In my opinion, Habiro and Goussarov's theory of claspers is a milestone in low dimensional topology [1, 2, 3]. Using this theory, they were able to solve important outstanding problems in quantum topology, and to build a bridge to connect the "quantum" with the "classical".

In this series of posts, I would like to introduce claspers to readers of this blog. I think that claspers are an handy tool for low-dimensional topologists, as a natural extension (or refinement) of Dehn surgery which is compatible with the lower central series of the Torelli group of a Heegaard surface. Claspers are already fairly mainstream in some circles (used or referenced in around 300 papers according to MathSciNet), but I think that they should be more popular with a wider audience (if gropes are familiar to you- in dimension 3 claspers and gropes are essentially equivalent).

In this first post, I would like to ignore the content of clasper calculus— all the beautiful general constructions and shimmering theoretical triumphs of the theory— and focus instead on just (a small part of) the form— claspers as notation. Just graphical notation and nothing else! Look how efficiently it presents mathematics in your head!

My example today is a proof of the following theorem (relevant terminology explained below the cut) [6]:

Theorem 1. *Two links are link homologous if and only if they are related by Δ–moves.*

Two oriented and ordered links $L = K_1 \cup \cdots \cup K_m$ and $L' = K'_1 \cup \cdots \cup K'_n$ are said to be *link homologous* if $m = n$ and $\mathrm{Link}(K_i, K_j) = \mathrm{Link}(K'_i, K'_j)$ for all i and j with $1 \leq i < j \leq n$. The terminology is explained by the fact that the linking number $\mathrm{Link}(K_i, K_j)$ is defined as the image of the ith longitude in $H_1(S^3 - K_j; \mathbb{Z}) \cong \mathbb{Z}$, so (loosely speaking) two links are link homologous if and only if their homologies agree. The Δ–move is a certain local move we will soon define. Thus the theorem characterizes Δ–move equivalence classes (geometric topological information) in terms of a readily computable algebraic invariant.

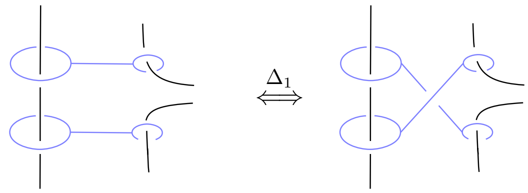
The proof which I present below, in bold font, was told to me by Andrew Kricker last year at his Nanyang Heights apartment in under 30 seconds. Kricker had heard it from Habiro, who was reformulating Murakami and Nakanishi's original argument, which was no doubt known to Matveev in his original paper where he introduce Δ–moves [5] (It may appear explicitly in [4]— I don't know because our library does not seem to have a copy. Please tell me if you know). This then is math folklore.

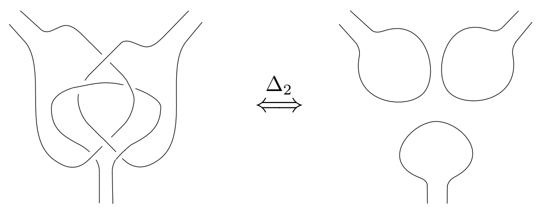
Let's introduce our graphic notation tool, which are a special kind of basic claspers, which I'll call "chord claspers" (just because I'm biased against non-descriptive adjectives like "basic"). A *chord clasper* $C = A_1 \cup B \cup A_2$ in the complement of $L \subset S^3$ is two annuli A_1 and A_2 called *leaves* whose deformation retracts (which are circles) bound discs each of which intersect L at one point, and which are connected by a band B called an *edge*. In this post I'm going to assume that the leaves are not twisted. Claspers can be drawn using their cores via the rewriting rules in the Figure.

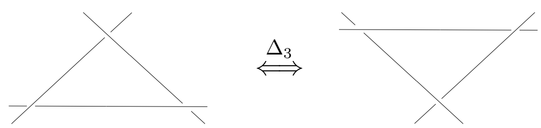
They look just like kayak paddles! Chord claspers are nothing more than graphical shorthand for linkage as described in Figure 1.

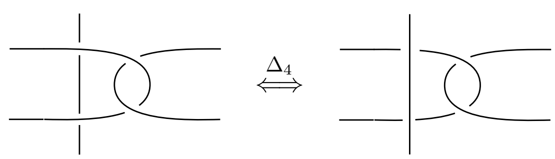
It's time now to introduce the Δ–move (in a few versions):

Proposition. *The following local moves are equivalent:*

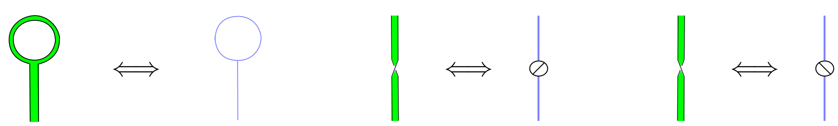
(1) 

(2) 

(3) 

(4) 

Define the Δ–move *to be any of the above.*



Rule 1 Rule 2 Rule 3

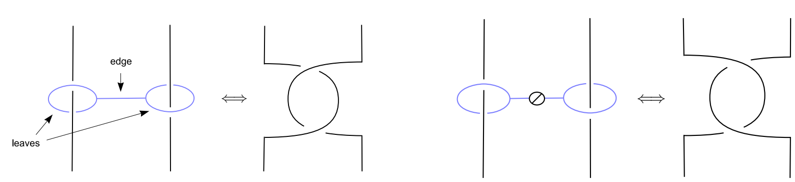
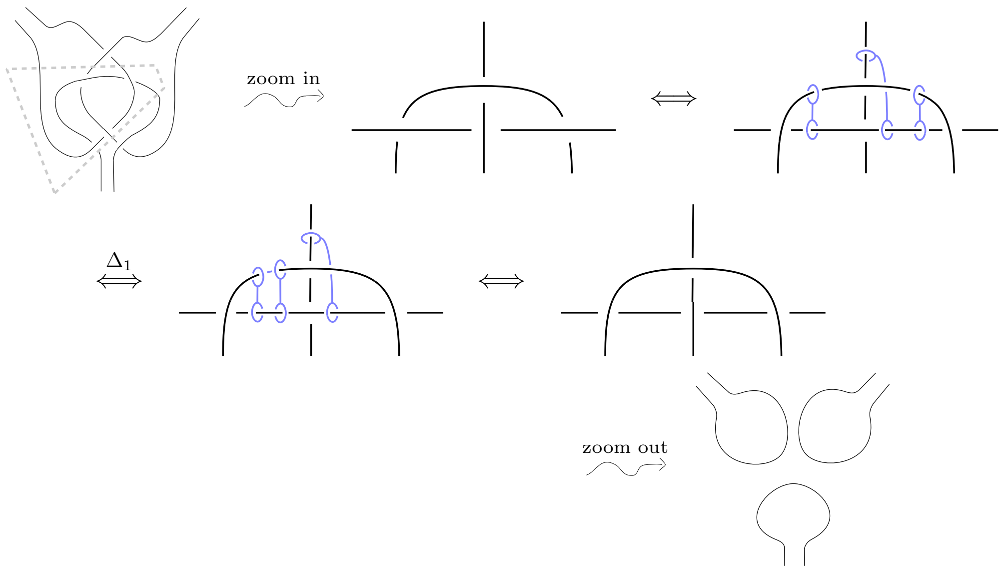
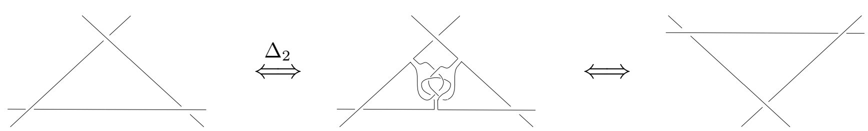


Figure 1. Clasper as graphical shorthand for linkage.

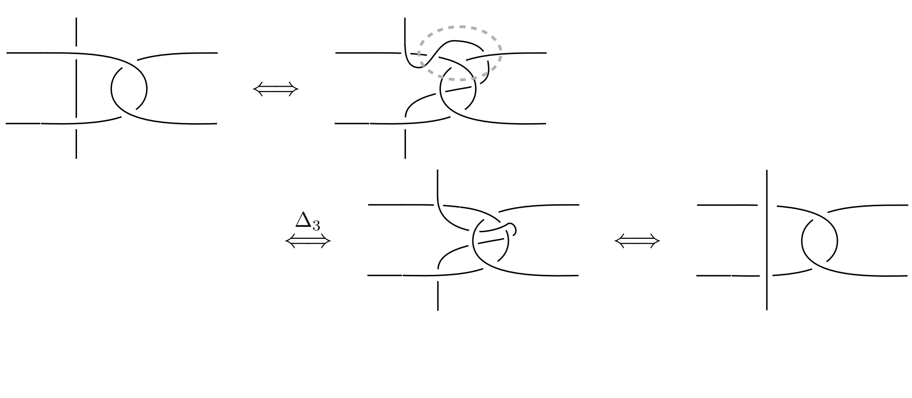
Proof. **$\Delta_1 \Rightarrow \Delta_2$:**



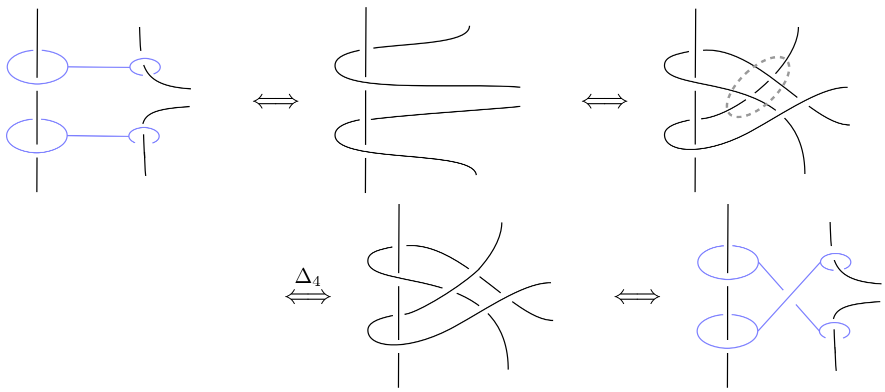
$\Delta_2 \Rightarrow \Delta_3$:



$\Delta_3 \Rightarrow \Delta_4$:



$\Delta_4 \Rightarrow \Delta_1$**:**



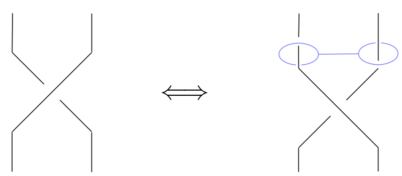
□

0.1. **The proof.** Here is the proof in bold font, with my explanations in normal font:

Undo the link, replacing it with an unlink and some messy web of chord claspers hanging between the components.

Any link may be undone by crossing changes, which in turn may be realized by introducing chord claspers.

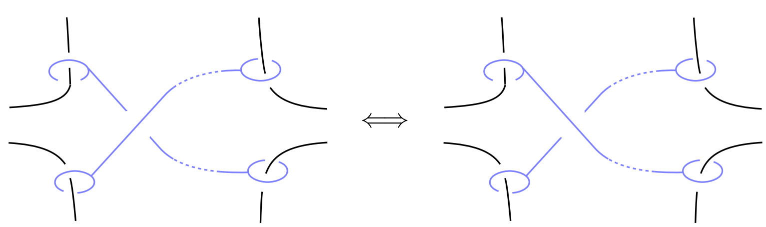
(5)



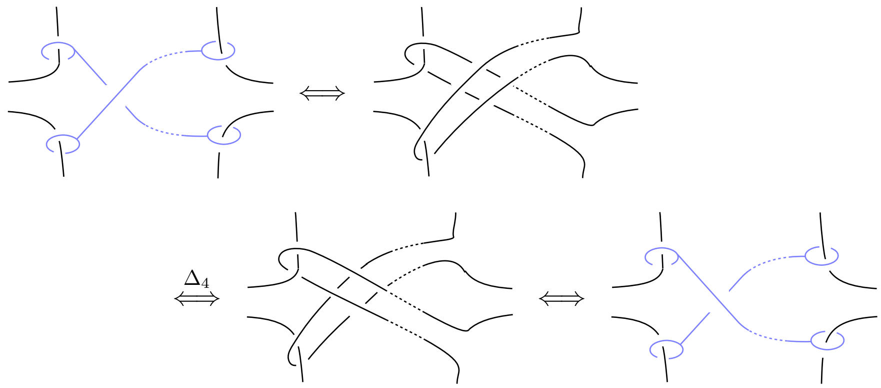
Realize "permuting leaves" and the clasp-pass move using Δ–moves; and untwist edges

Permuting leaves is Δ_1, which according to our definition **is** the delta move. The clasp-pass move is the following move:

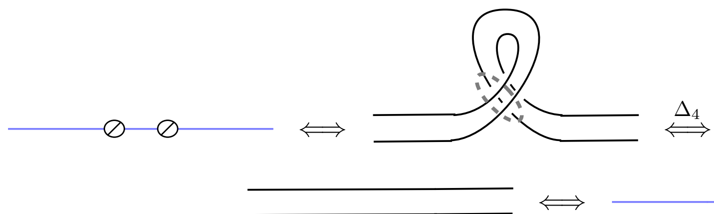
(6)



which is two Δ_4 moves one after the other.

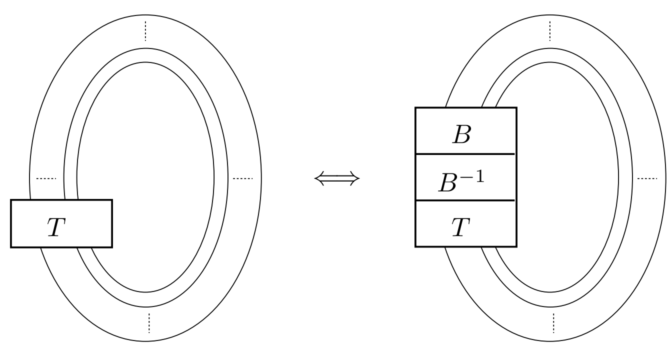
(7) 

We untwist edges as follows, using Δ_4:

(8) 

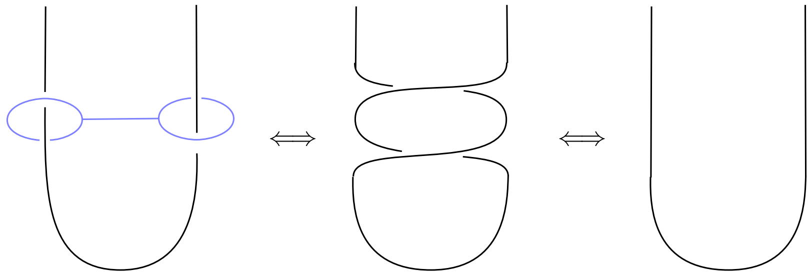
Now the whole messy web comes undone, and you are left with the "standard dude". Choose a "standard dude" B, which is a pure braid $B = S_1 \cup \cdots \cup S_n$ with the property that $\mathrm{Link}(S_i, S_j) = \mathrm{Link}(K_i, K_j)$ for each i and j with $1 \leq i < j \leq n$.

The link L may be presented as the closure of a tangle T. Stack B^{-1} and B above T:

(9) 

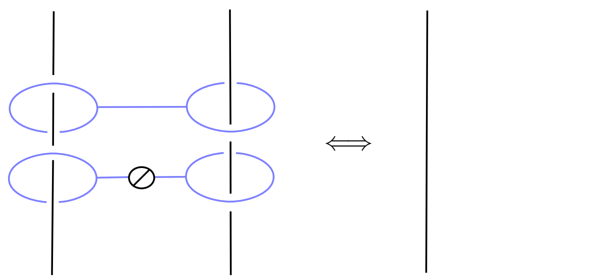
The linking number of any two arcs in $B^{-1}T$ vanishes. When working modulo Δ–moves, edges don't see other edges and leaves don't see other leaves. So we can rearrange claspers at will. If a both leaves of a chord clasper clasp the same component S_i, use Δ_1 and clasp-pass repeatedly to get the local picture below, and remove the clasper.

(10)



So all chord claspers both of whose leaves clasp the same component in $B^{-1}T$ vanish. What about those whose leaves clasp different components? Well, the number of half-twists in the edge of a clasper can be 2–reduced, and two chord claspers between S_i and S_j whose edges have numbers of half-twists with different parity cancel. This cancels all claspers between S_i and S_j in $B^{-1}T$ because of the condition $\mathrm{Link}(S_i, S_j) = 0$.

(11)



And voilá, $B^{-1}T$ comes undone, and you are left with the standard dude B. Thus any two link homologous links are Δ–move equivalent to B and therefore to one another.

We're done. Any questions?

References

1. M.N. Goussarov, *Finite type invariants and n-equivalence of 3-manifolds*, C. R. Acad. Sci. Paris Ser. I Math. 329(6) (1999), 517–522.
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6. H. Murakami and Y. Nakanishi, *On a certain move generating link–homology*, Math.Ann. **284** (1989), 75–89.