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A NEW ADMISSIBLE PIVOT METHOD FOR LINEAR PROGRAMMING

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We present a new admissible pivot method for linear programming that works with a sequence of improving primal feasible interior points and dual feasible interior points. This method is a practicable variant of the short admissible pivot sequence algorithm, which was suggested by Fukuda and Terlaky. Here, we also show that this method can be modified to terminate in finite pivot steps. Finally, we show that this method outperforms Terlaky's criss-cross method by computational experiments.

Keywords: Linear programming; admissible pivot method; criss-cross method.

1. Introduction

We consider primal and dual linear programming (LP) problems in the standard form:

$$(P)\colon \begin{array}{ll} \min & c^{\mathrm{T}}x \\ \text{s.t.} & Ax = b, \\ & x \geq 0, \end{array} \qquad (D)\colon \begin{array}{ll} \max & b^{\mathrm{T}}y \\ \text{s.t.} & A^{T}y + s = c, \\ & s \geq 0, \end{array}$$

where $c \in \Re^n$, $b \in \Re^m$, and $A \in \Re^{m \times n}$. Linear programming has been one of the most active areas of applied mathematics in the last fifty years (Park *et al.*, 2000; Weng and Wen, 2000). The simplex method has been studied extensively since its invention in 1947 by Dantzig and still remains one of the most efficient methods for solving a great majority of practical problems (Park, 1999; Lim and Park, 2002). Although a number of variants of the simplex method have been developed, none of them has polynomial time complexity (Terlaky and Zhang, 1993). Admissible pivot methods exploit the combinatorial structure of LP as the simplex method, but they allow more general pivot selection than the simplex method. Criss-cross methods can be considered as admissible pivot methods. The first criss-cross type method was designed by Zionts (1969). The finite criss-cross algorithm was presented independently by Terlaky (1985) and Wang (1987). Terlaky presented this algorithm for linear and oriented matroid programming, while Wang presented it only for oriented matroid programming. Unfortunately, the finiteness of Zionts' criss-cross method is not clear, in general, while Terlaky–Wang's criss-cross method is finite. Recently, Fukuda and Terlaky (2000) have proved the existence of a short admissible pivot sequence for linear programming problems, which offers many researchers a good incentive to look for a strongly polynomial LP algorithm. They proved their result by constructing an admissible pivot algorithm that takes at most n pivot steps, provided that an optimal primal–dual solution pair is given. In that algorithm, an optimal solution pair guides the sequence of admissible pivots. We will denote that algorithm as the short admissible pivot sequence algorithm. By contrast to this result, there is no easy way to prove the existence of a short simplex pivot sequence.

While developing an efficient variant of the simplex method, Hu (1992) devised a leaving variable selection rule for the dual simplex method, which utilizes primal feasible interior points. Also, Lim *et al.* (1999) introduced the idea to the primal simplex method, and showed its superior performance to Dantzig's rule by computational experiments. These methods are known as the primal interior dual simplex method (PIDS) and the dual interior primal simplex method (DIPS), respectively. The PIDS and the DIPS maintain a sequence of improving feasible interior solutions for (P) and (D) which allows the problems to be seen from a global standpoint. We think these interior points can play the role of a given optimal solution in the short admissible pivot sequence algorithm.

Motivated by the above results, we devised a new admissible pivot method for LP. The method is a practicable version of the short admissible pivot sequence algorithm.

Before we introduce our new method, we will define our notation and terminology first. We use standard simplex notation and terminology. Given a linear program (P), a basis B is a square, nonsingular submatrix of A. The columns and variables corresponding to B are called basic, and the remaining columns and variables are called nonbasic. The nonbasic columns of A are denoted by N. We denote E as the column index set of (P), and the basic column index set and the nonbasic column index set are denoted as E_{B} and E_{N}, respectively. A basis B is primal feasible if $\bar{b} = B^{-1}b \geq 0$, and in this case the basic feasible solution corresponding to B is $x^{\mathrm{T}} = \left(x_{\mathrm{B}}^{\mathrm{T}}, x_{\mathrm{N}}^{\mathrm{T}}\right) = (\bar{b}^{\mathrm{T}}, 0)$ where x_{B} and x_{N} denote the basic and nonbasic variables, respectively. For a given basis B, the reduced costs are $s = c - A^{\mathrm{T}}y$ where $y = B^{-\mathrm{T}}c_{\mathrm{B}}$ are the simplex multipliers and c_{B} are the basic components of c. A basis is dual feasible if $s \geq 0$, and in this case y and s are a feasible solution to (D).

For a given basis B, the simplex tableau is defined as shown in Figure 1. In the simplex tableau, each row of $D, \bar{b}$, and each column of D, s_{N} are indexed by E_{B} and

$s_N = c_N - N^{\mathrm{T}}B^{-\mathrm{T}}c_B$	
$D = (d_{ij}) = B^{-1}N$	$\bar{b}$

Fig. 1. Simplex tableau.

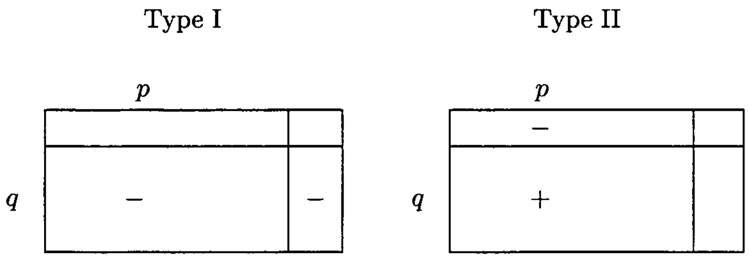

Fig. 2. Two types of admissible pivots.

E_{N}, respectively. A variable index $i \in E$ is said to be primal (dual) infeasible at a basis if the associated value in the current primal (dual) basic solution is negative.

For $q \in E_{\mathrm{B}}$ and $p \in E_{\mathrm{N}}$ with $d_{qp} \neq 0$, a pivot on (q, p) is said to be admissible if either (I) $\bar{b}_q < 0$ and $d_{qp} < 0$ or (II) $(s_N)_p < 0$ and $d_{qp} > 0$. Two types of admissible pivots are illustrated in Figure 2. Admissible pivots locally improve the infeasibility status of the solution. That is, both pivot variables, x_q and x_p, become primal feasible after an admissible pivot of type I, and dual feasible after an admissible pivot of type II. An admissible pivot method for LP is defined as a pivot method that uses only admissible pivots.

The primal simplex method is one of admissible pivot methods in which admissible pivots of type II are always performed. Similarly, the dual simplex method is a special case of admissible pivot methods in which admissible pivots of type I are always performed. Also, many variants of finite criss-cross methods use admissible pivots.

Admissible pivot methods detect the inconsistency or the dual inconsistency (primal unboundedness) of a given LP problem as in the simplex method. That is, if $\bar{b}_q < 0$ and $d_{qj} \geq 0$ for all $j \in E_{\mathrm{N}}$ in a simplex tableau, primal inconsistency will appear. Also, if $(s_{\mathrm{N}})_p < 0$ and $d_{ip} \leq 0$ for all $i \in E_{\mathrm{B}}$ in a simplex tableau, dual inconsistency will be found. These cases are illustrated in Figure 3.

Finally, we define lexicographic ordering. Let i, j be two elements in a linear ordering Γ of E. We say that i is lexicographically greater than j if i is placed on the left of j in Γ, and represent the case as $j \prec_\Gamma i$. Let γ, $\hat{\gamma}$ be two real-valued vectors indexed by E, and $\gamma(i)$ be the ith element of γ. We say that γ is lexicographically greater than $\hat{\gamma}$ if there exists an index $k \in E$ such that $\gamma(k) > \hat{\gamma}(k)$ and for all $j \succ_\Gamma k$, $\gamma(j) = \hat{\gamma}(j)$.

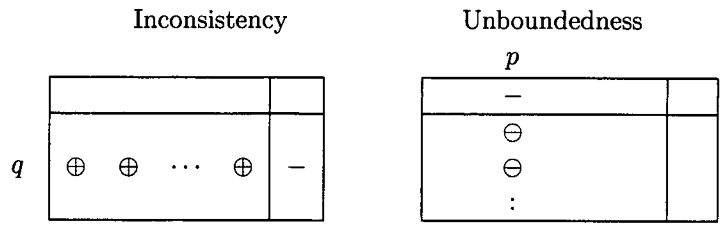

Fig. 3. Inconsistency and unboundedness.

2. Theoretical Background

2.1. *The existence of a short admissible pivot sequence*

Fukuda, Lűthi, and Namiki have proved the existence of a short admissible pivot sequence from an arbitrary basis to the unique optimal basis, under the assumption that the given LP problem is fully nondegenerate. A few years later, Fukuda and Terlaky proved the existence of a short admissible pivot sequence without any nondegeneracy assumptions by constructing an admissible pivot algorithm that proves their result. Their algorithm denoted as the short admissible pivot sequence algorithm terminates in at most n pivot steps, provided that an optimal primal–dual solution pair is given. The short admissible pivot sequence algorithm can be stated as follows.

Let B be any basis of A and $x, (y, s)$ be the corresponding primal–dual basic solution pair, and $x^*, (y^*, s^*)$ be a primal–dual optimal solution pair. Let $I^* = \{i \in E \,|\, x_i^* > 0\}$, $J^* = \{j \in E \,|\, s_j^* > 0\}$, $F = I^* \cup E_{\mathrm{B}}$ and $G = J^* \cup E_{\mathrm{N}}$. If $x \geq 0$ and $s \geq 0$, then an optimal basis is found, so stop. If there exists an index $q \in E_{\mathrm{B}} \setminus I^*$ with $x_q < 0$ or there exists an index $p \in E_{\mathrm{N}} \setminus J^*$ with $s_p < 0$, then perform REDUCE-F OR G procedure, which will be stated below. Otherwise, REDUCE-SOLUTIONS procedure, which is also stated below, is performed.

In REDUCE-F OR G procedure, an admissible pivot operation occurs. If there is an index $q \in E_{\mathrm{B}} \setminus I^*$ with $x_q < 0$, then it is proved in Fukuda and Terlaky (2000) that there exists an index $p \in E_{\mathrm{N}} \cap I^*$ such that pivot (q, p) is admissible. Also, if there is an index $p \in E_{\mathrm{N}} \setminus J^*$ with $s_{\mathrm{p}} < 0$, then it can be proved that there exists an index $q \in E_{\mathrm{B}} \cap J^*$ such that pivot (q, p) is admissible (Fukuda and Terlaky, 2000).

In REDUCE-SOLUTIONS procedure, a positive coordinate of x^* or s^* is reduced to zero without losing optimality. If $x_i < 0$ for some $i \in E_{\mathrm{B}} \cap I^*$, then let $\lambda = \min\left\{\frac{x_i^*}{x_i^* - x_i} \,\middle|\, x_i < 0\right\} = \frac{x_q^*}{x_q^* - x_q}$ and by defining the new optimal solution as $x^* := \lambda x + (1-\lambda)x^*$, we have $x_q^* = 0$ and x^* is still an optimal solution. If $s_j < 0$ for some $j \in E_{\mathrm{N}} \cap J^*$, then let $\lambda = \min\left\{\frac{s_j^*}{s_j^* - s_j} \,\middle|\, s_j < 0\right\} = \frac{s_p^*}{s_p^* - s_p}$ and by defining the new dual optimal solution as $(y^*, s^*) := \lambda(y, s) + (1 - \lambda)(y^*, s^*)$, we have $s_p^* = 0$ and (y^*, s^*) is still a dual optimal solution.

Finally, the sets I^*, J^*, F, and G are redefined and the above procedure is restarted. For a more detailed description of the algorithm, we refer the reader to Fukuda and Terlaky (2000).

It can be proved easily that the above algorithm terminates in at most n pivot steps. That is, given an optimal primal–dual solution pair, this algorithm finds an optimal basis in polynomial time bound. The sketch of the proof is that $n(F)$ or $n(G)$ is reduced by at least one after each pivot, where $n(S)$ is defined as the size of a set S, and the value $n(F) + n(G)$ is between n and $2n$.

It should be noted that this algorithm is only for the proof and we thought we could make an appropriate modification for the algorithm using the idea stated in the following subsection.

2.2. *Primal interior dual simplex and dual interior primal simplex*

In the PIDS, leaving variables are chosen by utilizing primal feasible interior solutions. Let x^{pf} be a primal feasible interior solution and x be the basic solution corresponding to a given dual feasible basis B. By the weak duality theorem, $c^{\mathrm{T}}x \leq c^{\mathrm{T}}x^{pf}$ is satisfied so that $x - x^{pf}$ is an improving direction for x^{pf}. If x^{pf} moves along the direction, it meets the sign restriction on a variable and the corresponding variable is chosen to leave the basis. Formally, a leaving variable x_q is determined by the following ratio test and an entering variable is chosen as in the usual dual simplex method.

$$q = \operatorname{argmin}\left\{ \frac{x_i^{pf}}{x_i^{pf} - x_i} \,\middle|\, x_i < 0 \right\}.$$

After the pivot, the primal feasible interior point x^{pf} is improved along the direction $x - x^{pf}$:

$$x^{pf} := x^{pf} + \alpha\lambda_1(x - x^{pf}), \quad \text{where } \lambda_1 = \frac{x_q^{pf}}{x_q^{pf} - x_q}, \quad 0 < \alpha < 1,$$

where α is a real number. This procedure continues until an optimal basis is found.

The DIPS can be easily derived from the idea of the PIDS. It chooses entering variables using dual feasible interior points. In the DIPS, an entering variable x_p is selected by the following ratio test given a dual feasible interior solution (y^{df}, s^{df}):

$$p = \operatorname{argmin}\left\{ \frac{s_i^{df}}{A_i^T(y - y^{df})} : A_i^T(y - y^{df}) > 0 \right\}$$

The dual feasible interior point (y^{df}, s^{df}) is improved as follows:

$$y^{df} := y^{df} + \alpha\lambda_2(y - y^{df}), \quad s^{df} := s^{df} - \alpha\lambda_2 A^T(y - y^{df}),$$
$$\text{where } \lambda_2 = \frac{s_p^{df}}{A_p^T(y - y^{df})}, \quad 0 < \alpha < 1,$$

where α is a real number.

3. A New Admissible Pivot Method

3.1. *Main features*

Our algorithm is a practicable version of the short admissible pivot sequence algorithm. In the algorithm, a sequence of improving feasible interior points plays the role of a primal–dual optimal solution pair in the short admissible pivot sequence algorithm. We adopt the idea of the DIPS and the PIDS in choosing admissible pivot elements and improving the feasible interior solutions. We think the use of feasible interior points can help view the given LP from a global standpoint. Let B be any basis of A and $x, (y, s)$ be the corresponding basic solutions. We suppose a primal feasible interior solution and a dual feasible interior solution are given as x^{pf} and (y^{df}, s^{df}), respectively. Also, we define the sets J_p, J_d as follows:

$$J_p = \{i \in E_{\mathrm{B}} \mid x_i < 0\}, \quad J_d = \{j \in E_{\mathrm{N}} \mid s_j < 0\}$$

If $J_p \neq \phi$ and $c^{\mathrm{T}}x < c^{\mathrm{T}}x^{pf}$, an admissible pivot of type I is performed. The leaving variable is chosen as in the PIDS. If $c^{\mathrm{T}}x < c^{\mathrm{T}}x^{pf}$, then $c^{\mathrm{T}}(x - x^{pf}) < 0$ is satisfied so that $x - x^{pf}$ is an improving direction for x^{pf}. Also, $x - x^{pf}$ is a feasible direction for x^{pf} because $A(x - x^{pf}) = 0$ is satisfied. So, the primal feasible interior point x^{pf} is improved along the direction $x - x^{pf}$.

If $J_d \neq \phi$ and $b^{\mathrm{T}}y > b^{\mathrm{T}}y^{df}$, an admissible pivot of type II is performed. The entering variable is chosen as in the DIPS. If $b^{\mathrm{T}}y > b^{\mathrm{T}}y^{df}$, then $b^{\mathrm{T}}(y - y^{df}) > 0$ is satisfied so that $(y - y^{df})$ is an improving direction for y^{df}. Also, $((y - y^{df}), -A^{\mathrm{T}}(y - y^{df}))$ is a feasible direction for (y^{df}, s^{df}) because $A^{\mathrm{T}}(y - y^{df}) - A^{\mathrm{T}}(y - y^{df}) = 0$ is satisfied. So the dual feasible interior points (y^{df}, s^{df}) are improved along the direction $((y - y^{df}), -A^{\mathrm{T}}(y - y^{df}))$.

We can summarize this algorithm as follows:

The new admissible pivot method

Step 0: Initialization

Let B be a basis and $x, (y, s)$ be the corresponding primal–dual basic solution pair.

Let x^{pf} be a primal feasible interior solution.

Let (y^{df}, s^{df}) be a dual feasible interior solution.

Let α be any real number such that $0 < \alpha < 1$.

Step 1: Optimality check

Set $J_p = \{i \in E_{\mathrm{B}} \mid x_i < 0\}$ and $J_d = \{j \in E_{\mathrm{N}} \mid s_j < 0\}$.

If $c^{\mathrm{T}}x^{pf} > c^{\mathrm{T}}x$ and $J_p \neq \phi$, go to Step 2.

If $b^{\mathrm{T}}y^{df} < b^{\mathrm{T}}y$ and $J_d \neq \phi$, go to Step 3.

Otherwise, an optimal basis B is found, so stop.

Step 2: Admissible pivot of type I

Let $\lambda_1 = \min\left\{ \frac{x_i^{pf}}{x_i^{pf} - x_i} \,\middle|\, x_i < 0 \right\} = \frac{x_q^{pf}}{x_q^{pf} - x_q}$.

If $d_{qj} \geq 0$ for all $j \in E_{\mathrm{N}}$, the problem is infeasible, so stop.

Let $p = \min\{j \in E_{\mathrm{N}} \mid d_{qj} < 0\}$ and set $E_{\mathrm{B}} := E_{\mathrm{B}} \cup \{p\} \setminus \{q\}$, $E_{\mathrm{N}} = E \setminus E_{\mathrm{B}}$.

Set $x^{pf} := x^{pf} + \alpha\lambda_1(x - x^{pf})$ and go to Step 1.

Step 3: Admissible pivot of type II

Let $\lambda_2 = \min\left\{ \frac{s_i^{df}}{A_i^T(y-y^{df})} \,\middle|\, s_i < 0 \right\} = \frac{s_p^{df}}{A_p^T(y-y^{df})}$.

If $d_{ip} \leq 0$ for all $i \in E_{\mathrm{B}}$, the problem is dual infeasible, so stop.

Let $q = \min\{i \in E_{\mathrm{B}} \,|\, d_{ip} > 0\}$ and set $E_{\mathrm{B}} := E_{\mathrm{B}} \cup \{p\} \setminus \{q\}$, $E_{\mathrm{N}} = E \setminus E_{\mathrm{B}}$.

Set $y^{df} := y^{df} + \alpha\lambda_2(y - y^{df}), s^{df} := s^{df} - \alpha\lambda_2 A^{\mathrm{T}}(y - y^{df})$ and go to Step 1.

Next, we show that Step 1 in the above algorithm covers all the possible cases.

Proposition 3.1. *For an arbitrary basis B which is not optimal, if $c^{\mathrm{T}}x^{pf} \leq c^{\mathrm{T}}x$ is satisfied, the following two cases never occur.*

Case 1: B is dual feasible.
Case 2: $b^{\mathrm{T}}y^{df} \geq b^{\mathrm{T}}y$ is satisfied.

Proof. First, we show that Case 1 never occurs. If Case 1 occurs, $b^{\mathrm{T}}y < c^{\mathrm{T}}x^{pf}$ is satisfied by the weak duality theorem. But, $b^{\mathrm{T}}y = c^{\mathrm{T}}x$ is satisfied so that $c^{\mathrm{T}}x = b^{\mathrm{T}}y < c^{\mathrm{T}}x^{pf}$, which is a contradiction. Therefore, Case 1 never occurs. Next, if Case 2 occurs, $c^{\mathrm{T}}x^{pf} \leq c^{\mathrm{T}}x = b^{\mathrm{T}}y \leq b^{\mathrm{T}}y^{df}$ is satisfied, which contradicts the weak duality theorem. Therefore, Case 2 also never occurs. □

Proposition 3.2. *For an arbitrary basis B which is not optimal, if $b^{\mathrm{T}}y^{df} \geq b^{\mathrm{T}}y$ is satisfied, the following two cases never occur.*

Case 1: B is primal feasible.
Case 2: $c^{\mathrm{T}}x^{pf} \leq c^{\mathrm{T}}x$ is satisfied.

Proof. First, we show that Case 1 never occurs. If Case 1 occurs, $c^{\mathrm{T}}x > b^{\mathrm{T}}y^{df}$ is satisfied by the weak duality theorem. But, $c^{\mathrm{T}}x = b^{\mathrm{T}}y$ is satisfied so that $b^{\mathrm{T}}y = c^{\mathrm{T}}x > b^{\mathrm{T}}y^{df}$, which is a contradiction. Therefore, Case 1 never occurs. Next, if Case 2 occurs, $b^{\mathrm{T}}y^{df} \geq b^{\mathrm{T}}y = c^{\mathrm{T}}x \geq c^{\mathrm{T}}x^{pf}$ is satisfied, which contradicts the weak duality theorem. Therefore, Case 2 also never occurs. □

3.2. *Finiteness*

Although we are unable to prove that our method is finite, we can construct a variant of this method so that it can terminate in finite pivot steps.

Fukuda and Matsui (1991) proved the finiteness of the following criss-cross method.

Fukuda-Matsui's criss-cross method

Step 0: Initialization

Let B be a basis of (P).

Let Γ be a linear ordering of the index set E.

Let γ be a $\{0,1\}$-valued vector indexed by E with all entries equal to 0.

Step 1: Optimality check
Set $I = \{i \mid i \in E_{\mathrm{B}}$ and $x_i < 0$, or $i \in E_{\mathrm{N}}$ and $s_i < 0\}$.
If $I = \phi$, then stop (the current basis is optimal).
Otherwise, go to Step 2.
Step 2: Infeasibility and dual infeasibility check
Set $q = \min_{\Gamma}\{I\}$.
Case I: If $q \in E_{\mathrm{B}}$, set $J = \{i \mid i \in E_{\mathrm{N}}$ and $d_{qi} < 0\}$.
If $J = \phi$, then stop ((P) is infeasible).
Otherwise, set $p = \min_{\Gamma}\{J\}$ and go to Step 3.
Case II: If $q \in E_{\mathrm{N}}$, set $J = \{i \mid i \in E_{\mathrm{B}}$ and $d_{iq} > 0\}$.
If $J = \phi$, then stop ((D) is infeasible).
Otherwise, set $p = \min_{\Gamma}\{J\}$ and go to Step 3.
Step 3: Pivoting
Set $t = \max_{\Gamma}\{p, q\}$.
Set $\gamma(t) = 1$. Set $\gamma(i) = 0$, $\forall i \prec_{\Gamma} t$.
If $q \in E_{\mathrm{B}}$, then set $E_{\mathrm{B}} := E_{\mathrm{B}} \cup \{p\} \setminus \{q\}$, $E_{\mathrm{N}} = E \setminus E_{\mathrm{B}}$.
If $q \in E_{\mathrm{N}}$, then set $E_{\mathrm{B}} := E_{\mathrm{B}} \cup \{q\} \setminus \{p\}$, $E_{\mathrm{N}} = E \setminus E_{\mathrm{B}}$.
Go to Step 1.

They proved the finiteness by showing that the vector γ monotonically increases in the sense of lexicographic ordering with respect to Γ at each iteration. Also, they showed the flexibility of choosing pivot elements of their criss-cross method without spoiling the finiteness, by proving the following lemma.

Lemma 3.1. *We define the* 0*-interval of* Γ *with respect to* γ, *as a consecutive sublist of* Γ *(sequence of consecutive indices) whose corresponding entries in* γ *are all* 0. *Any permutation in* 0*-interval in each iteration does not affect the finiteness.*

Proof. See Namiki and Matsui (1995). □

By utilizing the above result, Namiki and Matsui (1995) suggested a criss-cross method and showed its superior performance to Terlaky's criss-cross method by computational experiments. We utilize this result to construct a finite variant of our method. Now, we describe the variant as follows:

A finite variant of the new admissible pivot method

Step 0: Initialization
Let B be a basis and $x, (y, s)$ be the corresponding primal–dual basic solution pair.
Let x^{pf} be a primal feasible interior solution.
Let (y^{df}, s^{df}) be a dual feasible interior solution.
Let α be a real number such that $0 < \alpha < 1$.
Let Γ be a linear ordering of the index set E.
Let γ be a $\{0, 1\}$-valued vector indexed by E with all entries equal to 0.

Step 1: Optimality check

Set $J_p = \{i \in E_{\mathrm{B}} \mid x_i < 0\}$ and $J_d = \{j \in E_{\mathrm{N}} \mid s_j < 0\}$.

Set $k = \min_{\Gamma}\{i \mid i \in J_p \cup J_d\}$.

If $c^{\mathrm{T}}x^{pf} > c^{\mathrm{T}}x$ and $J_p \neq \phi$, go to Step 2,

If $b^{\mathrm{T}}y^{df} < b^{\mathrm{T}}y$ and $J_d \neq \phi$, go to Step 3.

Otherwise, an optimal basis B is found, so stop.

Step 2: Admissible pivot

Set $\lambda_1 = \min\left\{ \frac{x_i^{pf}}{x_i^{pf} - x_i} \;\middle|\; x_i < 0 \right\} = \frac{x_q^{pf}}{x_q^{pf} - x_q}$.

If $d_{qj} \geq 0$ for all $j \in E_{\mathrm{N}}$, the problem is infeasible, so stop.

Let $[f, g]$ be the longest 0-interval of Γ with respect to γ which contains k.

Case I: If $q \in [f, g]$ and $q \neq k$, then permute Γ so that q is placed on the right of k, and set $p = \min_{\Gamma}\{j \in E_{\mathrm{N}} : \; d_{qj} < 0\}$.

Case II: If $q \in [f, g]$ and q equals k, then set $p = \min_{\Gamma}\{j \in E_{\mathrm{N}} : \; d_{qj} < 0\}$.

Case III: If $q \notin [f, g]$ and $k \in J_p$, then set $q = k$ and $p = \min_{\Gamma}\{j \in E_{\mathrm{N}} \mid d_{qj} < 0\}$.

Case IV: If $q \notin [f, g]$ and $k \in J_d$, then set $p = k$ and $q = \min_{\Gamma}\{i \in E_{\mathrm{B}} \mid d_{ip} > 0\}$.

Set $E_{\mathrm{B}} := E_{\mathrm{B}} \cup \{p\} \setminus \{q\}$, $E_{\mathrm{N}} = E \setminus E_{\mathrm{B}}$. Set $x^{pf} := x^{pf} + \alpha\lambda_1(x - x^{pf})$.

Set $t = \max_{\Gamma}\{p,\; q\}$. Set $\gamma(t) = 1$. Set $\gamma(i) = 0,\; \forall i \prec_{\Gamma} t$.

Go to Step 1.

Step 3: Admissible pivot

Set $\lambda_2 = \min\left\{ \frac{s_i^{df}}{A_i^T(y - y^{df})} \;\middle|\; s_i < 0 \right\} = \frac{s_p^{df}}{A_p^T(y - y^{df})}$.

If $d_{ip} \leq 0$ for all $i \in E_{\mathrm{B}}$, the problem is dual infeasible, so stop.

Let $[f, g]$ be the longest 0-interval of Γ with respect to γ which contains k.

Case I: If $p \in [f, g]$ and $p \neq k$, then permute Γ so that p is placed on the right of k, and set $q = \min_{\Gamma}\{i \in E_{\mathrm{B}} \mid d_{ip} > 0\}$.

Case II: If $p \in [f, g]$ and p equals k, then set $q = \min_{\Gamma}\{i \in E_{\mathrm{B}} \mid d_{ip} > 0\}$.

Case III: If $p \notin [f, g]$ and $k \in J_p$, then set $q = k$ and $p = \min_{\Gamma}\{j \in E_{\mathrm{N}} \mid d_{qj} < 0\}$.

Case IV: If $p \notin [f, g]$ and $k \in J_d$, then set $p = k$ and $q = \min_{\Gamma}\{i \in E_{\mathrm{B}} \mid d_{ip} > 0\}$.

Set $E_{\mathrm{B}} := E_{\mathrm{B}} \cup \{p\} \backslash \{q\}$, $E_{\mathrm{N}} = E \backslash E_{\mathrm{B}}$.

Set $y^{df} := y^{df} + \alpha\lambda_2(y - y^{df})$, $s^{df} := s^{df} - \alpha\lambda_2 A^T(y - y^{df})$.

Set $t = \max_{\Gamma}\{p,\; q\}$. Set $\gamma(t) = 1$. Set $\gamma(i) = 0,\; \forall\, i \prec_{\Gamma} t$.

Go to Step 1.

Now, we prove the finiteness of the above variant.

Theorem 3.1. *The above variant terminates in finite pivot steps.*

Proof. By the finiteness of Fukuda–Matsui's criss-cross method and Lemma 3.1, it is easy to see that in the above algorithm, vector γ is monotonically increasing in the sense of lexicographic ordering with respect to Γ. So, the above algorithm terminates in finite pivot steps. □

4. Initialization

As suggested in Lim *et al.* (1999), an initial primal–dual feasible interior solution pair $(x^{pf}, (y^{df}, s^{df}))$ can be obtained by modifying the original problems (P) and (D) to the following $(\hat{P})$ and $(\hat{D})$.

$$(\hat{P}): \quad \begin{array}{ll} \min & c^{\mathrm{T}}x + Mx_a \\ \text{s.t.} & Ax + (b - Ax_0)x_a = b, \\ & x \geq 0,\ x_a \geq 0. \end{array}$$

$$(\hat{D}): \quad \begin{array}{ll} \max & b^{\mathrm{T}}y - My_a \\ \text{s.t.} & A^Ty + (c - A^Ty^0 - s^0)y_a + s = c, \\ & s \geq 0. \end{array}$$

It is obvious that $((x^0, 1), (y^0, 1, s^0))$ is a primal–dual feasible interior solution pair for (P) and (D), where x^0 and s^0 are arbitrary positive vectors and y^0 is an arbitrary vector.

5. Remark

5.1. *Relationships to the short admissible pivot sequence algorithm*

One can think that our algorithm is a modification of the short admissible pivot sequence algorithm. In this modification, a primal–dual feasible interior solution pair $x^{pf}, (y^{df}, s^{df})$ is used instead of an optimal solution pair $x^*, (y^*, s^*)$ in the original short admissible pivot sequence algorithm. Therefore, both I^* and J^* are always equal to E, so that the REDUCE-SOLUTIONS procedure is always performed. But, it is assumed that the interior point moves to touch the boundary of the feasible region, so that the REDUCE-F OR G procedure occurs after each REDUCE-SOLUTIONS procedure. That is, the variable which corresponds to the coordinate eliminated in the REDUCE-SOLUTIONS procedure leaves or enters the basis in the REDUCE-F OR G procedure. It should be noted that the REDUCE-SOLUTIONS procedure in this modification corresponds to the improvement of interior points in our algorithm. Also, the REDUCE-F OR G procedure corresponds to the admissible pivot operation in our algorithm.

5.2. *Properties*

It can be said that our algorithm has several problem reduction properties as in the DIPS and the PIDS (Lim *et al.*, 1999). This result is stated as follows.

Property 1. *If x_i is positive in every primal feasible solution, x_i never becomes nonbasic by any admissible pivot of type I, once it becomes basic.*

Proof. For $0 \leq \beta \leq \lambda_1$, $x(\beta) = x^{pf} + \beta(x - x^{pf})$ is a primal feasible solution. Also, if x_i is to be dropped from basis, $x(\lambda_1)_i$ should have the value of zero. This is a contradiction. □

Property 2. *Let $K = \{x \mid Ax = b, x \geq 0\}$ be the primal feasible solution set and K_i be the subset of K, in which x_i equals zero: $K_i = \{x \mid Ax = b, x \geq 0, x_i = 0\}$. If $c^{\mathrm{T}}x^{pf} \leq c^{\mathrm{T}}x^*(K_i) = \min\{c^{\mathrm{T}}x \mid x \in K_i\}$ is satisfied, x_i never becomes nonbasic by any admissible pivot of type I, once it becomes basic.*

Proof. If we assume x_i is to be dropped from basis, then $x(\lambda_1)_i$ has the value of zero, so that it belongs to the set K_i. It is also satisfied that $c^{\mathrm{T}}x(\lambda_1) < c^{\mathrm{T}}x^{pf}$. But, based on the assumption, $c^{\mathrm{T}}x^{pf} \leq c^{\mathrm{T}}x(\lambda_1)$ is satisfied, which is a contradiction. □

Property 3. *If s_i is positive in every dual feasible solution, x_i never becomes basic by any admissible pivot of type II, once it becomes nonbasic.*

Proof. For $0 \leq \beta \leq \lambda_2$, $s(\beta) = s^{df} - \beta A^{\mathrm{T}}(y - y^{df})$ is a dual feasible solution. Also, if x_i is to become basic, $s(\lambda_2)_i$ should have the value of zero. This is a contradiction. □

Property 4. *Let $K = \{(y,\ s) \mid A^{\mathrm{T}}y + s = c,\ s \geq 0\}$ be the dual feasible solution set and K_i be the subset of K, in which s_i equals zero: $K_i = \{(y,\ s) \mid A^{\mathrm{T}}y + s = c,$ $s \geq 0,\ s_i = 0\}$. If $b^{\mathrm{T}}y^{df} \geq b^{\mathrm{T}}y^*(K_i) = \max\{b^{\mathrm{T}}y \mid (y,\ s) \in K_i\}$ is satisfied, x_i never becomes basic by any admissible pivot of type II, once it becomes nonbasic.*

Proof. If we assume x_i is to become basic, then $s(\lambda_2)_i$ has the value of zero, so that it belongs to the set K_i. $b^{\mathrm{T}}y(\lambda_2) > b^{\mathrm{T}}y^{df}$ is also satisfied. But, based on the assumption, $b^{\mathrm{T}}y^{df} \geq b^{\mathrm{T}}y(\lambda_2)$ is satisfied, which is a contradiction. □

6. Computational Experiments

It is well known that the number of pivots required in admissible pivot methods, such as criss-cross methods, is larger than that in the simplex method using Dantzig's pivot selection rule. In this section, we show that our idea can reduce the number of pivots in Terlaky's criss-cross method. We test our method for two kinds of linear programming problems. Those are transportation problems, and real-world problems. For the experiments, we implemented Terlaky's criss-cross method and our method. The programs were written completely in C, and ran on the Sun Ultra Sparc 170.

6.1. *Transportation problems*

In the transportation model we have a set of nodes or places called sources, which have a commodity available for shipment, and another set of places called demand centers, which require this commodity. The amount of commodity available at each source and the amount required at each demand center are specified, as well as the cost per unit of transporting the commodity from each source to each demand center. The problem is to determine the quantity to be transported from each source to each demand center, so as to meet all the requirements at minimum total shipping cost.

We consider the following uncapacitated balanced transportation problem:

$$
\begin{aligned}
& & \min \ & \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \\
&\text{(TP):} & \text{s.t.} \ & \sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1, 2, \ldots, m \\
& & & \sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1, 2, \ldots, n \\
& & & x_{ij} \geq 0, \quad \forall\ i,\ j,
\end{aligned}
$$

where $a_i > 0$, $b_j > 0$ for all i, j, and $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$, and c_{ij} may be arbitrary real numbers. Here a_i, b_j are, respectively, the amount of some commodity available at source i, and required at demand center j.

We solve the problems for $m = n = 30, 60, 120, 200$ and the values a_i, b_j are generated randomly within the interval $[10, 1000]$ so that $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$ is satisfied. Forty problems are solved for each $m(= n)$. Table 1 shows the average of the number of iterations and CPU-time for each m.

As the experimental result shows, the new method requires a fewer number of pivot steps and less CPU-time than Terlaky's criss-cross method for all m.

6.2. *Real-world problems*

We also test our method for some problems taken from the NETLIB set. The characteristics of the chosen problems are summarized in Table 2. Table 3 shows the number of iterations and the CPU-time for each problem.

As in the case of transportation problems, the new method requires fewer number of iterations and less CPU-time than Terlaky's criss-cross method.

Table 1. Result of solving transportation problems.

m	Terlaky's criss-cross method		The new method	
	Iterations	CPU-time	Iterations	CPU-time
30	68	0.03	41	0.03
60	184	0.11	103	0.07
120	313	0.29	257	0.23
200	678	1.04	418	0.69

Table 2. The characteristics of test problems.

Problem name	Characteristics		
	Number of constraints	Number of variables	Number of nonzeros
ADLITTLE	57	97	465
AFIRO	28	32	88
AGG	489	163	2541
BANDM	306	472	2659
BLEND	75	83	521
BOEING2	167	143	1339
BORE3D	234	315	1525

Table 3. Result of solving real-world problems.

Problem name	Terlaky's criss-cross method		The new method	
	Iterations	CPU-time	Iterations	CPU-time
ADLITTLE	595	0.17	317	0.11
AFIRO	48	0.04	31	0.03
AGG	684	0.56	556	0.41
BANDM	737	1.03	549	0.76
BLEND	158	0.27	134	0.24
BOEING2	571	0.48	489	0.31
BORE3D	198	0.12	107	0.09

7. Conclusion

We introduced a new admissible pivot method for linear programming. It should be noted that this method is a practicable version of the short admissible pivot sequence algorithm. In this method, a sequence of improving the primal–dual feasible interior solution pair takes the role of a given primal–dual optimal solution pair in the short admissible pivot sequence algorithm. Also, we showed that our method can be modified to attain the finiteness property. Moreover, our method has problem reduction properties by which some redundancy in the original problem is removed from consideration. Finally, we showed the superiority of our method to Terlaky's criss-cross method by computational experiments.

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