THREE DIMENSIONAL TRIGONOMETRY

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1 Sine rule and cosine rule

Sine rule: \( \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \)

Cosine rule: 
\[
\begin{align*}
\cos A &= \frac{b^2 + c^2 - a^2}{2bc} \\
\cos B &= \frac{a^2 + c^2 - b^2}{2ac} \\
\cos C &= \frac{a^2 + b^2 - c^2}{2ab}
\end{align*}
\]

Example 1: N2002/1/9(i)

In triangle ABC, angle B = \( \theta \), angle C = \( \theta + \alpha \), AB = 2 and AC = 1. Show that \( \alpha - \alpha = \theta \cos^2 \sin \tan \).

Solution:

\[
\begin{align*}
\theta + \alpha &= \theta \\
2 \sin \theta &= \sin \theta \cos \alpha + \cos \theta \sin \alpha \\
2 \tan \theta &= \tan \theta \cos \alpha + \sin \alpha \\
\tan \theta (2 - \cos \alpha) &= \sin \alpha \\
\tan \theta &= \frac{\sin \alpha}{2 - \cos \alpha}
\end{align*}
\]

Example 2: J95/1/4

The diagram shows triangle OAB in which OA = \( \sqrt{2} \) units, OB = 1 unit and angle OAB = 30\(^\circ\). The point C is the foot of the perpendicular from O to the line AB produced.

(a) Find angle ABO, and show that AB = 2 sin 15\(^o\).
(b) Hence, by using triangle OAC to find AC, show that the exact value of sin 15\(^o\) is \( \frac{\sqrt{6} - \sqrt{2}}{4} \).

Solution:

(a) Let angle ABO = \( \theta \)
\[
\frac{1}{\sin 30^\circ} = \frac{\sqrt{2}}{\sin \theta} \\
\sin \theta = \sqrt{2} \sin 30^\circ = \frac{\sqrt{2}}{2} \\
\therefore \theta = 135^\circ
\]

angle AOB = 180^\circ - 135^\circ - 30^\circ = 15^\circ

\[
\frac{AB}{\sin 15^\circ} = \frac{\sqrt{2}}{\sin \theta} \\
AB = \frac{\sqrt{2}}{\sqrt{2}} \times 2 \times \sin 15^\circ = 2 \sin 15^\circ
\]

(b) \[
\cos 30^\circ = \frac{AC}{\sqrt{2}} \\
AC = \sqrt{2} \times (\sqrt{3}/2) = \sqrt{6}/2 \\
BC = AC - AB = \sqrt{6}/2 - 2 \sin 15^\circ \\
\text{But } \cos \angle OBC = BC/1 \\
\therefore BC = \cos 45^\circ = \sqrt{2}/2 \\
\text{Hence, } \sqrt{2}/2 = \sqrt{6}/2 - 2 \sin 15^\circ \\
2 \sin 15^\circ = \frac{\sqrt{6} - \sqrt{2}}{2} \\
\therefore \sin 15^\circ = \frac{\sqrt{6} - \sqrt{2}}{4}
\]

2 Angle between two lines

2.1 Lines in common plane
These are either parallel, non-parallel or intersecting

Angle between L_1 and L_2 is \(\alpha\) or \(\beta\).

Angle between AC and AB is \(\beta\).
Angle between CA and AB is \(\alpha\).
2.2 Lines lying in different planes

These are non-parallel and non-intersecting. They are called skew lines.

[Diagram showing skew lines AG and EF, DH and AB]

AG and EF is a pair of skew lines.
DH and AB is also a pair of skew lines.

**Definition:**
Angle between two skew lines is the angle between two straight lines drawn parallel to the two skew lines through any point in space.

Angle between AG and EF
= Angle between AG and AB
= \( \alpha \) (since AB is parallel to EF)

Angle between DH and AB
= Angle AE and AB = \( 90^\circ \)

**Example 3:**
Consider the box given below. Find
(i) angle between BC and BG
(ii) angle between AD and AG
(iii) angle between DG and EF.

**Solution:**
(i) \( \tan \alpha = \frac{3}{4} \)
\( \alpha = 36.87^\circ \)

(ii) \( DG^2 = 3^2 + 12^2 \)
\( DG = 12.35 \)
\( \tan \beta = \frac{12.35}{4} \)
\( \beta = 72.08^\circ \)

(iii) Angle between DG and EF
= Angle between DG and HG (since HG parallel to EF)
= \( \gamma \)
\( \tan \gamma = \frac{3}{12} \)
\( \gamma = 14.04^\circ \)
Angles between a line and a plane

Line QP meets plane \( \pi \) at P. Drop a perpendicular from Q onto \( \pi \). Call this QR. Then, RP is the projection of QP onto \( \pi \).

**Definition:**
Angle between a line and a plane
= Angle between a line and its projection onto the plane

Therefore, angle between QP and \( \pi \)
= angle between QP and PR = \( \alpha \)

**Example 4:**
Consider given tetrahedron. Find the angle between AD and plane BCD.

Let OD be the projection of AD onto BCD.
Therefore, AO is perpendicular to plane BCD
Angle between AD and BCD
= Angle ADO = \( \cos^{-1}(OD/AD) \)
= \( \cos^{-1}(OD/6) \)

To find OD consider triangle BCD

Since this is a regular pyramid, O will coincide with the centroid of triangle BCD
OD = 2/3 DE
=2/3\((5 \sin 60^\circ)\) = 2.89
Therefore, angle ADO = \( \cos^{-1}(2.89/6) \) = 61.2°
4 Angle between two planes

Definition:
The common line of 2 non-parallel planes \( \pi_1 \) and \( \pi_2 \) is the line where 2 planes meet.

Consider any point P on the common line CD. PA and PB are drawn at right angles to CD such that PA is in \( \pi_1 \) and PB is in \( \pi_2 \). Then angle between \( \pi_1 \) and \( \pi_2 \) is angle APB = \( \alpha \).

Example 5:
A triangular pyramid with vertex V and base ABC has VA = VB = VC = 5 cm and AB = BC = CA = 6 cm.
(i) Prove that angle between edge VA and base ABC is \( \cos^{-1}(2\sqrt{3}/5) \).
(ii) Calculate the length of perpendicular from B to VA.
(iii) Prove that angle between plane faces VAB and VAC is \( \cos^{-1}(7/32) \).

Solution:
(i) Angle between VA and base ABC
  = Angle VAO = \( \alpha \)
  \( \cos \alpha = OA/5 \)

Consider triangle ABC
  \( AE = 6 \sin 60^\circ = 3\sqrt{3} \)
O is the centroid of triangle ABC
  \( OA = 2/3 \ AE = 2\sqrt{3} \)
Therefore, \( \cos \alpha = 2\sqrt{3}/5 \)
  \( \alpha = \cos^{-1}(2\sqrt{3}/5) \)

(ii)
Consider triangle VAB
Let perpendicular from B to VA be BP.
\[
\cos \beta = \frac{AQ}{AV} = \frac{3}{5} \quad \sin \beta = \frac{4}{5} \quad BP = 6 \sin \beta = \frac{24}{5}
\]

(iii) Angle between VAB and VAC is angle BPC = \(\gamma\)
By symmetry, CP = BP = \(\frac{24}{5}\)
Considering triangle BPC, by cosine rule,
\[
\cos \gamma = \frac{BP^2 + CP^2 - BC^2}{2 \times BP \times CP}
= \frac{\left(\frac{24}{5}\right)^2 + \left(\frac{24}{5}\right)^2 - 6^2}{2 \left(\frac{24}{5}\right)^2} = \frac{7}{32}
\]
\[
\therefore \gamma = \cos^{-1}\left(\frac{7}{32}\right)
\]

Example 6:
Consider the regular pyramid with square base ABCD, (dim. 5 \times 5 cm) and vertex P, length of each sloping edge 7 cm. Find,
(i) angle between any sloping plane to the base.
(ii) angle between any 2 sloping faces.

Solution:
Angle between any sloping face and base = angle between BCP and ABCD
= Angle PEO = \(\alpha\)
Consider triangle BEP,
\[
PE^2 = PB^2 - BE^2 = 7^2 - (5/2)^2
PE = \sqrt{(171)/4}
OE = ½ AB = 5/2
\alpha = \cos^{-1}(OE/PE) = 67.5^\circ
\]

(ii) Angle between 2 sloping faces = angle between PCD and PDA
= Angle AFC = \(\beta\)
We need to find FC, FA, AC.

To find FC, consider triangle PDC
\[
\cos(\angle PDC) = \frac{DG}{DP} = \frac{5}{14}
\sin(\angle PDC) = \frac{\sqrt{171}}{14}
FC = 5 \sin \angle PDC = 4.7
\]
By symmetry, FA = FC
Considering triangle ADC,
\[
AC^2 = 5^2 + 5^2 = 50
AC = \sqrt{50}
\]
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Considering triangle AFC, by cosine rule
\[ \cos \beta = \frac{(4.7)^2 + (4.7)^2 - 50}{2(4.7)^2} \]
\[ \beta = 98.4^\circ \]

Example 7:

A square PQRS of side 10 cm lies on a horizontal plane. A second square PQXY of side 10 cm also lies in a plane inclined at an angle of 40° to the square PQRS. Find the angles made with the square PQRS by (i) PX and (ii) the plane PXS.

\[ \begin{array}{c}
\text{Solution:} \\
\text{Angle between PX and PQRS = angle XPA} \\
\text{Considering square PQXY,} \\
\quad \text{PX}^2 = 10^2 + 10^2 \\
\quad \text{PX} = 10\sqrt{2} \\
\text{Consider triangle QAX, AX = 10 sin 40^\circ} \\
\quad \text{sin } \alpha = \frac{AX}{PX} = \frac{10 \sin 40^\circ}{10\sqrt{2}} \\
\quad \therefore \alpha = 27.03^\circ \\
\text{(ii) Angle between PQRS and PXS = angle ABX = } \beta \\
\quad \text{tan } \beta = \frac{AX}{AB} = \frac{10 \sin 40^\circ}{10} \\
\quad \beta = 32.73^\circ \\
\end{array} \]

Example 8:

Each cross-section of a prism is a sector of a circle, of radius 4 cm, with angle at the centre equal to 45°. Two cross-sections are OAB and PDC where A, B, C, D lie on the curved surface of the prism and the vertical line OP is the intersection of the vertical plane face OADP and OBCP, as shown in the figure. The lines AD and BC are vertical. The cross-sections OAB and PDC are horizontal and 5 cm apart. Giving each answer correct to 3 significant figures, find
(i) the area of the curved surface ABCD.
(ii) the angle between AC and the plane OAB.
(iii) the angle between the planes OAC and OADP.
(iv) the area of the triangle OAC.
Solution:

EG // DA

(i) Area of curved surface = $5 \times \text{length of arc AB}$
   
   \[ = 5 \times 4 \left( \frac{\pi}{4} \right) = 15.7 \text{ cm}^2 \]

(ii) Angle between AC and plane OAB = angle BAC = $\alpha$

   \[ \tan \alpha = \frac{BC}{AB} = \frac{5}{AB} \]

   Consider triangle OAB,
   
   \[ AB = 2 \times BF \]
   
   \[ = 2(4 \sin 22.5^\circ) \]
   
   \[ = 3.06 \text{ cm} \]

   \[ \tan \alpha = \frac{5}{3.06} \Rightarrow \alpha = 58.5^\circ \]

(iii) Angle between OAC and OADP = angle CEG = $\beta$

   \[ \tan \beta = \frac{CG}{EG} = \frac{CG}{5} \]

   Considering triangle PCG,
   
   \[ CG = 4 \sin 45^\circ \]
   
   \[ \tan \beta = 4 \sin 45^\circ/5 \]

   \[ \beta = 29.5^\circ \]

(iv) Area of triangle OAC = $\frac{1}{2} \times OA \times EC$

   Consider triangle CEG, EC = $5/\cos \beta = 5.74 \text{ cm}$

   Area of triangle OAC = $\frac{1}{2} \times 4 \times 5.74 = 11.48 \text{ cm}^2$

Example 9:

The sides of the square ABCD are each of length a. The rectangle BKLC lies in a plane perpendicular to the plane ABCD and BK = CL = 2a. Find each of the following angles, giving your answers to the nearest tenth of a degree.

(a) the angle between the line AL and the plane AKB

(b) the angle AKC

(c) the angle between the planes ACK and ABCD

(d) the angle between the skew lines AD and KC.

Solution:

(a) $AK^2 = AB^2 + BK^2 = a^2 + (2a)^2$
   
   \[ = 5a^2 \]

   \[ AK = a\sqrt{5} \]

   Angle between line AL and plane AKB
   
   = Angle between line AL and line AK

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=angle LAK
=\tan^{-1}(LK/AK) = \tan^{-1}(a/a\sqrt{5}) = 24.1^\circ

\begin{align*}
\cos \angle AKC &= \frac{AK^2 + KC^2 - AC^2}{2 \times AK \times KC} \\
&= \frac{5a^2 + 5a^2 - 2a^2}{2 \times 5a^2} = 0.8 \\
\angle AKC &= 36.9^\circ
\end{align*}

(c) Let $\alpha$ be the angle between plane ACK and ABCD
$\alpha = \tan^{-1}(BK/\frac{1}{2} BD)$
$=\tan^{-1}(2a/\frac{1}{2}a\sqrt{2}) = 70.5^\circ$ (since $BD^2 = a^2 + a^2 = 2a^2$)

(d) Angle between AD and KC
=Angle between BC and KC
$=\tan^{-1}(2a/a) = 63.43^\circ$

Example 10:

A pyramid has a rectangular base ABCD and vertex V; VN is perpendicular to the base ABCD while VP, VQ, VR and VS respectively are perpendicular to the sides AB, BC, CD and DA of base. PR and QS meet at right angles at N. If AB = 18 units, BC = 16 units, VA = 22 units, VB = 14 units, VC = 6 units
(i) Calculate the lengths of BP and BQ
(ii) Prove that VD = 18 units and VN = $\sqrt{26}$ units.
(iii) Calculate the angle between the plane VDA and base ABCD, correct to the nearest 0.1°

Solution:
(i)  
\[22^2 - (18 - x)^2 = 14^2 - x^2\]
\[484 - 324 + 36x - x^2 = 14^2 - x^2\]
\[x = 1 = BP\]

\[14^2 - y^2 = 6^2 - (16 - y)^2\]
\[= 36 - 256 + 32y - y^2\]
\[y = 13 = BQ\]

(ii)  
\[DR = AP = 17\]
\[z^2 - 17^2 = 6^2 - 1^2\]
\[z = 18\]
\[VD = 18 \text{ units}\]

\[DS = CQ = 3\]
\[DN^2 = 3^2 + 17^2 = 298\]

\[VN^2 = 18^2 - 298\]
\[VN = \sqrt{26} \text{ units}\]

(iii)  
Angle between plane VDA and base ABCD = \(\theta\)
\[= \tan^{-1}(\sqrt{26}/17) = 16.7^\circ\]
Example 11:

The horizontal base of a pyramid is a rectangle ABCD, where AB = 6 cm and BC = 8 cm. The vertex V of the pyramid is such that VA = VB = VC = VD = 12 cm. Giving each answer to the nearest 0.1°, find

(i) the angle between the skew lines VA and BC
(ii) the inclination of the plane VAB to the horizontal
(iii) the angle between the two planes VAD and VBC

Find the length of the perpendicular from D to the line VC, correct to 3 significant figures.

Solution:

(i) Angle between VA and BC
   = Angle between VA and AD
   = ∠VAD
   Let M be the mid-point of AD
   ∠AMV = 90°, AM = 4 cm
   ∠VAD = ∠VAM = \cos^{-1}(4/12) = 70.5°

(ii) Let O be the centre of the base and N be the mid-point of AB
   ∠VNA = ∠VON = 90°
   VN = \sqrt{(VA^2 - AN^2)} = \sqrt{(12^2 - 3^2)} = \sqrt{135}
   Angle of inclination of VAB to horizontal = ∠VNO = \cos^{-1}(4/\sqrt{135}) = 69.6°

(iii) Angle between VAD and VBC = 2 × √MVO
    VM = \sqrt{(VA^2 - AM^2)} = \sqrt{(12^2 - 4^2)} = \sqrt{128}
    2 × √MVO = 2 \sin^{-1}(OM/VM)
    = 2 \sin^{-1}(3/\sqrt{128}) = 30.8°

Area of ΔVCD = Area of ΔVAB
   = \frac{1}{2} \times AB \times VN
   = \frac{1}{2} \times 6 \times \sqrt{135} = 3\sqrt{135}

Let F be foot of perpendicular from D to VC
   \frac{1}{2} \times DF \times VC = Area of ΔVCD
   \frac{1}{2} \times DF \times 12 = 3\sqrt{135}
   DF = 5.86 cm
Example 12:

The diagram shows a cube with all edges of unit length. P is a point on the edge FB such that PF = a. Find, in terms of a, the perpendicular distance from E to the line HP. If $\theta$ is the obtuse angle between the planes EHP and GHP, show that $\cos \theta = -1/(1 + a^2)$

If $a = 1/3$, find to the nearest 0.1º,

(a) the angle between the line HP and the plane ADHE,
(b) the angle between the planes EHP and EHC.

Solution:

EP = $\sqrt{(1 + a^2)}$  
HF = $\sqrt{2}$

HP = $\sqrt{(HE^2 + a^2)} = \sqrt{2 + a^2}$

Consider area of $\triangle$HEP: 
$\frac{1}{2} \times HP \times EX = \frac{1}{2} \times EH \times EP$

$\sqrt{(2 + a^2)} \times EX = \sqrt{(1 + a^2)}$

Perpendicular distance from E to HP = EX = $\frac{\sqrt{(1 + a^2)}}{\sqrt{(2 + a^2)}}$

$$\cos \theta = \frac{EX^2 + GX^2 - EG^2}{2 \times EX \times GX}$$

$$= 2 \left( \frac{1 + a^2}{2 + a^2} \right) - 2$$

$$= 2 \left( \frac{1 + a^2}{2 + a^2} \right)$$

$$= \frac{1 + a^2 - 2 - a^2}{2 + a^2}$$

$$= \frac{1}{1 + a^2}$$

(a) Angle between HP and ADHE = $\angle$ PHM

HM = $\sqrt{(HE^2 + EM^2)}$

$$= \sqrt{[1 + (1/3)^2]} = \sqrt{10/3}$$

$\angle$ PHM = $\tan^{-1}(PM/HM) = \tan^{-1}[1/(\sqrt{10}/3)] = 43.5^\circ$

(b) Angle between EHP and EHC

= Angle between EP and HC

= Angle between EP and EB = $\gamma$

$$\cos \gamma = \frac{EP^2 + EB^2 - PB^2}{2 \times EP \times EB}$$

$$= \frac{10 + 2 - \left( \frac{2}{3} \right)^2}{2 \sqrt{\frac{10}{9}} \sqrt{2}}$$

$$\gamma = 26.6^\circ$$