ALEXANDER POLYNOMIALS AND NONCOMMUTATIVE LOCALIZATION

Kent Orr at Indiana University taught me a gorgeous, intrinsic (but not well-known) homological proof due to Pierre Vogel that the Alexander module of a knot is annihilated by an Alexander polynomial. The only fact it uses about a knot is that its complement is a homology circle. The starting point is the low-dimensional topologist's definition of a Cohn localization.

Definition. Let R, S be rings with unity, let $R \stackrel{\epsilon}{\longrightarrow} S$ be a ring homomorphism, and let C_* be the chain complex

$$\cdots \stackrel{\partial_2}{\longrightarrow} C_2 \stackrel{\partial_1}{\longrightarrow} C_1 \stackrel{\partial_0}{\longrightarrow} C_0$$

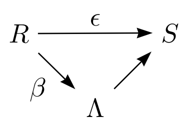
be a chain complex such that C_n is a finitely-generated free R module for all n. The *Cohn localization* Λ of $R \stackrel{\epsilon}{\longrightarrow} S$ is the initial ring with the property

$$C_* \otimes S \text{ is acyclic} \Longleftrightarrow C_* \otimes \Lambda \text{ is acyclic.}$$

Recall that a chain complex is acyclic if it has no homology in any degree.

The algebraist's definition of Cohn localization looks a bit different.

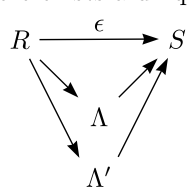
Definition. The *Cohn localization* of $R \stackrel{\epsilon}{\longrightarrow} S$ is a ring Λ and a diagram or ring homomorphisms



such that

- If (α_{ij}) is a square matrix over R such that $(\epsilon(\alpha_{ij}))$ is nonsingular, then $(\beta(\alpha_{ij}))$ is nonsingular over Λ.

Moreover, Λ is the initial ring with this property, meaning that for any ring Λ' satisfying the above condition, there exists a unique diagram of ring homomorphisms



Note that these properties imply that $(\beta(\alpha_{ij}))$ is invertible over Λ.

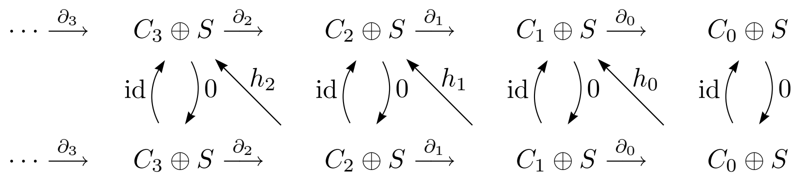
Proof that the definitions of Cohn localization coincide.

⇒: Consider the length 2 chain complex $R^n \stackrel{(\alpha_{ij})}{\longrightarrow} R^n$. Then $S^n \stackrel{(\epsilon(\alpha_{ij}))}{\longrightarrow} S^n$ is acyclic if and only if $\epsilon\,(\alpha_{ij}))$ is nonsingular. The low-dimensional topologist's definition of Cohn localization says that $\Lambda^n \stackrel{(\beta(\alpha_{ij}))}{\longrightarrow} \Lambda^n$ is acyclic, meaning that $\beta\,(\alpha_{ij}))$ is nonsingular, implying the algebraist's definition of Cohn localization.

⇐: Let

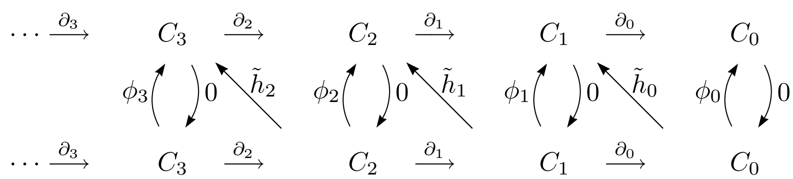
$$C_* = \quad \cdots \xrightarrow{\partial_2} C_2 \xrightarrow{\partial_1} C_1 \xrightarrow{\partial_0} C_0$$

be a chain complex such that C_n is a finitely-generated free R module for all n, and such that $C_* \otimes S$ is acyclic. Homological algebra implies that $C_* \otimes S$ is contractible, which means that there exist homomorphisms $h_0, h_1, \ldots$



with $\partial_i h_{i+1} + h_i \partial_{i+1} = \mathrm{id}$ for $i = 0, 1, \ldots$.

Choose arbitrary lifts $\tilde{h}_0, \tilde{h}_1, \ldots$ for $h_0, h_1, \ldots$ and let $\phi_i = \partial_i \tilde{h}_{i+1} + \tilde{h}_i \partial_{i+1}$.



Now, because $\phi_i \otimes \mathrm{id}_S \colon C_i \otimes S \to C_i \otimes S$ is the identity, ϕ_i is a ring homomorphism of finite-dimensional free R modules which is represented by a finite square matrix which becomes non-singular under the ring homomorphism to S. Thus by algebraist's definition of Cohn localization hold for ϕ_i, and $\phi_i \otimes \mathrm{id}_\Lambda \colon C_i \underset{R}{\otimes} \Lambda \to C_i \underset{R}{\otimes} \Lambda$ it is an isomorphism (represented by an invertible matrix). Therefore $\tilde{h}_* \otimes \Lambda$ is a contraction of $C_* \otimes \Lambda$. Because $C_* \otimes \Lambda$ is contractible (we have just constructed a contraction for it), it is acyclic.

□

The goal of this post is to show that $H_1(E_K; \mathbb{Z}[t^{\pm 1}])$ is Λ-torsion, where E_K denotes the complement of a knot $K \subset S^3$, and $\Lambda = \{1 + \ker \epsilon\}$ and $\mathbb{Z}[t^{\pm 1}] \xrightarrow{\epsilon} \mathbb{Z}$ is the augmentation map, sending t to 1.

Let m be a meridian of K and let $\tilde{E}_K$ be the infinite cyclic cover of E_K in which m (a loop) lifts to $\tilde{m}$ (an infinite line). Consider the chain complex $C_*(\tilde{E}_K, \tilde{m})$ of free $\mathbb{Z}[t^{\pm 1}]$ modules. Then $H_1(C_*(\tilde{E}_K, \tilde{m}) \underset{\mathbb{Z}[t^{\pm 1}]}{\otimes} \mathbb{Z}) \cong H_1(C_*(E_K, m)) \cong 0$ by Alexander duality or Mayer–Vietoris (the point here is that E_K is a homology circle generated by m). The following purely algebraic fact is given without proof.

Fact. The Cohn localization of $\mathbb{Z}[t^{\pm 1}] \xrightarrow{\epsilon} \mathbb{Z}$ is $\Lambda = \{1 + \ker \epsilon\}$. It is a flat $\mathbb{Z}[t^{\pm 1}]$ module.

Now by the low-dimensional topologist's definition of Cohn localization

$$0 \cong H_1(C_*(\tilde{E}_K, \tilde{m}) \underset{\mathbb{Z}[t^{\pm 1}]}{\otimes} \Lambda) \cong H_1(C_*(\tilde{E}_K, \tilde{m})) \underset{\mathbb{Z}[t^{\pm 1}]}{\otimes} \Lambda.$$

Since $\tilde{m}$ is contractible, we get that $H_1(C_*(\tilde{E}_K, \tilde{m})) = H_1(\tilde{E}_K)$ is Λ torsion, which is QED.

Simple, elegant, and general!

There is a point I'm uncomfortable with. You see, Λ is strictly larger than the set of all Alexander polynomials. This is not surprising, because the proof is purely homological, and thus works for CW-complexes which may not be manifolds, and where Poincaré duality may not hold. But it is still disturbing.