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2. Logic and Proof

2.1 Introduction

Theorems of any mathematical system and the algorithms that make up a program system are developed and put together using rules of logic. Thus, in order to understand mathematical proofs and the development of program systems, it is important to be comfortable with these rules. It also has practical applications to the design of computing systems.

2.2 Logic of Compound Statements

2.2.1 Propositions

A proposition (simple statement) is a verbal or written assertion which can be determined to be either true (T) or false (F), but not both. The truth or falsity of a statement is called its truth value. Therefore, the truth value of a proposition is either true or false but not both.

Example 2.2-1

1)	1 + 1 = 2	(Truth value = T)
2)	1 + 1 = 3	(Truth value = F)
3)	Hong Kong is in Asia.	(Truth value = T)
4)	Where are you going ?	
5)	Keep quiet please.	

Notice that both (4) and (5) are not propositions since they are neither true nor false.

2.2.2 Compound Propositions

A compound proposition is formed by connecting simple propositions using logical operators. The truth value of a compound proposition is determined by the truth value of its substatements together with the way in which they are connected to form the compound statement.

2.2.3 Connectives

We use truth table to display the relationship between truth value of statements. We also use lower case letters p, q, r ..., to denote propositions.

2.2.4 Conjunction (AND) $p \wedge q$

Any two statements can be combined by the word "and" to form a compound statement called <u>conjunction</u> of the original statements. It is denoted by $p \wedge q$, read "p and q". The truth value of $p \wedge q$ is given by the following truth table.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Example 2.2-2

1) Hong Kong is in Asia and 2+3 = 5. (Truth value = T)
2) Hong Kong is in Asia and 2+3 = 6. (Truth value = F)
3) Hong Kong is in Europe and 2+3 = 5. (Truth value = F)

2.2.5 Disjunction (OR) $p \vee q$

Any two statements can be combined by the word "or" to form a compound statement called <u>disjunction</u> of the original statements. It is denoted by $p \vee q$, read "p or q". The truth value of $p \vee q$ is given by the following truth table.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example 2.2-3

1) London is in England or 2+3 = 5. (Truth value = T)
2) London is in France or 2+3 = 5. (Truth value = T)
3) London is in France or 2+3 = 6. (Truth value = F)

2.2.6 **Negation** (NOT) $\sim p$

Given any statement p, another statement called the negation of p, can be formed by inserting in p the word "not". It is denoted by $\sim p$, read "not p". The truth value of $\sim p$ is given by the following truth table.

p	$\sim p$
T	F
F	T

Example 2.2-4 Consider the following statements.

	p		$\sim p$
1)	London is in England.	1')	London is not in England.
2)	$2+2=5$	2')	$2+2\neq 5$

By repetitive use of logical connective $(\wedge, \vee, \sim)$, we can construct complicated compound statements.

Example 2.2-5 For the compound statement $\sim(p\vee q)\wedge\sim r$, we can construct the following truth table:

p	q	r	$(p\vee q)$	$\sim(p\vee q)$	$\sim r$	$\sim(p\vee q)\wedge\sim r$
T	T	T	T	F	F	F
T	T	F	T	F	T	F
T	F	T	T	F	F	F
T	F	F	T	F	T	F
F	T	T	T	F	F	F
F	T	F	T	F	T	F
F	F	T	F	T	F	F
F	F	F	F	T	T	T

2.2.7 Conditional $\rightarrow$

Many propositions, particularly in mathematics, are of the form "If *p* then *q*". Such propositions are called conditional propositions. It is denoted by $p \rightarrow q$, read "*p* implies *q*" or "*q* only if *p*". In this proposition *p* is called the hypothesis and *q* is called the conclusion. The truth value of $p \rightarrow q$ is given by the following truth table:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Example 2.2-6 "If it is sunny today, then we will go to the beach."

This proposition is considered valid unless it is indeed sunny today, but we do not go to the beach.

2.2.8 Biconditional $\leftrightarrow$

Another common proposition is of the form "*p* if and only if *q*". Such propositions are called biconditional propositions. It is denoted by $p \leftrightarrow q$, read "*p* if and only if *q*". i.e. the biconditional $p \leftrightarrow q$ is true only if $p \rightarrow q$ and $q \rightarrow p$ are true. The truth value of $p \leftrightarrow q$ is given by the following truth table:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

2.2.9 Algebra of Propositions

A proposition is a statement or compound statements made up of substatements *p*, *q*, *r*, ... For example, the statement $\sim (p \vee q) \wedge \sim r$ is a proposition made up of substatements *p*, *q* and *r*. In a parenthesis-free statement, the logical connectives are applied in the following order :

$$\sim \; \wedge \; \vee \; \rightarrow \; \leftrightarrow$$

2.2.10 Tautologies and Contradictions

A proposition is called tautologies if its truth value is always true regardless of the truth values of its substatements. Similarly, a proposition is called contradiction if its truth value is always false regardless of the truth values of its substatements.

Example 2.2-7 Show that $(p \wedge q) \vee \sim (p \wedge q)$ is a tautology.

p	q	$p \wedge q$	$\sim (p \wedge q)$	$(p \wedge q) \vee \sim (p \wedge q)$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

2.2.11 Logical Equivalence $\equiv$

Two propositions P and Q are said to be <u>logical equivalent</u>, or equivalent or equal, if $P \leftrightarrow Q$ is a tautology or simply they have the same truth tables. It is denoted by $P \equiv Q$.

Example 2.2-8 Show that $p \to q$ and $\sim p \vee q$ are logical equivalent.

p	q	$p \to q$	$\sim p$	$\sim p \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Propositions satisfy many logical equivalences or laws. Some of the more important ones are listed below. Note that in the following equivalences, T denotes a tautology and F denotes a contradiction.

Equivalence	Name
$p \wedge T \equiv p$ $p \vee F \equiv p$	Identity laws
$p \vee T \equiv T$ $p \wedge F \equiv F$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\sim (\sim p) \equiv p$	Double negation laws
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\sim (p \vee q) \equiv \sim p \wedge \sim q$ $\sim (p \wedge q) \equiv \sim p \vee \sim q$	De Morgan's laws

2.3 Rules of Inference

2.3.1 Introduction

In artificial intelligence, the execution of either deductive or inductive reasoning needs several basic reasoning rules to allow the manipulation of the logical expressions to create new expressions.

2.3.2 Modus Ponens

The most important rule is called *modus ponens*. According to this procedure, if there is a rule "if p, then q" and if we know that p is true, then it is valid to conclude that q is also true. In the terminology of logic, we express this as $((p \to q) \wedge p) \to q$ and is written in the following way :

$$\begin{array}{l} p \to q \\ \underline{p \qquad\quad} \\ \therefore q \end{array}$$

Using this notation, the hypotheses are written in a column and the conclusion is below a bar. The symbol $\therefore$ read "therefore" is normally placed just before the conclusion.

Example 2.3-1 Consider the following propositions:

$p \to q$: If you studied hard, then you will pass the examination.
p : You studied hard.
q : You pass the examination.

Using modus ponens you can then deduce that you will pass the examination if both hypotheses $p \to q$ and p are true.

The following table summarizes the two important inference rules Modus ponens and Modus tollens.

Modus ponens	$\begin{array}{l} p \to q \\ \underline{p \qquad} \\ \therefore q \end{array}$	Example: p: "You studied hard" q: "You pass the examination"
Modus tollens	$\begin{array}{l} p \to q \\ \underline{\sim q \qquad} \\ \therefore \sim p \end{array}$	Example: p: "You studied hard" q: "You pass the examination"

Example 2.3-2

If Tom is not in team A, then Peter is in team B.
If Peter is not in team B, then Tom is in team A.
Therefore, Tom is not in team A or Peter is not in team B.

(i) Use symbols to write the logical form of the argument.
(ii) Use a truth table to test for validity of the argument.

(i) p : Tom is on team A
q : Peter is on team B
$\therefore$: Therefore

$$\begin{array}{l} \sim p \to q \\ \sim q \to p \\ \therefore \sim p \vee \sim q \end{array}$$

(ii)

p	q	$\sim p$	$\sim q$	$\sim p \to q$	$\sim q \to p$	$\sim p \vee \sim q$	$[(\sim p \to q) \wedge (\sim q \to p)] \to (\sim p \vee \sim q)$
T	T	F	F	T	T	F	F
T	F	F	T	T	T	T	T
T	T	T	F	T	T	T	T
T	F	T	T	F	F	T	T

The statement form $[(\sim p \to q) \wedge (\sim q \to p)] \to (\sim p \vee \sim q)$ is not a tautology, hence the argument is invalid.

2.4 Logic of Quantified Statements

2.4.1 Predicates

Statements involving variables, such as "$x > 3$", "$x = y + 3$" and "$x = x + 1$" are often found in mathematical assertions and in computer programs. These statements are neither true nor false when the values of the variables are not specified.

The statement "*x* is greater than 3" has two parts. The first part, the variable *x*, is the subject of the statement. The second part, the predicate "is greater than three", refers to a property that the subject of the statement can have. We denote the above statement by $P(x)$, where *P* denotes the predicate "is greater than three" and *x* is the variable. The statement $P(x)$ is also said to be the value of the propositional function *P* at *x*. Once a value has been assigned to the variable *x*, the statement $P(x)$ has a truth value.

Example 2.4-1

Let $P(x)$ denote the statement "$x > 3$", then $P(4)$, which is the statement "$4 > 3$" is true. However, $P(2)$ is false.

Example 2.4-2

Let $Q(x, y, z)$ denote the statement "$z = 3x + 2y$", then $Q(1,2,3)$ which is the statement $3 = 3(1) + 2(2)$ is false while $Q(2,2,10)$ which is the statement $10 = 3(2) + 2(2)$ is true.

In general, $P(x_1, x_2, \ldots, x_n)$ is called a propositional function or a predicate.

2.4.2 Quantifiers

A quantifier is a rule that assigns a simple proposition to a propositional function. There are two quantifiers that play a major role in mathematical logic: the universal quantifier and the existential quantifier.

2.4.3 Universal Quantifier

Many mathematical statements assert that a property is true for all values of a variable. For instance, $x + 1 > x$ is always true for all values of *x*. The universal quantification of $P(x)$ is the proposition "$P(x)$ is true for all values of *x* in the universe of discourse". It is denoted by $\forall x\ P(x)$, read "for all x $P(x)$ is true" or "for every x $P(x)$ is true".

Example 2.4-3

Let the universe of discourse be $\{1, 2, 3, 4\}$. Determine the truth values of the following propositions :

(a) $\forall x,\ x < 6$ (Truth value = T)

(b) $\forall x,\ 2x < 6$ (Truth value = F)

2.4.4 Existential Quantifier

Many mathematical statements assert that there is an element with a certain property. For instance, $x = 2$ is the only solution to the equation $x - 2 = 0$. The existential quantification of $P(x)$ is the proposition "There exists an element x in the universe of discourse such that $P(x)$ is true". It is denoted by $\exists x \ P(x)$, read "There exist an x such that $P(x)$ is true" or "There is at least one x such that $P(x)$ is true".

Example 2.4-4

(a) Let $P(x)$ be the statement "$x > 5$", then the quantification $\exists x \ P(x)$ is true. (Consider the case when $x = 10$.)

(b) Let $Q(x)$ be the statement "$x = x + 1$", then the quantification $\exists x \ P(x)$ is false. (Consider the case when $x = 10$.)

2.5 Methods of Proof

2.5.1 Introduction

A theorem is a statement that can be shown to be true by a sequence of statements that form a valid argument called a proof. To construct proofs, methods are needed to derive new statements from old ones. The statements used in a proof can include axioms or postulates, which are the underlying assumptions about mathematical structures, the hypotheses of the theorem to be prove and previously proved theorems.

2.5.2 Direct Proof

The implication $p \to q$ can be proved by showing that if p is true, then q must also be true. A proof of this kind is called a direct proof. To carry out such a proof, assume that p is true and use rules of inference and theorems already proved to show that q must also be true.

Example 2.5-1 Prove that "If n is odd, then n^2 is odd".

Prove : Suppose that n is odd, i.e. $n = 2k+1$ where k is an integer, then

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$$

Since $2k^2 + 2k$ is an integer and $2(2k^2 + 2k)$ is twice an integer, which is always even. Therefore, n^2 is odd because it is 1 more than an even number.

2.5.3 Indirect Proof

Since $p \to q \equiv \sim q \to \sim p$, the implication $p \to q$ can be proved by showing that its contrapositive, $\sim q \to \sim p$, is true.

Example 2.5-2 Prove that "If $3n+2$ is odd, then n is odd".

Prove : Assume that the conclusion of this implication is false, i.e. $n = 2k$ (an even number) where k is an integer, then

$$3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k+1)$$

Since $3k+1$ is an integer and twice of an integer is an even number. Therefore, $3n+2$ is odd because the negation of the conclusion of the implication implies that the hypothesis is false, then the original implication is true.

2.5.4 Proof by Contradiction

Instead of proving a proposition p is true, we prove that $\sim p$ is false.

Example 2.5-3 Prove that the square of an even number is an even number.

Prove : Assume that *n* is even but n^2 is odd. However, we have proved that if *n* is odd, then n^2 is odd from example of direct proof. Therefore, this contradicts our assumption that *n* is even. Since we have derived a contradiction, the original proposition has been proved to be true.

Alternative method:
The statement is equivalent to "If *n* is even, then n^2 is even."

Proof:
Suppose that *n* is even and n^2 is odd.
Since the product of two odd numbers must be odd and n^2 is odd, so *n* is odd.
This contradicts to *n* is even.
Therefore the supposition that *n* is even and n^2 is odd is wrong.
That is the original statement is right.