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REDUCING INCONSISTENCY IN FUZZY AHP BY MATHEMATICAL PROGRAMMING MODELS

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The inconsistency of judgments in the fuzzy Analytic Hierarchy Process (AHP) is a crucial issue. To make the appropriate decision, the inconsistency in decision maker's (DM) judgments needs to be eliminated or reduced. This paper proposes two mathematical models to deal with inconsistency in fuzzy AHP. In the first model, the DM's judgments are modified where the preference order of the DM's judgments remained unchanged. The second model allows reversing the preference orders of judgments. The proposed models aim to eliminate or reduce the inconsistency of fuzzy AHP by changing judgments. The models cause fewer changes for the high certain judgments. Two examples solved by the proposed models are included for purposes of illustration.

Keywords: Analytic Hierarchy Process (AHP); fuzzy set; inconsistency.

1. Introduction

The Analytic Hierarchy Process (AHP) method was introduced by Saaty (1980). This method elicits preferences through pairwise comparisons in which the decision maker (DM) considers the relative importance of two factors at a time with respect to a common higher level criterion. For each comparison, the DM indicates the intensity of preference of one factor over another as a point estimate (in classical AHP) or a fuzzy term/number (in fuzzy AHP) on an appropriate scale. In general, evaluation and calculation in AHP can be divided into four stages: (1) scoring the alternatives under each criterion; (2) weighting the criterion; (3) calculating the final score; and (4) ranking and final decision.

Van Laarhoven and Pedrycs (1993) used the triangular membership function for the imprecise elicitation of pairwise comparisons and the computation of corresponding fuzzy weights. Buckley (1985) and Boender *et al.* (1989) extended these results to a more general membership function, and employed the logarithmic least square method to compute the local priorities. Salo (1996) was the first to investigate the issue of consistency in fuzzy AHP. He applied an auxiliary programming formulation with constraints on the membership values of local priorities. He assumed that a fuzzy matrix is consistent if there is a set of crisp relative weights within the feasible region defined by the ranges of mean values. To obtain feasible weights when the matrix is inconsistent, Salo developed a simple heuristic that enlarges the feasible region. Arbel (1989) showed the feasible region of relative weights in terms of linear inequalities. Salo and Hämäläinen (1995) extended this work such that it requires only quasi-concave and continuous fuzzy comparison ratios. Leung and Cao (2000) proposed a fuzzy consistency test, which is based on the feasible region of relative weights. They used information contained in the region for analysis. Allowing a certain tolerance deviation to the bounds of mean membership values, they modified the feasible region and defined fuzzy consistency as the existence of information within it. They provided a basis for the exclusion of irrelevant information and the inclusion of relevant information.

The most important criticism on the aforementioned works is their failure to address the issue of modifying judgments in the inconsistent comparison matrix. In this paper, we adopt the feasible region of relative weights as a framework and apply the definition of fuzzy consistency as the existence of relative weights within the region. Using mathematical programming as a tool, we alter the inconsistent ratios.

This paper is organized as follows. Section 2 defines the problem. Section 3 discusses the consistency calculation in the fuzzy AHP. Providing the necessary definitions and assumptions, Section 4 develops two mathematical models to eliminate/reduce inconsistency in fuzzy comparison matrices. Two illustrative examples are provided in Section 5. Section 6 makes some concluding remarks.

2. Problem Definition

One of the main problems in (the crisp and fuzzy) AHP method is the issue of inconsistency in the DM's opinions. Saaty (1980) referred to weaknesses resulting from this problem and provided a consistency test for crisp AHP. Leung and Cao (2000) developed a model to test the consistency in fuzzy AHP.

Working with inconsistent comparison matrices may provide inappropriate solutions. Therefore, the inconsistency should be eliminated or at least be reduced. For this purpose, we may change some or all the judgments in the earlier matrix. Thus, we need to know the ratios that might change and the amount of their changes in the earlier comparison matrix to prepare a new comparison matrix.

"Expert Choice" as a tool for crisp AHP can determine the contribution of each ratio in creating inconsistency and the ratio with the most contribution. Then, it helps the user by providing new values for crisp ratios that need to be changed. But there is no such tool for fuzzy AHP. In this case, due to fuzziness of ratios, it is not easy to determine the ratios needed to be changed and their new values. This paper develops an approach to eliminate the inconsistency of the fuzzy comparison matrix. The consistency test developed by Leung and Cao is employed as the basis of our work, and hence their work is described in Section 3.

3. Consistency Calculation in Fuzzy AHP

Leung and Cao developed a mathematical model to check the consistency in fuzzy AHP. They assumed that the elements of the fuzzy comparison matrix are fuzzy numbers with continuous and quasi-concave membership functions. In this situation, considering the normalized weights, the α-level cut feasible region (S_α) for a fuzzy comparison matrix is defined as follows:

$$S_\alpha = \left\{ W\colon l_{ij\alpha} \le \frac{w_i}{w_j} \le u_{ij\alpha},\ i \ne j = 1, 2, \ldots, n,\ w_j \ge 0,\ \sum_{j=1}^{n} w_j = 1 \right\} \tag{3.1}$$

where w_i is the normalized weight of criterion i, and $l_{ij\alpha}$ and $u_{ij\alpha}$ are the upper and lower bounds of the α-level cut for the fuzzy number R_{ij}. In fact, the region S_α gives feasible weights whose membership values are not less than α. Letting $\alpha = 1$, Leung and Cao interpreted the 1-level cut as the most stringent requirement for fuzzy consistency. They also argued that a fuzzy comparison matrix is consistent if and only if $S_1 \ne \emptyset$.

This analysis is given for perfect consistency, based on which α-level cut the feasible region (S_α) is constructed. Since, in practice, a level of judgment deviation should be allowed, they incorporated a tolerance deviation to their consistency analysis. Assuming that a tolerance deviation (δ) is allowed, they expanded the feasible relative region (S'_α) as follows:

$$S'_\alpha = \left\{ W\colon (1-\delta) \cdot l_{ij\alpha} \le \frac{w_i}{w_j} \le (1+\delta) \cdot u_{ij\alpha},\ i \ne j = 1, 2, \ldots, n, \right.$$
$$\left. w_j \ge 0, \sum_{j=1}^{n} w_j = 1 \right\} \tag{3.2}$$

where δ represents the deviation of the ratios from the upper bound $u_{ij\alpha}$ and the lower bound $l_{ij\alpha}$.

Leung and Cao proved that the size of tolerance deviation is identical to the size of consistency tolerance. Based on Saaty's consistency index the maximum consistency tolerance allowed is $0.1 \times \text{RI}$ (where RI is the ratio of inconsistency). In this approach the consistency test is based on the most stringent case, which is $\alpha = 1$ for quasi-concavity. It means that if a solution exists in S_1, then a solution will exist in S'_α, for all lower levels of α. They developed the following auxiliary linear programming to test the consistency of the fuzzy comparison matrix within the tolerance deviation δ.

$$\min \ \beta = \beta_1 + \beta_2 \tag{3.3}$$

$$\ln(1-\delta)l_{ij\alpha} \leq \ln w_i - \ln w_j + \beta_{ij}^1 - \beta_{ij}^2 \leq \ln(1+\delta)u_{ij\alpha} \qquad \forall i \neq j \tag{3.4}$$

$$\beta_1 \geq \beta_{ij}^1 \qquad \forall i \neq j \tag{3.5}$$

$$\beta_2 \geq \beta_{ij}^2 \qquad \forall i \neq j \tag{3.6}$$

$$\beta_{ij}^1, \beta_{ij}^2 \geq 0 \qquad \forall i \neq j \tag{3.7}$$

where $\ln w_i$, β_{ij}^1, β_{ij}^2, β_1, and β_2, are decision variables.

The fuzzy comparison matrix is consistent within deviation tolerance δ, if $\beta = 0$. Otherwise there are no feasible weights ($S'_1 = \emptyset$), and so the fuzzy comparison matrix is not consistent within δ. In this case, the decision maker would need to redo his/her judgments in the comparison matrix.

4. Description of the Proposed Models

As mentioned in the previous section, DM would need to redo the matrix when the fuzzy comparison matrix is not consistent. Extending the fuzzy consistency test developed by Leung and Cao, this paper proposes two models to find the new ratios. The key point is that the minimum changes in earlier judgments should occur to construct a consistent matrix. The models suggested in this paper are based on the following definition and assumptions.

4.1. *Definitions and assumptions*

In order to provide an appropriate context for the model development, we make the following definitions and assumptions.

1. The DM' judgments are ratios stated using fuzzy numbers with quasi-concave and continuous membership functions. These fuzzy numbers are shown as (m, l, u, d) in which l and u are the min and max of the FN for $\alpha = 1$ and m and d are the min and max of the fuzzy number (FN) for $\alpha = 0$, respectively. Values m, l, u, and d are called the "*parameters*" of a fuzzy number. In a fuzzy number, if $l \geq 1$, then FN is considered larger than 1. When $u \leq 1$, the FN is considered to be less than 1. The set of ratios in which l is larger than 1 is shown as Ω.
2. The parameters of new fuzzy numbers indicating the revised judgments should have appropriate values. The parameters are to be selected from the set Ψ as follows:

$$\Psi = \left\{\frac{1}{9}, \frac{1}{8}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1, 2, 3, 4, 5, 6, 7, 8, 9\right\}$$

The above values are called *allowed points* (AP). In fact, the parameters of fuzzy numbers can only be selected from the *AP set.*

3. *The distance* between two adjacent allowed points is called *a sub-interval* (SI). For example, the distance between allowed points $\frac{1}{6}$ and $\frac{1}{5}$ consists of one SI. The *distance* between two nonadjacent APs equals the number of SIs between those points. For example, the distance between APs $\frac{1}{3}$ and 2 consists of three SIs.
4. *The structure* of a FN, shown as a triple (λ, θ, π), is defined as the number of SIs between m to l, l to u, and u to d, respectively. Based on this definition, two different cases may occur during the process of changing the judgments. In the first case, new FNs have the same structure as the earlier FNs (unchanged structure) as shown in Figure 1. But in the second case, new FNs have different structures compared to the earlier ones (changed structure) as shown in Figure 2.

   Our main assumption is that the structure of FNs during the process of eliminating inconsistency is fixed. In other words, the new FN (say R') is obtained by moving the earlier FN (say R) toward the left or right side, as shown in Figure 1, while its structure is kept unchanged ($\lambda = \lambda'$, $\theta = \theta'$, $\pi = \pi'$). For example, suppose that the earlier FN is $R = (5, 6, 7, 8)$ and the new FN is $R' = (\frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4})$; therefore, the structures of R and R' are the same, i.e., $(1, 1, 1)$.
5. Two FNs are said to have the *same preference order* if both are greater than one or both are less than one. Otherwise FNs have the *opposite preference order.* Based on this concept, two different cases may occur during the process of changing the judgment to eliminate the inconsistency. In the first case, new FNs have the same preference order as the earlier FNs with smaller or larger values (unchanged preference order). But in the second case, new ratios are in orders opposite to that of earlier ones (changed preference order).

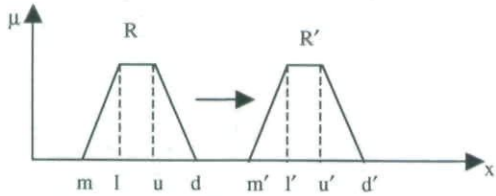

Fig. 1. The earlier (R) and new (R') ratios with the same structures.

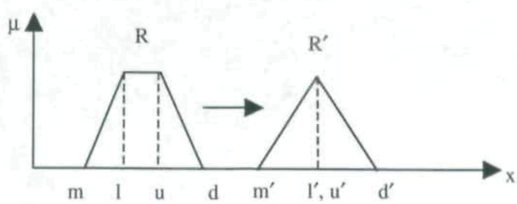

Fig. 2. The earlier (R) and new (R') ratios with different structures.

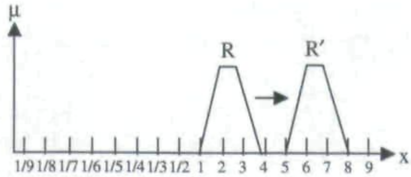

Fig. 3. The earlier (R) and new (R') ratios with the same preference order.

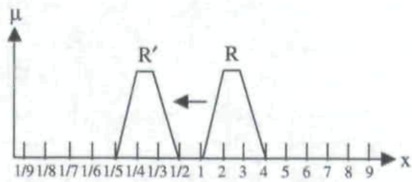

Fig. 4. The earlier (R) and new (R') ratios with the opposite preference order.

By change of preference order, we mean that if the earlier judgment states: "*a is better than b*," the new judgment says: "*b is better than a*" and vice versa. Converting the earlier FN (say R) to the new FN (say R') without the change of preference order is shown in Figure 3. Figure 4 indicates the change of preference order of earlier FN.

6. The *amount of change* of converting an earlier FN (say R) to a new FN (say R') is defined as the number of SIs between l and l'. For example, the amount of changes caused by converting R to R' in Figure 4 is four SIs. It is obvious that the reduction of inconsistency should involve the minimum total amount of changes.
7. The DM has different levels of confidence in his/her judgments and a certainty factor (CF) index is used to show his/her confidence. CF is a real number in $[0, 1]$, where $\text{CF} = 1$ shows a complete confidence in the judgments and $\text{CF} = 0$ shows no confidence in the judgments. The values from 1 to 0 reflect a gradual decrease in confidence.

   Usually DMs would not like to get many changes in earlier judgments when they have higher CF indexes. It is assumed that the DM can specify the CFs of all judgments. If the DM does not specify the CFs, they are all considered to be equal.
8. Due to the invertibility of the comparison matrix, only half of the judgments are investigated. For ease of computation, only ratios stated with FNs greater than one are selected.

Based on these definitions and assumptions two mathematical models to eliminate/reduce inconsistency in fuzzy AHP are proposed.

4.2. *The unchangeable preference orders model*

In this model, it is assumed that the preference order changes of ratios are not allowed, which is compatible with the first case discussed in assumption 5. For this case, the following mathematical programming model is developed.

Model 1

$$\text{Min } Z = \sum_i \sum_j P_{ij}(l'_{ij} - l_{ij})^2 \qquad \forall (i,j) \in \Omega \tag{4.1}$$

$$\ln(1-\delta) l'_{ij} \leq \ln w_i - \ln w_j \leq \ln(1+\delta) u'_{ij} \qquad \forall i \neq j \tag{4.2}$$

$$l'_{ji} = \frac{1}{u'_{ij}} \qquad \forall (i,j) \in \Omega \tag{4.3}$$

$$u'_{ji} = \frac{1}{l'_{ij}} \qquad \forall (i,j) \in \Omega \tag{4.4}$$

$$1 \leq l'_{ij} \leq 9 - (\pi_{ij} + \theta_{ij}) \qquad \forall (i,j) \in \Omega \tag{4.5}$$

$$u'_{ij} = l'_{ij} + \theta_{ij} \qquad \forall (i,j) \in \Omega \tag{4.6}$$

$$l'_{ij} = \text{integer} \qquad \forall (i,j) \in \Omega \tag{4.7}$$

where $\ln w_i$, l'_{ij}, and u'_{ij} are decision variables. The parameters of the earlier FN in row i and column j (say R_{ij}) are denoted by $(m_{ij}, l_{ij}, u_{ij}, d_{ij})$. Other notations are as follows:

CF_{ij}: the certainty factor of R_{ij}. In fact, CF_{ij} is the certainty factor of judgment stated for row i and column j in the comparison matrix.

P_{ij}: the penalty coefficient of changes in R_{ij}, which is based on the CF_{ij} of the ratio. It is assumed that $\mathrm{CF}_{ij} = \mathrm{CF}_{ji}$. The following linear function may be applied to calculate the penalty value:

$$P_{ij} = P_{\min} + (P_{\max} - P_{\min})\mathrm{CF}_{ij} \tag{4.8}$$

where $P_{\min}$ and $P_{\max}$ are values pre-determined by the DM.

The equations in Model 1 are described as follows: Equation (4.1) is the objective function, which aims to minimize the amount of changes created from converting the earlier judgments into new ones. As mentioned already the amount of changes is calculated based on the number of SIs between l (in the earlier FN) and l' (in the new FN).

Equations (4.2) are constraints to get a fuzzy consistent comparison matrix, which are adopted from Leung and Cao.

Equations (4.3) and (4.4) are constraints that make the new comparison matrix invertible.

Equation (4.5) ensures that new ratios have appropriate values, i.e., parameters of new FNs belong to the allowed points set Ψ.

Equation (4.6) controls the structures of FNs to remain fixed. This equation is adopted from assumption 4. As a result, the structures of new FNs are the same as the structure of earlier FNs.

Equation (4.7) is a constraint related to the domain of variation for variables.

As mentioned already, Model 1 assumes that the preference orders are fixed. Another model for changing of preference orders is provided in Section 4.3.

4.3. *The changeable preference orders model*

In this model, the preference orders of FNs can be altered which is compatible with the second case discussed in assumption 5. For this case, the following mathematical programming model is proposed:

Model 2

$$\begin{aligned}\text{Min } Z = \sum_i \sum_j P_{ij} \Bigg\{ & (1 - t_{ij})(l'_{ij} - l_{ij})^2 \\ & + t_{ij} \left((l_{ij} - 1) + \left(\frac{1}{l'_{ij}} - 1 \right) \right)^2 \Bigg\} \qquad \forall (i,j) \in \Omega \quad (4.9)\end{aligned}$$

$$\ln(1 - \delta) l'_{ij} \leq \ln w_i - \ln w_j \leq \ln(1 + \delta) u'_{ij} \qquad \forall i \neq j \qquad (4.10)$$

$$l'_{ji} = \frac{1}{u'_{ij}} \qquad \forall (i,j) \in \Omega \qquad (4.11)$$

$$u'_{ji} = \frac{1}{l'_{ij}} \qquad \forall (i,j) \in \Omega \qquad (4.12)$$

$$1 \leq S_{ij} \leq 9 \qquad \forall (i,j) \in \Omega \qquad (4.13)$$

$$l'_{ij} = S_{ij}(1 - t_{ij}) + \frac{1}{S_{ij}}(t_{ij}) \qquad \forall (i,j) \in \Omega \qquad (4.14)$$

$$\begin{aligned}&(1 - t_{ij}) + t_{ij} \left[\frac{1}{9 - \lambda_{ij}} \right] \\ &\quad \leq l'_{ij} \leq (1 - t_{ij})[9 - (\pi_{ij} + \theta_{ij})] + t_{ij} \left[\frac{1}{\theta_{ij} + 1} \right] \qquad \forall (i,j) \in \Omega \qquad (4.15)\end{aligned}$$

$$u'_{ij} = (1 - t_{ij})[l'_{ij} + \theta_{ij}] + t_{ij} \left[\frac{1}{(1/l'_{ij}) - \theta_{ij}} \right] \qquad \forall (i,j) \in \Omega \qquad (4.16)$$

$$S_{ij} = \text{integer} \quad t_{i,j} \in (0,1) \qquad \forall (i,j) \in \Omega \qquad (4.17)$$

The notations in Models 1 and 2 are the same. The additional decision variables are as follows:

S_{ij}: an integer variable in the interval [1–9],
t_{ij}: a 0–1 variable that determines the preference order of the new ratio (say R'_{ij}). If $t_{ij} = 0$, then $R'_{ij} \geq 1$, otherwise $R'_{ij} < 1$.

The equations of Model 2 are described as follows: Equation (4.9) is the objective function and aims to minimize the total amount of changes. This objective function is obtained by revising the objective function of the first model so as to calculate the amount of changes based on the number of SIs when a change in preference orders is allowed.

Equations (4.10)–(4.12) have the same descriptions as in the first model.

Equation (4.13) generates integer variables in the interval [1–9].

Equations (4.14) and (4.15) are constraints that allow new ratios to have appropriate values from Ψ.

Equation (4.16) controls the structure of FNs to be fixed.

Equation (4.17) is the constraint of domain of variation for variables.

In applying these models the following points should be considered:

1. Comparing the first model to the second one, it is obvious that the first model with less variables and constraints is simpler to solve. But, it should be noted that some inconsistent fuzzy comparison matrices may not be converted to consistent ones without many changes in the earlier judgments. This is because the first model does not allow a change in preference orders. In this case, the use of the second model may cause a few changes.
2. Some DMs may not accept changes in their judgments suggested by the proposed models. If the DM refuses to accept the changes and tends to provide his/her own new matrix, he/she can use the matrix suggested by models as a guideline. In other words, the suggested matrix gives new vision to the DM and helps him/her to revise his/her judgments appropriately. Of course, the intervention of the DM in the construction of new matrix does not guarantee the elimination of inconsistency, but it may reduce inconsistency of the earlier matrix.
3. Regarding the parameters of new FNs, the proposed models specify only l' and u' as decision variables. The other two parameters, i.e., m' and d' are computed based on the values l' and u' (from the new FN) and values of λ and π (from the earlier FN). This is based on assumption 4 which says the structures of earlier and new FNs are similar; i.e., $\lambda = \lambda'$, $\theta = \theta'$, $\pi = \pi'$.

5. Illustrative Examples

In this section, we illustrate our approach by solving two examples. Both examples are solved using the unchangeable preference orders model (the first model) and the changeable preference orders model (the second model).

5.1. *The first example*

Consider the first example again and suppose that the decision maker gives the fuzzy comparison matrix as shown in Table 1. Suppose also that the DM gives the following CFs as his/her certainty to ratios as shown in Table 2. Moreover, assume that the DM uses $P_{\min} = 0$ and $P_{\max} = 100$ in the penalty function to calculate the penalty values as follows:

$$P_{ij} = 100 \times \mathrm{CF}_{ij}$$

Applying the consistency test (with $\delta = 0.112$), we found out that the fuzzy comparison matrix is not consistent ($\beta = 1.01$). To get a consistent fuzzy comparison matrix, we have applied both models. For this example, both models give the same results as shown in Table 3.

Table 1. The earlier fuzzy comparison matrix for the first example.

X_i \ X_j	X_1	X_2	X_3	X_4	X_5
X_1	1	$(1, 2, 4, 5)$	$(\frac{1}{9}, \frac{1}{8}, \frac{1}{7}, \frac{1}{6})$	$(\frac{1}{9}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5})$	$(\frac{1}{8}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5})$
X_2	$(\frac{1}{5}, \frac{1}{4}, \frac{1}{2}, 1)$	1	$(\frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9})$	$(\frac{1}{7}, \frac{1}{6}, \frac{1}{6}, \frac{1}{5})$	$(\frac{1}{4}, \frac{1}{2}, \frac{1}{2}, 1)$
X_3	$(6, 7, 8, 9)$	$(9, 9, 9, 9)$	1	$(1, 2, 4, 5)$	$(1, 2, 3, 4)$
X_4	$(5, 6, 7, 9)$	$(5, 6, 6, 7)$	$(\frac{1}{5}, \frac{1}{4}, \frac{1}{2}, 1)$	1	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{2}, \frac{1}{2})$
X_5	$(5, 6, 7, 8)$	$(1, 2, 2, 4)$	$(\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1)$	$(2, 2, 3, 3)$	1

Table 2. CF matrix for DM's judgments.

X_i \ X_j	X_1	X_2	X_3	X_4	X_5
X_1	—	0.9	1	0.8	0.9
X_2	0.9	—	1	1	1
X_3	1	1	—	1	0.9
X_4	0.8	1	1	—	0.8
X_5	0.9	1	0.9	0.8	—

Table 3. The new fuzzy comparison matrix for the first example.

X_i \ X_j	X_1	X_2	X_3	X_4	X_5
X_1	1	$(\frac{1}{2}, 1, 3, 4)$	$(\frac{1}{9}, \frac{1}{8}, \frac{1}{7}, \frac{1}{6})$	$(\frac{1}{8}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4})$	$(\frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4})$
X_2	$(\frac{1}{4}, \frac{1}{3}, 1, \frac{1}{2})$	1	$(\frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9})$	$(\frac{1}{6}, \frac{1}{5}, \frac{1}{5}, \frac{1}{4})$	$(\frac{1}{6}, \frac{1}{4}, \frac{1}{4}, \frac{1}{3})$
X_3	$(6, 7, 8, 9)$	$(9, 9, 9, 9)$	1	$(1, 2, 4, 5)$	$(1, 2, 3, 4)$
X_4	$(4, 5, 6, 8)$	$(4, 5, 5, 6)$	$(\frac{1}{5}, \frac{1}{4}, \frac{1}{2}, 1)$	1	$(\frac{1}{2}, \frac{1}{2}, 1, 1)$
X_5	$(4, 5, 6, 7)$	$(3, 4, 4, 6)$	$(\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1)$	$(1, 1, 2, 2)$	1

As results show, judgments R_{12}, R_{41}, R_{42}, R_{51}, R_{52}, and R_{54} (and their inverses) have changed. The new matrix is consistent ($\beta = 0$) and the amount of objective function is $Z = 840$.

5.2. *The second example*

As the second example, suppose that a decision maker gives the fuzzy comparison matrix as shown in Table 4. Suppose also that the DM gives the following CFs as his/her certainty to ratios as shown in Table 5. Moreover, assume that the DM uses $P_{\min} = 0$ and $P_{\max} = 100$ in the penalty function to calculate the penalty values as follows:

$$P_{ij} = 100 \times \mathrm{CF}_{ij}$$

Applying the consistency test (with $\delta = 0.09$), we found out that the fuzzy comparison matrix is not consistent ($\beta = 2.28$). To get a consistent fuzzy comparison matrix, we applied the unchangeable preference orders model. The results of this model are shown in Table 6.

Table 4. The earlier fuzzy comparison matrix for the second example.

$X_i \backslash X_j$	X_1	X_2	X_3	X_4
X_1	1	$(3, 4, 6, 7)$	$(\frac{1}{7}, \frac{1}{7}, \frac{1}{5}, \frac{1}{5})$	$(\frac{1}{6}, \frac{1}{5}, \frac{1}{5}, \frac{1}{4})$
X_2	$(\frac{1}{7}, \frac{1}{6}, \frac{1}{4}, \frac{1}{3})$	1	$(1, 2, 3, 4)$	$(\frac{1}{9}, \frac{1}{8}, \frac{1}{7}, \frac{1}{5})$
X_3	$(5, 5, 7, 7)$	$(\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1)$	1	$(\frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2})$
X_4	$(4, 5, 5, 6)$	$(5, 7, 8, 9)$	$(2, 3, 4, 6)$	1

Table 5. CF matrix for DM's judgments.

$X_i \backslash X_j$	X_1	X_2	X_3	X_4
X_1	—	0.9	1	0.8
X_2	0.9	—	1	1
X_3	1	1	—	1
X_4	0.8	1	1	—

Table 6. The new fuzzy comparison matrix using the first model.

$X_i \backslash X_j$	X_1	X_2	X_3	X_4
X_1	1	$(\frac{1}{2}, 1, 3, 5)$	$(\frac{1}{3}, \frac{1}{3}, 1, 1)$	$(\frac{1}{6}, \frac{1}{5}, \frac{1}{5}, \frac{1}{4})$
X_2	$(\frac{1}{5}, \frac{1}{3}, 1, 2)$	1	$(\frac{1}{2}, 1, 2, 3)$	$(\frac{1}{8}, \frac{1}{7}, \frac{1}{6}, \frac{1}{4})$
X_3	$(1, 1, 3, 3)$	$(\frac{1}{3}, \frac{1}{2}, 1, 2)$	1	$(\frac{1}{7}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3})$
X_4	$(4, 5, 5, 6)$	$(4, 6, 7, 8)$	$(3, 4, 5, 7)$	1

Table 7. The new fuzzy comparison matrix using the second model.

$X_i \backslash X_j$	X_1	X_2	X_3	X_4
X_1	1	$(\frac{1}{2}, 1, 3, 5)$	$(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, \frac{1}{2})$	$(\frac{1}{6}, \frac{1}{5}, \frac{1}{5}, \frac{1}{4})$
X_2	$(\frac{1}{5}, \frac{1}{3}, 1, 2)$	1	$(\frac{1}{3}, \frac{1}{2}, 1, 2)$	$(\frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2})$
X_3	$(2, 2, 4, 4)$	$(\frac{1}{2}, 1, 2, 3)$	1	$(\frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2})$
X_4	$(4, 5, 5, 6)$	$(4, 6, 7, 8)$	$(2, 3, 4, 6)$	1

As the results show, judgments R_{12}, R_{23}, R_{31}, R_{42}, and R_{43} (and their inverses) have changed. The new matrix is consistent ($\beta = 0$) and the amount of objective function is $Z = 2710$.

Applying the changeable preference orders model another new matrix is found, as shown in Table 7.

As the results show, R_{12}, R_{23}, R_{31}, and R_{42} (and their inverses) have changed. It should be noted that the preference order of R_{23} has also changed. The new matrix is consistent ($\beta = 0$) and the amount of objective function is $Z = 2210$. A comparison of the results shows that the amount of changes using the second model is less than the first one.

6. Conclusion

In this paper, we have addressed the issue of eliminating/reducing the inconsistency in fuzzy AHP. Considering the framework of the feasible region of relative weights, we have proposed two models to give a consistent fuzzy comparison matrix. These models are based on the fuzzy consistency test defined by Leung and Cao.

In the first model, the ratios can become smaller or larger compared to their earlier values, but their preference orders are not changed. The second model looks for a new consistent matrix while changes of preference orders of ratios are allowed. For both models, we keep the structure of FNs unchanged. Two examples are solved to show the performance of the models.

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