 1\times 3 + 5\times 8 & 1\times 1 + 5\times 6 \\ 2\times 8 + 7\times 2 & 2\times 4 + 7\times 5 & 2\times 3 + 7\times 8 & 2\times 1 + 7\times 6 \\ 3\times 8 + 4\times 2 & 3\times 4 + 4\times 5 & 3\times 3 + 4\times 8 & 3\times 1 + 4\times 6 \end{pmatrix}$$

$$= \begin{pmatrix} 18 & 29 & 43 & 31 \\ 30 & 43 & 62 & 44 \\ 32 & 32 & 41 & 27 \end{pmatrix}$$

Note that multiplying a (3×2) matrix and a (2×4) matrix gives a product matrix of order (3×4).

same

i.e. order (3×2) * order (2×4) ⟶ order (3×4).

In general, the product of an $(\mathbf{c} \times \mathbf{d})$ matrix and an $(\mathbf{d} \times \mathbf{e})$ matrix has order $(\mathbf{c} \times \mathbf{e})$.

3.5.1 Matrix Multiplication is not Commutative i.e. AB ≠ BA

Example 3.5-5 If $A = \begin{pmatrix} 5 & 2 \\ 7 & 4 \\ 3 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 9 & 2 & 4 \\ -2 & 3 & 6 \end{pmatrix}$,

then $AB = \begin{pmatrix} 41 & 16 & 32 \\ 55 & 26 & 52 \\ 25 & 9 & 18 \end{pmatrix}$ and $BA = \begin{pmatrix} 71 & 30 \\ 29 & 14 \end{pmatrix}$.

3.5.2 A Matrix can be Squared only if it is a Square Matrix i.e. number of rows = number of columns.

A (3×2) matrix B multiples with a (3×2) matrix (i.e. B^2) yields nothing since such matrix can not be multiplied; whereas a (2×2) or (3×3) or $(n \times n)$ matrix can be squared with meaningful result.

3.5.3 Example in More Advanced Topics

Example 3.5-6

Consider if we were to write an algorithm for matrix multiplication to multiple two $(m \times m)$ matrices;

(a) How many steps of addition and multiplication of integers are used?

(b) Given matrix multiplication is associative $[\text{i.e. } (B_1B_2)B_3 = B_1(B_2B_3)]$, would such preference in the order of multiplication affects the number of steps taken?

Solution

(a) Multiplying two $(m \times m)$ matrices C and D, there would be m^2 entries in the product.

Total numbers of multiplication $= m^3$

Total numbers of addition $= m^2(m - 1)$

(b) If B_1 is a (30×20) matrix, B_2 is a (20×40) matrix, and B_3 is a (40×10) matrix, then steps involved for $(B_1B_2)B_3$:

Total numbers of multiplication $= 30 \times 20 \times 40 + 30 \times 40 \times 10 = 36000$.

and steps involved for $B_1(B_2B_3)$:

Total numbers of multiplication $= 20 \times 40 \times 10 + 30 \times 20 \times 10 = 14000$.

Clearly, we can see that the second method is much more efficient.

3.6 Transpose of a Matrix

If the rows and columns of a matrix are *interchanged*,

i.e. the first row becomes the first column,

the second row becomes the second column,

the third row becomes the third column, etc.,

the new matrix so formed is the *transpose* of the original matrix.

If M is the original matrix, its transpose is denoted by M^T.

Example 3.6-1 If $M = \begin{pmatrix} 4 & 6 \\ 8 & 3 \\ 5 & 1 \end{pmatrix}$, then $M^T = \begin{pmatrix} 4 & 8 & 5 \\ 6 & 3 & 1 \end{pmatrix}$.

3.7 Special Matrices

(a) A square matrix M is *symmetric* if $M = M^T$.

$$M = \begin{pmatrix} 1 & 2 & 5 \\ 2 & 8 & 9 \\ 5 & 9 & 4 \end{pmatrix} \text{ and } M^T = \begin{pmatrix} 1 & 2 & 5 \\ 2 & 8 & 9 \\ 5 & 9 & 4 \end{pmatrix}.$$

(b) A *diagonal matrix* M is a square matrix with all elements 0 except those on the leading diagonal.

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

(c) A *unit matrix* I is a diagonal square matrix with all elements 0 except those on the leading diagonal and elements of leading diagonal are 1.

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Unit matrix behaves very much like the unit factor in ordinary algebra where

$$I\,M = M\,I = M$$

(d) A *null matrix* N is a matrix with all elements zero.

$$N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Relations

3.8 Introduction to Relations

Relationships between elements of sets occur in many contexts. We have many examples in everyday life such as those between a companies, a schools and their telephone numbers, a person and a relative, a student and his/her student number.

In mathematics we study relationships such as those between a positive integer and one that it divides, an integer and one that it is congruent to modulo 5, a real number and one that is larger than it, and so on.

Relationships such as that between a program and a variable it uses and that between a computer language and a valid statement in this language often arise in computer science.

Relationships between elements of sets are represented using the structure called a relation. Relations can be used to solve problems such as determining which pairs of cities are linked by airline flights in a network, finding a viable order for the different phases of a complicated project, or producing a useful way to store information in computer datebases.

3.9 Relations

Cartesian Product $A \times B$

The Cartesian product of A and B, $A \times B$, is the set of all ordered pairs (a, b) such that the first element of the ordered pair, a, is form A and the second element of the ordered pair, b, is from B.

Example 3.9-1

Let $A = \{1, 2\}$ and $B = \{\text{a, b, c}\}$, then

$$A \times B = \{1, 2\} \times \{\text{a, b, c}\}$$
$$= \{(1, \text{a}), (1, \text{b}), (1, \text{c}), (2, \text{a}), (2, \text{b}), (2, \text{c})\}$$

Example 3.9-2

Let Instructor = {John Chan, Peter Sun, May Lau}

and Course = {CS 105, CS 170, CS 240, CS 243}

then Instructor $\times$ Course Code

= {(John Chan, CS 105), (John Chan, CS 170), (John Chan, CS 240), (John Chan, CS 243), (Peter Sun, CS 105), ……, (May Lau, Cs243)}

A relation, ***R***, with domain *A* and range *B*, is any subset of $A \times B$.
If (a, b) is in ***R***, we often write $a\boldsymbol{R}b$.

Example 3.9-3

Instructor-Teaching-Course = {(John Chan, CS 105), (John Chan, CS 243), (Peter Sun, CS 170), (May Lau, CS 240), (May Lau, CS 243)}

Example 3.9-4

Let domain = range = {1, 2, 3, 4}.

$R(a < b) = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$

$R(a = b) = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$

Relations from a set *A* to itself are of special interest.

Definition: A *relation on* the set *A* is a relation from *A* to *A*.

i.e. a relation on a set *A* is a subset of $A \times A$.

Example 3.9-5

Let $A = \{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b)\} \mid a$ divides $b\}$?

Solution

Since (a, b) is in R if and only if a and b are positive integers not exceeding 4 such that a divides b, we see that

$$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$$

Example 3.9-6

Consider the following relations on the set of integers:

$R_1 = \{(a, b) \mid a \leq b\}$

$R_2 = \{(a, b) \mid a > b\}$

$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$

$R_4 = \{(a, b) \mid a = b\}$

$R_5 = \{(a, b) \mid a = b + 1\}$

$R_6 = \{(a, b) \mid a + b \leq 3\}$.

Which of these relations contain each of the pairs (1, 1), (1, 2), (2, 1), (1, -1), and (2, 2)?

The pair (1, 1) is in R_1, R_3, R_4 and R_6; (1, 2) is in R_1 and R_6; (2, 1) is in R_2, R_5, and R_6; (1, -1) is in R_2, R_3, and R_6; and finally, (2, 2) is in R_1, R_3 and R_4.

3.10 Combining Relations

Since relations from A to B are subsets of $A \times B$, two relations from A to B can be combined in any way two sets can be combined. Consider the following examples.

Example 3.10-1

Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$. The relations $R_1 = \{(1, 1), (2, 2), (3, 3)\}$ and $R_2 = \{(1, 1), (1, 2), (1, 3), (1, 4)\}$ can be combined to get

$R_1 \cup R_2 = \{(1, 1), (1, 2), (1, 3), (1,4), (2, 2), (3, 3)\}$

$R_1 \cap R_2 = \{(1, 1)\}$

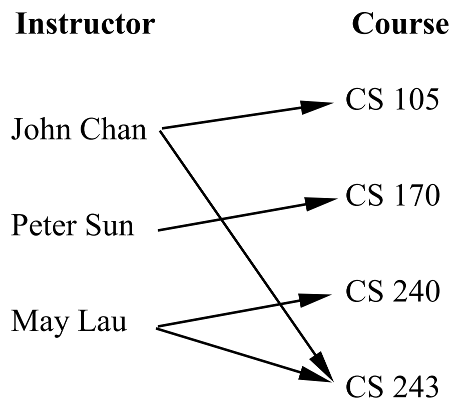
$R_1 \setminus R_2 = \{(2, 2), (3, 3)\}$

$R_2 \setminus R_1 = \{(1, 2), (1, 3), (1,4)\}$

3.11 Representing Relations

(a) Graphical Representation

Relation can be represented graphically, as shown below, using arrows to represent ordered pairs.



Graphical Representation for Relation in Example 3.9-3

(b) Tabular Representation

Another way is to use a table to represent relations.

Instructor-Teaching-Course	**CS 105**	**CS 170**	**CS 240**	**CS 243**
John Chan	X			X
Peter Sun		X		
May Lau			X	X

Tabular Representation for Relation in Example 3.9-3

(c) Representing Relations using Matrices

A relation between finite sets can be represented using a zero-one matrix.

Suppose that R is a relation from $A = \{a_1, a_2, \ldots\ldots a_m\}$ to $B = \{b_1, b_2, \ldots\ldots b_n\}$.

The relation R can be represented by the matrix $M_R = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 \text{ if } (a_i, b_j) \in R \\ \\ 0 \text{ if } (a_i, b_j) \notin R \end{cases}$$

In other words, the zero-one matrix representing R has a 1 as its (i, j) entry when a_i is related to b_j, and a 0 in this position if a_i is not related to b_j.

(Such a representation depends on the orderings used for A and B.)

The use of matrices to represent relations is illustrated in the examples as follow.

Example 3.11-1

Suppose that $A = \{1, 2, 3\}$ and $B = \{1, 2\}$. Let R be the relation from A to B containing (a, b) if $a \in A$, $b \in B$, and $a > b$. What is the matrix representing R if $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, $b_1 = 1$ and $b_2 = 2$?

Since $R = \{(2, 1), (3, 1), (3, 2)\}$, the matrix for R is $M_R = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$

The 1s in M_R show that the pairs (2, 1), (3, 1) and (3, 2) belong to R. The 0s show that no other pairs belong to R.

Example 3.11-2

Suppose that $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4, b_5\}$. Which ordered pairs are in the relation R represented by the matrix $M_R = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix}$.

$R = \{(a_1, b_2), (a_2, b_1), (a_2, b_3), (a_2, b_4), (a_3, b_1), (a_3, b_3), (a_3, b_5)\}$.

3.12 Properties of Binary Relations

The most direct way to express a relationship between of two sets was to use ordered pairs, which we have already seen, made up of two related elements. For this reason, sets of ordered pairs are called *binary relations*.

Definition: Let A and B be sets. A ***binary relation*** *from* A to B is a subset of $A \times B$.

Example 3.12-1

Let $A = \{x: x \text{ is all positive integers}\}$. We may define a binary relation R on A such that (a, b) is in R if $a - b \geq 10$. Thus, (12, 1) is in R, but (12, 3) is not; neither is (1, 12).

Example 3.12-2

Let B = {*CSC121, CSC221, CSC257, CSC264, CSC273, CSC281*}.

A binary relation on B = {(*CSC121, CSC221*), (*CSC121, CSC257*), (*CSC257, CSC281*)} might describe the prerequisite structure of these courses in that an ordered pair in the binary relation means the first course in the pair is a prerequisite of the second course in the pair.

3.13 A Relational Model for Data Bases

As binary relations describe the relationship between pairs of objects, we would like to definite ternary relations to describe the relationship among triples of objects, and n-ary relations to describe the relationship among n of objects.

Let A be a set of students, B be a set of courses, and C be a set of all possible grades. Then a ternary relation among A, B, C can be defined to describe the grades the students obtained in the courses they took.

In the relational model of databases such n-ary relations are used as the standard method for describing the relationships in the data. So that the data can be handled effectively.

Example 3.13-1

For example, let SUPPLIER = $\{s_1, s_2, s_3, s_4\}$ be the set of suppliers of parts,
PART = $\{p_1, p_2, p_3, p_4, p_5, p_6, p_7\}$ be the set of parts, and
PROJECT = $\{j_1, j_2, j_3, j_4, j_5\}$ be the set of projects, and
QUANTITY be the set of positive integers.

We may have a relation named SUPPLY on the above sets describing the names of suppliers who supply parts to the various projects and the quantities they supply.

Thus, SUPPLY = {(s_1, p_2, j_5, 5), (s_1, p_3, j_5, 17), (s_2, p_3, j_5, 9), (s_2, p_1, j_5, 5), (s_4, p_1, j_1, 4)}

We can also represent the SUPPLY relation in tabular form.

SUPPLY

SUPPLIER	PART	PROJECT	QUANTITY
s_1	p_2	j_5	5
s_1	p_3	j_5	17
s_2	p_3	j_5	9
s_2	p_1	j_5	5
s_4	p_1	j_1	4

Projection is an operation that selects specified subcomponents from a relation.

Example 3.13-2
Consider a relation ASSEMBLE which is the projection of SUPPLY on PART, PROJECT and QUANTITY.
Thus ASSEMBLE = {(p_1, j_5, 5), (p_1, j_1, 4), (p_2, j_5, 5), (p_3, j_5, 26)}

3.14 Composite Relation

Given two relations such that

R is a relation from set *A* to set *B* and
S is a relation from set *B* to set *C*.

Then the composition of R and S is defined as the relation

$$SoR = \{(a, c) \in A \times C : (a, \mathrm{b}) \in R \text{ and } (b, c) \in S, \text{ for some } b \in B\}$$

Example 3.14-1
Consider the following relations:
Student-Subject = {(John, DMS), (Mary, DMS), (John, programming), (Paul, programming), (Mary, SAD), (John, SAD)}

Subject-Teacher = {(DMS, Jim Chan), (SAD, Peter Cheung), (Programming, Steve Chow)}.

Thus Student-Teacher = Subject-Teacher o Student-Subject
= {(John, Jim Chan),(John, Steve Chow), (John, Peter Cheung), (Paul, Steve Chow), (Mary, Jim Chan), (Mary, Peter Cheung)}

Example 3.14-2

Given the relation

Father-Son = {(John, Peter), (John, David), (Eric, Brian), (David, Paul), (Paul, Stephen), (Brian, Patrick)}

Thus, the relation between Grandfather and Son is

Grandfather-Son = Father-Son o Father-Son
= {(John, Paul), (David, Stephen), (Eric, Patrick)}

3.15 Reflexive, Symmetric and Transitive Property of a Binary Relation

(a) Reflexive Property of a Binary Relation

In some relations an element is always related to itself. For example, let R be the relation on the set of all people consisting of pairs (x, y) where x and y have the same mother and the same father, then xRx for every person x.

Definition:

A relation R on a set A is called ***reflexive*** if $(a, a) \in R$ for every element $a \in A$.

We see that a relation on A is reflexive if every element of A is related to itself. The following examples demonstrate the concept of a reflexive relation.

Example 3.15-1

Consider the following relations on {1, 2, 3, 4}:

R_1 = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)}
R_2 = {(1, 1), (1, 2), (2, 1)}
R_3 = {(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)}
R_4 = {(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)}
R_5 = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)}
R_6 = {(3, 4)}.

Which of these relations are reflexive?

The relations R_3 and R_5 are reflexive since they both contain all pairs of the form (a, a), namely, (1, 1), (2, 2), (3, 3) and (4, 4). The other elements are not reflexive since they do not contain all of these ordered pairs. In particular, R_1, R_2, R_4, and R_6 are not reflexive since (3, 3) is not in any of these relations.

Example 3.15-2
Which of the relations from example 3.9-6 are reflexive?

The reflexive relations from this example are R_1 (since $a \leq a$ for every integer a), R_3 and R_4. For each of the other relations in this example it is easy to find a pair of the form (a, a) that is not in the relation.

(b) Symmetric Property of a Binary Relation

In some relations an element is related to a second element if and only if the second element is also related to the first element. The relation consisting of pairs (x, y) where x and y are students at the college with at least one common class has this property. Other relations have the property that if an element is related to a second element, then this second element is not related to the first. The relation consisting of the pairs (x, y) where x and y are students at the college where x has a higher grade point average than y has this property.

Definition:
A relation R on a set A is called ***symmetric*** if $(b, a) \in R$ whenever $(a, b) \in R$, for $a, b \in A$.
A relation R on a set A such that $(a, b) \in R$ and $(b, a) \in R$ only if $a = b$, for $a, b \in A$, is called ***antisymmetric***.

Example 3.15-3
Which of the relations below are symmetric and which are antisymmetric?

$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$
$R_2 = \{(1, 1), (1, 2), (2, 1)\}$
$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$
$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$
$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$
$R_6 = \{(3, 4)\}$.

The relation R_2 and R_3 are symmetric, because in each case (b, a) belongs to the relation whenever (a, b) does.

For R_2, the only thing to check is that both (2, 1) and (1, 2) are in the relation.

For R_3, it is necessary to check that both (1, 2) and (2, 1) belong to the relation, and (1, 4) and (4, 1) belong to the relation.

R_4, R_5, R_6 are all antisymmetric. For each of these relations there is no pair of elements a and b with $a \neq b$ such that both (a, b) and (b, a) belong to the relation.

Example 3.15-4

Which of the relations below are symmetric and which are antisymmetric?

$R_1 = \{(a, b) \mid a \leq b\}$
$R_2 = \{(a, b) \mid a > b\}$
$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$
$R_4 = \{(a, b) \mid a = b\}$
$R_5 = \{(a, b) \mid a = b + 1\}$
$R_6 = \{(a, b) \mid a + b \leq 3\}$.

The relations R_3, R_4 and R_6 are symmetric.

R_3 is symmetric, for if $a = b$ or $a = -b$, then $b = a$ or $b = -a$.

R_4 is symmetric since $a = b$ implies that $b = a$.

R_6 is symmetric since $a + b \leq 3$ implies that $b + a \leq 3$.

R_1, R_2, R_4, and R_5 are all antisymmetric.

R_1 is antisymmetric because the inequalities $a \leq b$ and $b \leq a$ imply that $a = b$.

R_2 is antisymmetric since it is impossible for $a > b$ and $b > a$.

R_4 is antisymmetric, since two elements are related with respect to R_4 if they are equal. R_5 is antisymmetric since it is impossible that $a = b + 1$ and $b = a + 1$.

(c) Transitive Property of a Binary Relation

Let R be the relation consisting of pairs (x, y) of students at the college where x has taken more credits than y. Suppose that x is related to y and y is related to z. This means that x has taken more credits than y and y has taken more credits than z. We can conclude that x has taken more credits than z, so that x is related to z. What we have shown is that R has the transitive property, which is defined as follows.

Definition:

A relation R on a set A is called ***transitive*** if whenever $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$, for $a, b, c \in A$.

Example 3.15-5

Which of the relations bellow are transitive?

$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$
$R_2 = \{(1, 1), (1, 2), (2, 1)\}$
$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$
$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$
$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$

R_4 and R_5 are transitive.

For each of these relations, we can show that it is transitive by verifying that if (a, b) and (b, c) belong to this relation, then (a, c) also does.

For instance, R_4 is transitive, since (3, 2) and (2, 1), (4, 2) and (2, 1), (4, 3) and (3, 1), and (4, 3) and (3, 2) are the only such sets of pairs, and (3, 1), (4, 1), and (4, 2) belong to R_4.

R_1 is not transitive since (3, 4) and (4, 1) belong to R_1, but (3, 1) does not.

R_2 is not transitive since (2, 1) and (1, 2) belong to R_2, but (2, 2) does not.

R_3 is not transitive since (4, 1) and (1, 2) belong to R_3, but (4, 2) does not.

3.16 Reference

1. Discrete Mathematics and Its Applications McGRAW-HILL Kenneth H. Rosen
2. Essential Computing Mathematics McGRAW-HILL Seymour Lipschutz