Asia-Pacific Journal of Operational Research 20 (2003) 41-56

TSORO AND HUNGARIAN APPROACHES: A HYBRID ALGORITHM FOR AN ASSIGNMENT PROBLEM

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Tsoro is a two-person game which is widely played in Southern African countries. The idea of the game is to optimize (maximize or minimize) the number of playing stones depending on the objective of the game. There are many versions of this game. In this paper, we describe briefly these games and give details of the Matabele version, as it has a winning strategy that can be modified and applied to solve an assignment problem. We have known that the Hungarian method of assignment always results in an optimal solution to an assignment problem, and its iterative attempts are such that it moves from one infeasible to another infeasible solution such that it is moving towards a feasible optimal solution, until it has found one. Although optimal solution is guaranteed, it may be accomplished in some cases after many iterations. An approach based on the Tsoro strategy gives a feasible solution, but does not guarantee that solution is optimal. Hence, the Tsoro feasible solution can be regarded as an upper bound on the assignment optimal solution and the Hungarian method solution as its lower bound. If the difference between these two bounds is not significant, Tsoro solution may be accepted as a good working solution. In this paper, we first describe the Tsoro game in general and its Matabele version in particular, and based on the Matabele version of Tsoro, we present an hybrid algorithm for an assignment problem.

Keywords: Tsoro game, Hungarian method of assignment, upper and lower bounds.

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Received August 2001, revised version received December 2001.

1. Introduction

Tsoro is a two-person rural game played in Southern African countries. The game is played with either stones or hard amurula nuts which are distributed unevenly in two or four rows of playing holes. There are many versions of the game, which vary from region to region. The players attempt to accumulate the playing stones to maximize or minimize the number of playing stones as determined by the objective of the game. The objective of this paper is to

(1) briefly explain some versions of the Tsoro games,
(2) select the Matabele version of Tsoro game and investigate its winning strategy, and
(3) based on the winning strategies of Tsoro and the Hungarian method of assignment, develop a hybrid algorithm for solving an assignment problem.

Section 2 deals with various versions of the Tsoro game, the Matabele version, its winning strategy, and some other variants of the game. In Section 3, we make some useful observations, and present the proposed hybrid algorithm in Section 4. Section 5 outlines a flowchart needed for execution of the algorithm. In Section 6, we discuss an illustrative example, along with a summary of computational experience we have had with the proposed algorithm.

2. Tsoro: The Game

2.1. History of the game

Tsoro is an African indigenous (probably Nyanja or Shona) word which means brain war between two parties. It also refers to a trick used by one party to win over the other party. The game is popular in the rural areas of Zimbabwe, Zambia, Malawi, South Africa and Mozambique. The game is believed to have originated in Malawi. It was then spread by Malawian migrant workers throughout Southern Africa in their quest for jobs in mines of its neighboring countries. The spread became more pronounced during the days of the Federation of Rhodesia and Nyasaland. It is believed that the game was exported to the neighboring countries by Nyasa's when they sought jobs in Zambian copper mines (then northern Rhodesia), Zimbabwe's coal, asbestos and gold mines (then Southern Rhodesia), and South Africa's gold and diamond mines (under the WENELA scheme). Tsoro game has acquired many different forms, for example, the authors are aware of (1) Matabele, (2) Zezuru Zigzag, (3) Manyika, (4) Karanga Big City, and (5) Nyanja versions of Tsoro game. Since we are going to use strategies of the Matabele version of Tsoro, we present a detailed description of this version of the game and mention others briefly.

2.2. Matabele version of Tsoro game

The game – Matabele is a district in Zimbabwe. Matabele version of Tsoro consists of two rows. Players i and j play the game with an agreed objective, that is, either maximize or minimize the total number of stones collected. Further, for this version of Tsoro game, it is assumed that

1. the number of holes in each row is equal,
2. the number of columns is a variable,
3. the number of playing stones in each hole is variable, and
4. each player has full knowledge of the number of stones in each hole.

Assuming that player i is the first to make a move, this player selects a hole in any column and picks all the playing stones in the selected hole. The competing player j is forced to pick stones in the other hole in the same column. When all stones have been taken out from the two holes in the same column, one iteration of the game is complete. If in the previous iteration player i has made the first selection, the next iteration allows player j to make the first selection from the holes which are still open for play and player i has no choice. The game ends when all playing stones have been picked by the two players i and j.

Winning strategy for the Matabele version of Tsoro game – Since the game can be played with an objective, either to maximize or minimize the total number of stones, let us consider without any loss of generality, that our objective is to minimize the total number of stones. In other words, the winning player is going to be the one that picks the smallest total number of stones. The winning strategy was developed by a certain Manyika boy, who whenever he played first, never lost a match. The strategy is as follows:

1. Select column k such that the difference between the number of playing stones in the holes in that column k is largest. That is, $\max_k |N_{1k} - N_{2k}|$, where $|N_{1k} - N_{2k}|$ is the positive difference between the number of stones in holes in column k. Here N_{ik} denotes the number of stones in row i and column k, $i = 1, 2;\ k = 1, 2, ..., n$.
2. Choose a hole having the smallest number of stones in column k selected above, if column k is unique.
3. If a tie exists between two columns, that tie is broken arbitrarily.

Illustration of the game by using the winning strategies – Consider a 8 column game with number of stones distributed as given in Table 1.

Table 1. Data for the game

k	1	2	3	4	5	6	7	8
Row 1	3	6	2	4	7	11	10	1
Row 2	9	1	4	3	10	2	4	3
$\lvert N_{1k} - N_{2k}\rvert$	6	5	2	1	3	9	6	2

Solution

The detailed solution is given in Table 2.

Table 2. Solution indicating game in favor of player i

k	i	j	Remarks
6	(2,6) = 2_1	(1,6) = 11_2	unique
1	(2,1) = 9_2	(1,1) = 3_1	*tie*, $k = 1, 7$
7	(2,7) = 4_1	(1,7) = 10_2	unique
2	(1,2) = 6_2	(2,2) = 1_1	unique
5	(1,5) = 7_1	(2,5) = 10_2	unique
8	(2,8) = 3_2	(1,8) = 1_1	*tie*, $k = 3, 8$
3	(1,3) = 2_1	(2,3) = 4_2	unique
4	(1,4) = 4_2	(2,4) = 3_1	unique
Total	37	43	

Note that the number 2_1 in row 1 of the above table indicates that player i had the first move and he selected the hole containing two stones in the cell (2, 6). The interpretation of the other numbers is similar. Further, it is evident from the above table that player i wins because he moves first. If player j had moved first, he would have won the game.

For the maximization game, the strategies are similar. Instead of picking the hole with minimum number of stones, the player who has a choice of selection would have gone for the hole with maximum number of stones in the selected column, k.

2.3. Other variants of the Tsoro game in Southern African countries

This rural game Tsoro is believed to have many different variants in different countries in the southern African region. Some popular variants, other than the Matabele version, are (a) Zezuru Zigzag, (b) Manyika continuous pick and drop, (c) Karanga Big City Pick and Spend, and (d) Nyanja. Some of these variants have four playing rows, with some partial control on two rows by each player and some interference by the opponents. These variants allow stones to be transferred from one hole to another, and also spend the stones as money units. As far as known to the authors, the Matabele version of the game is the only one with a winning strategy, and all other variants have no well defined preferred strategy for winning the game.

Hence our interest, in this paper, is confined only to the Matabele version of the game.

2.4. Other possible extensions of the Matabele version Tsoro game

Some possible extensions of the Matabele version of Tsoro game, which as far as known to the authors are not played in rural Africa, may be stated as follows:

2.4.1. Three person Tsoro game

Tsoro is a two person game. Consider that we have three rows and three players. Is it possible to establish a winning strategy for each player? Will the game once again be in favor of the person to make the first move? This idea can be generalized to n person n row Tsoro game.

2.4.2. Random termination game

The game may end randomly. The decision to end may be known to an external source, such as a referee but not known to the players. It is clear that the Matabele winning strategy will remain applicable to win the game.

2.4.3. Random number of columns to be added or deleted

Suppose after an iteration, a random number of columns may be added or deleted with known information about the stones contained in them, what is going to be the best strategy for this situation?

2.4.4. Stochastic version

Suppose we do not know the exact number of stones in each hole but we are given that they follow a distribution which is known to each player. Is it possible to establish a winning strategy?

Analysis of these various games may prove mathematically challenging and may have interesting applications. However, in this paper, we limit our discussion to the Matabele version which has a winning strategy. We study this strategy and apply it to solve an assignment problem.

3. Some Useful Observations

The mathematical model of an assignment problem is given by

$$\text{Minimize } \; x_0 = \sum_i \sum_j c_{ij} x_{ij}$$

subject to $\sum_i x_{ij} = 1, \; \forall j$ and $\sum_j x_{ij} = 1, \; \forall i$, where $x_{ij} = \{0, 1\}$ and $i, j = 1, 2, ..., n$.

The following observations are immediate.

3.1. Hungarian method of assignment

(1) The Hungarian method (HM) guarantees optimal solution. The method first writes equivalent cost matrix such that each row and column has at least one zero element. Total sums of these elements which were used

to create zero elements is a lower bound of the total cost associated with the optimal solution. Thus in the transformed matrix, if a feasible solution is possible in the cells with cost elements equal to zero, that assignment constitutes an optimal solution.

(2) However, if a feasible solution confined to zero elements in the transformed matrix is not possible, the HM creates more independent zeros by increasing the lower bound. A finite number of iterations guarantees the optimal solution as in each iteration the number of independent zero value cells are greater than or equal to the number of independent zero value cells compared to the previous iteration. Thus when a feasible solution in the transformed matrix is found, it corresponds to the optimal solution in the original cost matrix.

3.2. Tsoro strategy for assignment problems

It is known that the Matabele version of the Tsoro game deals with two rows only. However, the strategy used in playing the game can be extended for the case of more than two rows by taking the difference of the two smallest elements in each row and column, then choosing the smallest element in the row or column with the largest difference. The row and column corresponding to the chosen element are deleted from further consideration. The same process is repeated until n assignees have been determined. The sum of n elements chosen gives total cost of assignment. The assignment thus obtained may or may not be optimal. Since the solution is always feasible, but not necessarily optimal, it may be regarded as an upper bound on the optimal solution. There are a number of factors why the Tsoro strategy leads to an optimal solution in the Tsoro game, but not necessarily optimal for an assignment problem.

Alternatively, the Tsoro strategy for solving the AP can be observed as an application of Vogel's approximation in the transportation method, where for a balanced problem, we assume $s_i = 1,\ \forall i,\ d_j = 1,\ \forall j$ and $m = n$.

3.2.1. Why Tsoro strategy results in an optimal solution for the game?

Let the number of columns in the game be n. Let columns $k_1, k_2, ..., k_r$, $..., k_n$ be the order of selected columns such that

$$|N_{1k_1} - N^*_{2k_1}| \geq |N^*_{1k_2} - N_{2k_2}| ... \geq |N_{1k_r} - N^*_{2k_r} ... \geq |N^*_{1k_n} - N_{2_k n}| \tag{3.1}$$

where N_{ik_i} denotes the number of stones in row i and column k, and $N^*_{ik_i}$ denotes the number which is smaller among the two in that column. Let player i make the first move, hence player j will make the next move. The collection of playing stones for player i is $(N^*_{2k_1} + N_{2k_2})$ over the first two moves. Similarly for the player j, the collection in first two moves is given by $(N_{1k_1} + N^*_{1k_2})$. For player i to be better off than player j, we have

$$(N^*_{2k_1} + N_{2k_2}) \leqq (N_{1k_1} + N^*_{1k_2}) \tag{3.2}$$

or

$$-N^*_{1k_2} + N_{2k_2} \leqq N_{1k_1} - N^*_{2k_1} \ . \tag{3.3}$$

By assumption both the LHS and the RHS of (3.3) are positive quantities. Hence,

$$|N^*_{1k_2} - N_{2k_2}| \leqq |N_{1k_1} - N^*_{2k_1}| \tag{3.4}$$

which satisfies (3.1). Similarly, we can say that the sum of the third and fourth move will favor player i, that is, total of first four moves will still be less than or equal to the total for the player j. Thus for any even number of moves, player i will always be in a position to win the game. If the number of columns is odd, that is, $n = 2p + 1, p = 0, 1, ...,$ it is clear that the first $2p$ moves will be in favor of player i, provided he made the first move. The $(2p+1)$th move will be made by the player i, hence again will be in his favor. Hence independent of the total number of columns, the playing strategy of the game is such that the game will be won by the player who makes the first move.

3.2.2. Why Tsoro strategy is not necessarily optimal for AP?

In the Tsoro game, decisions are made under a real life situation, that is, once they are made, they cannot be reversed. Since by assumption player i has the first move, the game favors the player i. In the AP approach, we are given a set of outcomes, and the problem is how best can we optimize the total assignment? Although the game consists of two rows and AP consists of n rows, the analogy is that we still calculate our penalties using the difference of the two smallest entrants in each row and each column. Hence as a result, in Tsoro strategy for AP we end up having a collection of assignments which results in a feasible solution which has to qualify for global optimality as compared to the game where local optimality in each iteration is sufficient. Since Tsoro approach results in a feasible solution, it can easily serve as upper bound U of the optimal solution V to the AP .

3.3. Remarks for developing the hybrid algorithm

(1) The Hungarian method acts in the infeasible space. The first time it encounters a feasible solution, that solution becomes an optimal solution. Thus iterations are made towards obtaining an optimal solution,

the method is unable to indicate the quality of the current solution, and does not provide any idea on how far are we from the optimal solution.

(2) In the Tsoro approach, we have a feasible solution which acts as an upper bound and the HM gives the lower bound. Thus from these two values, the quality of Tsoro solution can be judged.

(3) In the HM approach, since there is no upper bound, maximum error in the current solution cannot be estimated. Since Tsoro provides an upper bound, we propose an alternative approach as an hybrid algorithm.

(4) In the HM, we improve the value of the lower bound in each iteration, similarly the upper bound obtained by Tsoro approach can also be improved towards an optimal solution. This improvement is achieved by an exchange process explained in Section 4.3.

4. Proposed Hybrid Algorithm

To explain the proposed hybrid algorithm, we need to develop some notation, Tsoro strategy for the AP, exchange process for decreasing the upper bound, and steps of the hybrid algorithm. These aspects are discussed in Sections 4.1 to 4.4.

4.1. Notation

The following notation are used in developing the hybrid algorithm for the AP.

(1) ρ_i – difference between the two smallest costs (numbers) in row i of the square cost matrix.

(2) δ_j – difference between the two smallest numbers in column j of the square cost matrix.

(3) n – dimension of the cost matrix, that is, $n \times n$.

(4) c_{ij} – cost associated with assignee i performing job j.

(5) x_{ij} – this is a number equal to either 1 or 0. If assignee i is assigned the job j, it is equal to one, otherwise equal to zero.

(6) $\rho^{n-k} = \max\{\rho_1, \rho_2, ..., \rho_{n-k}\}$, where k is the number of deleted rows in Tsoro solution to AP, since after each iteration a row and a column is deleted.

(7) $\delta^{n-k} = \max\{\delta_1, \delta_2, ..., \delta_{n-k}\}$, where k is the number of deleted columns in Tsoro solution to AP.

(8) The set T denotes the feasible solution as determined by Tsoro strategy, where T is given by

$$T = \{x_{i_1 j_1,1}, x_{i_2 j_2,2}, ..., x_{i_n j_n,n}\} \ .$$

Here, $x_{i_k j_k,k}$ denotes the value one in the kth cell, $k = 1, 2, ..., n$. All other x_{ij} are zero.

(9) C denotes a set corresponding to T,

$$C = \{c_{i_1 j_1,1}, c_{i_2 j_2,2}, ..., c_{i_n j_n,n}\} \; .$$

Here, $c_{i_k j_k,k}$ denotes the cost associated with $x_{i_k j_k,k} = 1$ in the kth cell, $k = 1, 2, ..., n$, as described by the set T.

(10) $\overline{T} = [n \times n] - T$, this is a set of all those elements which are not in the Tsoro solution set. The number of elements in the set $\overline{T}$ is $||n(\overline{T})|| = n^2 - n$. This is an even number, independent of the value of n.

(11) $\overline{C} = [n \times n] - C$, this is a set of costs of all those elements which are not in the Tsoro solution set. The number of elements in the set $\overline{C}$ is $||n(\overline{C})|| = n^2 - n$.

(12) C_u = maximum cost element in C.

(13) $\overline{C}_s = \{c_{ij} \in \overline{C} : c_{ij} \leq C_u\}$ is the set of selected elements for a possible exchange in the set $\overline{C}$. Note that the set $\overline{C}_s$ is a subset of $\overline{C}$.

(14) $\Delta_{ij} = \{c_{ij} + c_{lk} - \{c_{ik} + c_{lj}\}$ defines an exchange for a given element $c_{ij} \in \overline{C}_s$ where the elements c_{ij}, c_{lk}, c_{ik} and c_{lj} form a rectangle such that $c_{ik}, c_{lj} \in C$. Similarly, exchange can have six elements, three belongs to $\overline{C}$.

(15) A_v – acceptable variance of a feasible solution from optimal solution.

4.2. Tsoro strategy for AP

Tsoro strategy for solving assignment problems has been described in Section 3.2. Note this is similar to Vogel's approximation in transportation problem, however, relatively more affective here.

4.3. Exchange factor Δ_{ij}

Select an element $x_{ij} : c_{ij} \in \overline{C}_s$ as defined above in the section on notation, item (13). Since c_{lj} of the set $\overline{C}_s$, it is certain that in row i we have $c_{ik} \in C$, and in column j we have $c_{lj} \in C$ for some k and l. Thus, an exchange is said to occur if $\Delta_{ij} = \{c_{ij} + c_{lk}\} - \{c_{ik} + c_{lj}\} < 0$. The element x_{ij} and x_{lk} enter the set T, while x_{ik} and x_{lj} enter the complimentary set, $\overline{T}$. The new upper bound is $U' = U + \Delta_{ij}$. It is clear that exchange loop could be comprised of six elements, and exchange need not be performed on an element $x_{lj} : c_{ij} > C_u$, since for $\Delta_{ij} < 0$, $\{c_{ij} + c_{lk}\} < \{c_{ik} + c_{lj}\}$. This is possible only if at least one of the cost between c_{ij} and c_{lk} is less than $\max\{c_{ik}, c_{lj}\} \leq C_u$.

4.4. Hybrid algorithm

The following is the proposed hybrid algorithm to solve a balanced assignment problem.

(1) Enter the value of n. Let $k = 0$, then

(a) Calculate $\rho_i : i = 1, 2, ..., n-k$ and $\delta_j : j = 1, 2, ..., n-k$
(b) Determine ρ^{n-k} and δ^{n-k} from the following formulae:

$$\rho^{n-k} = \max_{\{i=1,2,...,n-k\}} (\rho_i) \tag{4.1}$$

$$\delta^{n-k} = \max_{\{i=1,2,...,n-k\}} (\delta_j) \tag{4.2}$$

(c) If $\rho^{n-k} > \delta^{n-k}$, select the row in cost matrix corresponding to ρ^{n-k}. Choose the smallest cost in that row and delete this row and corresponding column, or otherwise.
(d) If $\rho^{n-k} < \delta^{n-k}$, select the column corresponding to δ^{n-k}. Choose the minimum cost element in this column, delete this column and corresponding row from further consideration, or otherwise.
(e) If $\rho^{n-k} = \delta^{n-k}$, choose arbitrarily either the row or the column corresponding to tied ρ^{n-k} and δ^{n-k}. In the chosen row or column, select the minimum cost element in that row or column and delete both the row and the column corresponding to the chosen cell for further consideration.
(f) Let $k = k + 1$. If $n - k > 0$ go to (a), otherwise if $n - k = 0$ go to (g).
(g) Call the corresponding solution set, comprised of n elements as T the associated set of costs C, and the value of the upper bound U. Find C_u.

(2) Find the lower bound L by the usual approach using the HM of creating a zero in each row and each column.
(3) If $U = L$, we have obtained an optimal solution, go to step 8. If $U > L$, go to step 4.
(4) (a) Check feasibility of the HM solution in zero elements. If feasible, then L provides an optimal solution. Go to step 8.
   (b) If not feasible, and since $U > L$, check the value of 'h', that is, the smallest non-zero element not crossed out in the application of the Hungarian method solution. Let the new lower bound be $L \equiv L + h$. If $L = U$, conclude that Tsoro feasible solution, T, is optimal. Go to step 8, otherwise go to step 5.
(5) Develop the new equivalent matrix by the HM approach and go to step 6.
(6) Carry out exchange operation on each element of $\overline{C_s}$. For $i = 1, 2, ..., n$ check elements in row i. If for some element the exchange is successful, and the exchange loop was comprised of a four-sided loop, that is $\Delta_{ij} < 0$, then the new upper bound is $U \equiv U + \Delta_{ij}$. The sets T and C will change as two new elements will enter these sets and some two existing elements go out of these sets. However, if the successful exchange was

a six-sided loop, three new elements will enter and some existing three will go out of these sets, T and C. If $\frac{(U-L)}{L} \leq A_v$, go to step 8 to conclude that near optimal solution has been found, else go to step 5. Otherwise go to step 7.

(7) If exchange is not possible, Tsoro feasible solution T is optimal. Go to step 8.

(8) Print optimal solution.

5. Execution of the Method

5.1. Hungarian approach

This is a well documented approach. It is not discussed here. The reader is referred to any textbook, for example, Taha (1992).

5.2. Tsoro algorithm

A flow chart depicting the sequence of events in the application of the Tsoro algorithm to assignment problems is shown in Figure 1.

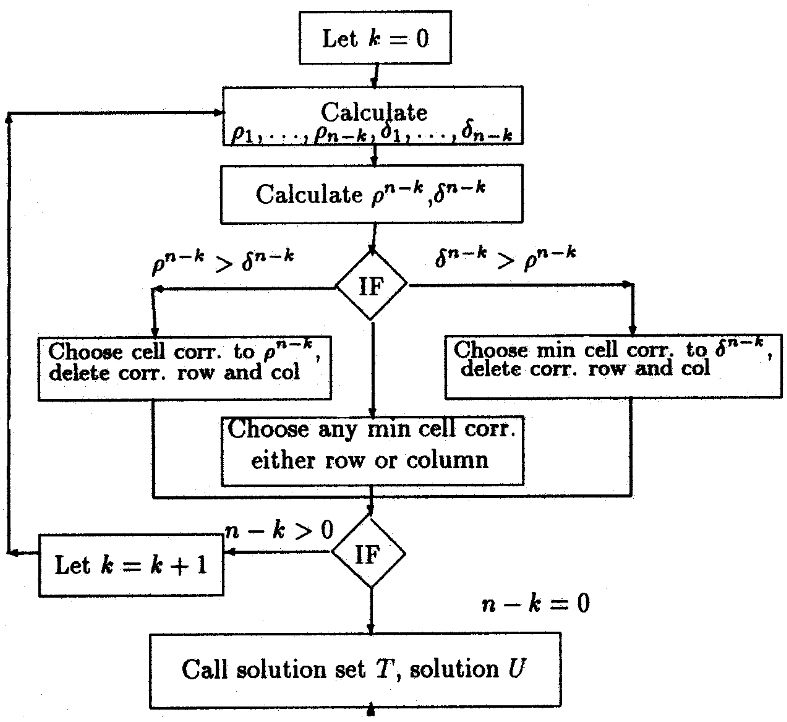

Figure 1.

5.3. Hybrid algorithm

A flow chart of the steps in the hybrid algorithm is given in Figure 2.

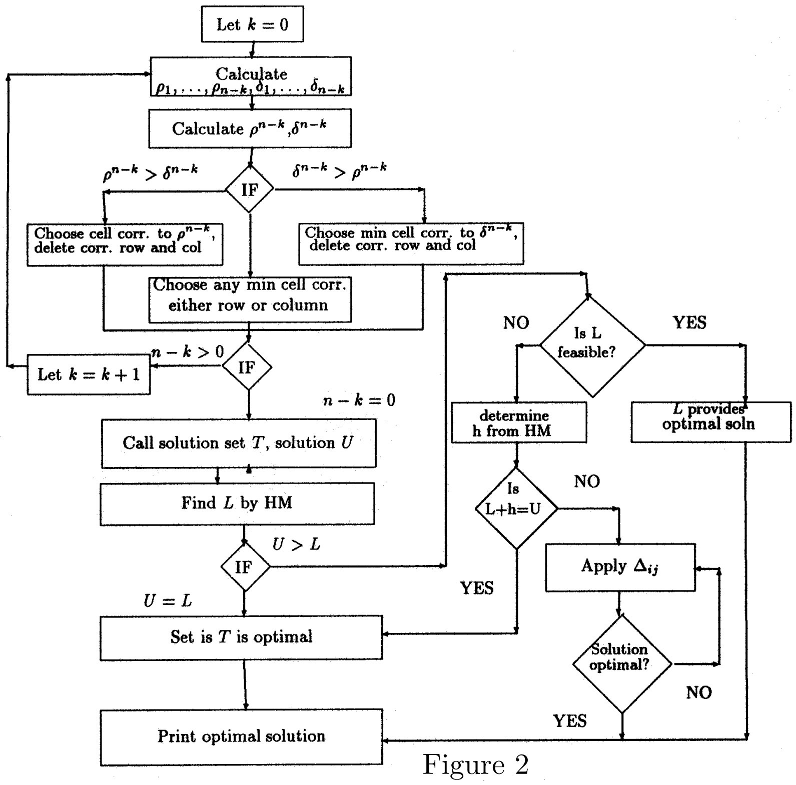

Figure 2

6. An Illustrative Example and Computational Experience

6.1. An illustrative example

Consider the following 5×5 cost matrix.

r_i/c_j	1	2	3	4	5
1	28	25	32	28	28
2	8	2	54	12	34
3	47	26	53	28	60
4	26	18	44	24	50
5	34	4	50	12	26

Solution

Step 1. (a)

r_i/c_j	1	2	3	4	5	ρ_i
1	28	25	32	28	28	3
2	8	2	54	12	34	6
3	47	26	53	28	60	2
4	26	18	44	24	50	6
5	34	4	50	12	26	8
δ_j	18	2	12	0	2	

$\rho^5 = \max_{\rho_i} = \rho_5 = 8$ and $\delta^5 = \max_{\delta_j} = \delta_1 = 18$. Let $x_{21} = 1$, since $\delta^5 > \rho^5$. Thus, row 2 and column 1 will be deleted from further consideration and the element x_{21} will be an element of the set T.

(b)

r_i/c_j		2	3	4	5	ρ_i
1		25	32	28	28	3
3		26	53	28	60	2
4		18	44	24	50	6
5		4	50	12	26	8
δ_j		14	12	0	2	

$\rho^4 = \max_{\rho_i} = \rho_5 = 8$ and $\delta^4 = \max_{\delta_j} = \delta_2 = 14$. Let $x_{52} = 1$, since $\delta^4 > \rho^4$. Thus, row 5 and column 2 will be deleted from further consideration and the element x_{52} will be the second element to enter the set T.

(c)

r_i/c_j			3	4	5	ρ_i
1			32	28	28	0
3			53	28	60	25
4			44	24	50	20
5						
δ_j			12	4	22	

$\rho^3 = \max_{\rho_i} = \rho_3 = 25$ and $\delta^3 = \max_{\delta_j} = \delta_5 = 22$. Let $x_{34} = 1$, since $\delta^3 > \rho^3$. Thus, row 3 and column 4 will be deleted from further consideration and the element x_{34} will be the third element to enter the set T.

(d)

r_i/c_j			3	4	5	ρ_i
1			32		28	4
3						
4			44		50	6
5						
δ_j			12		22	

$\rho^2 = \max_{\rho_i} = \rho_4 = 6$ and $\delta^2 = \max_{\delta_j} = \delta_5 = 22$. Let $x_{15} = 1$, since $\delta^2 > \rho^2$. Thus, row 1 and column 5 will be deleted from further consideration and the element x_{15} will be the fourth element to enter the set T. Finally, the element $x_{43} = 1$.

Thus, the set $T = \{x_{21}, x_{52}, x_{34}, x_{15}, x_{43}\}$ and corresponding to $C = \{8, 4, 28, 28, 44\}$ and the associated upper bound $U = \{8+4+28+28+44\} = 112$.

Now, we find the lower bound by the HM.

Step 2.

r_i/c_j	1	2	3	4	5	$\min_j(c_{ij})$
1	28	25	32	28	28	25
2	8	2	54	12	34	2
3	47	26	53	28	60	26
4	26	18	44	24	50	18
5	34	4	50	12	26	4

r_i/c_j	1	2	3	4	5
1	3	0	7	3	3
2	6	0	52	10	32
3	21	0	27	2	34
4	8	0	26	6	32
5	30	0	46	8	22
$\min_i(c_{ij})$	3	0	7	2	3

Hence, the lower bound $L = \{25+2+26+18+4+3+0+7+2+3\} = 90$. Thus, $U \neq L$. We can prepare a modified cost table such that each row and each column has a zero element. It can be easily seen that we do not get a feasible solution in the modified table. Hence to improve the value of the lower bound, we get the value of $h = 3$. Thus, the new lower bound will become 93. We note that $(U - L)/L = (112 - 93)/93 = 0.20 > A_v$. Hence, we must try to improve the upper bound by the exchange process. Thus from step 1(f) and step 6, we get $C = \{8, 4, 28, 28, 44\}$.

$$\overline{C} \ = \ \begin{array}{|c|c|c|c|c|} \hline 28 & 25 & 32 & 28 & \\ \hline & 2 & 54 & 12 & 34 \\ \hline 47 & 26 & 53 & & 60 \\ \hline 26 & 18 & & 24 & 50 \\ \hline 34 & & 50 & 12 & 26 \\ \hline \end{array} \text{ and } C_u = 44, \text{ implying that}$$

$$\overline{C_s} = \begin{array}{|c|c|c|c|c|} \hline 28 & 25 & 32 & 28 & \\ \hline & 2 & - & 12 & 34 \\ \hline - & 26 & - & & - \\ \hline 26 & 18 & & 24 & - \\ \hline 34 & & - & 12 & 26 \\ \hline \end{array}$$

The values of exchange factors are calculated. For example, consider $x_{11} \in \overline{C_s}$. The exchange value will be given by $\triangle_{11} = 28 + 34 - (8 + 28) = 2 > 0$. Similarly all other values are calculated and presented in the table below.

$\triangle_{ij} =$

Row\column	1	2	3	4	5
1	26	19	10	32	*
2	*	24	-	23	26
3	-	6	-	*	-
4	28	20	*	44	-
5	47	*	-	6	19

Note that * indicates an assigned cell, and - indicates $c_{ij} > C_u (= 44)$. Hence, it is an optimal solution.

6.2. Computational experience

A program was written to apply the hybrid algorithm in Matlab. This was used to solve 25 randomly generated problems in each size. An attempt was made to compare the efficiency of this method as compared to other available software. We had access to software, such as TORA, LINDO, CALISPSO and those available in (3.2), but they do not state time and number of iterations made to come up with an optimal solution. Hence, the table below presents a self-evaluation results of the proposed algorithm.

Size ($r \times c$)	% of problems reaching optimality at stage 4	% of problems reaching optimality at stage 6	% or problems reaching optimality after stage 6
5×5	48	27	25
8×8	40	30	30
10×10	45	25	30
15×15	30	34	36
20×20	36	16	48

From the computational experience we have gathered, we believe that the hybrid approach performs better as the optimal solution is guaranteed and the user can control error through the parameter value A_v. The coefficient $(U - L)/L$ always gives a relative forecast of the error in the current Tsoro feasible solution from the available lower bound.

Acknowledgement

The authors are thankful to two anonymous referees for their comments which have improved this paper. Thanks are also due to the Research Board, NUST, Zimbabwe for their financial support for this work.

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