Vectors summary

Quantities which have only magnitude are called <u>scalars</u>. Quantities which have magnitude and direction are called <u>vectors</u>.

$\overrightarrow{AB}$ is the <u>position vector</u> of B relative to A and is the vector which emanates from A and terminates at B and has length $|\overrightarrow{AB}|$.

Two vectors are equal if they have the same magnitude and direction. They will be parallel and have the same length.

$\overrightarrow{BA} = -\overrightarrow{AB}$ has the same magnitude but opposite direction to $\overrightarrow{AB}$.

Other notations: If $O = (0, 0, 0), A = (a_1, a_2, a_3)$ then $\mathbf{a} = \overrightarrow{OA} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \tilde{a} = \underset{\sim}{a} = \vec{a}$

Operations with vectors

Addition: To construct $\mathbf{a} + \mathbf{b}$ draw $\mathbf{b}$ at the arrowhead end of $\mathbf{a}$. Join beginning of $\mathbf{a}$ to arrowhead end of $\mathbf{b}$.

Vector subtraction: $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$

Zero vector: $\mathbf{a} - \mathbf{a} = \mathbf{0}$

Scalar multiplication: $k\mathbf{a}$ stretches $\mathbf{a}$ by factor k where k is a scalar.

$k > 0 \Rightarrow k\mathbf{a}$ is in the same direction as $\mathbf{a}$.

$k < 0 \Rightarrow k\mathbf{a}$ is in the opposite direction as $\mathbf{a}$.

2-D vectors in component form

If the horizontal and vertical components of $\mathbf{v}$ are x and y respectively, $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ as a column vector.

Addition: $\mathbf{a} + \mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1+b_1 \\ a_2+b_2 \end{pmatrix}$

Negative vectors: $-\mathbf{a} = -\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -a_1 \\ -a_2 \end{pmatrix}$

Zero vector: $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathbf{a} - \mathbf{a}$.

Vector subtraction: $\mathbf{a} - \mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} - \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1-b_1 \\ a_2-b_2 \end{pmatrix}$.

Vectors between two points: $A = (a_1, a_2), B = (b_1, b_2) \Rightarrow \overrightarrow{AB} = \binom{b_1 - a_1}{b_2 - a_2}$.

Scalar multiplication: $k\mathbf{a} = k\binom{a_1}{a_2} = \binom{ka_1}{ka_2}$

Length of a vector: $\mathbf{a} = \binom{a_1}{a_2} \Rightarrow |\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$.

These can be generalised to higher dimensions.

3-D coordinate geometry

If $O = (0,0,0), P = (x,y,z)$ then $|\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$ and $\overrightarrow{OP} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

Likewise, if $A = (x_1, y_1, z_1), B = (x_2, y_2, z_2)$ then $|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$ and $\overrightarrow{AB} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix}$

$M = \text{midpt}_{AB} = \text{midpoint of } AB = (\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2})$

Operations with 3-D vectors

$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \Rightarrow \mathbf{a} + \mathbf{b} = \begin{pmatrix} a_1+b_1 \\ a_2+b_2 \\ a_3+b_3 \end{pmatrix}, \mathbf{a} - \mathbf{b} = \begin{pmatrix} a_1-b_1 \\ a_2-b_2 \\ a_3-b_3 \end{pmatrix}$

$k = \text{a scalar} \Rightarrow k\mathbf{a} = \begin{pmatrix} ka_1 \\ ka_2 \\ ka_3 \end{pmatrix}$.

$\overrightarrow{OA} = \mathbf{a}, \overrightarrow{OB} = \mathbf{b} \Rightarrow \overrightarrow{AB} = \mathbf{b} - \mathbf{a}$.

Division of a line segment

Suppose A, B, X lie on a straight line. Then X divides AB in ratio $a : b$ if $|\overrightarrow{AX}| : |\overrightarrow{XB}| = a : b$.

$(a > 0, b > 0) \Rightarrow$ internal division

$(a > 0, b < 0) \Rightarrow$ external division & $|\overrightarrow{AX}| : |\overrightarrow{XB}| = a : -b$.

Parallelism

$\mathbf{a} \parallel \mathbf{b}$ if $\mathbf{a} = k\mathbf{b}$ for some scalar k.

A, B, C are collinear if $\overrightarrow{AB} = k\overrightarrow{BC}$ for some scalar k.

Unit vectors

A unit vector is a vector of length 1. Base unit vectors are unit vectors which can be used as linear combinations to write any other vector. The most commonly used base unit vectors in 3-D are $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

The unit vector in direction of $\mathbf{u}$ is denoted $\hat{\mathbf{u}}$ and is equal to $\frac{\mathbf{u}}{|\mathbf{u}|}$.

Scalar product

(Also called dot product, or inner product)

$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \Rightarrow \mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$.

Angle between vectors: If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ then $\theta = \cos^{-1} \hat{\mathbf{a}} \cdot \hat{\mathbf{b}}$.

Proof. By the cosine rule, $|\mathbf{b} - \mathbf{a}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}| \cos\theta$. But $\mathbf{b} - \mathbf{a} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \\ b_3 - a_3 \end{pmatrix}$

$\therefore (b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2 = a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2 - 2|\mathbf{a}||\mathbf{b}| \cos\theta$

$\therefore a_1 b_1 + a_2 b_2 + a_3 b_3 = \mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta \therefore \cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} \therefore \theta = \cos^{-1} \hat{\mathbf{a}} \cdot \hat{\mathbf{b}}$ □

Note that this can also be used to generalise the concept of angle to higher dimensions.

Perpendicular vectors

For nonzero vectors $\mathbf{a}, \mathbf{b}, \mathbf{a} \perp \mathbf{b} \Leftrightarrow \mathbf{a} \cdot \mathbf{b} = 0$ (since $\cos 90° = 0$)

Parallel vectors

For nonzero vectors $\mathbf{a}, \mathbf{b}, \mathbf{a} \parallel \mathbf{b} \Leftrightarrow \mathbf{a} \cdot \mathbf{b} = \pm|\mathbf{a}||\mathbf{b}|$ (since $\cos 0° = 1$ and $\cos 180° = -1$).

Vector Product

If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$, the vector product (or cross product) of $\mathbf{a}$ and $\mathbf{b}$ is

$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$

$\mathbf{a} \times \mathbf{b} \perp \mathbf{a}$ and $\mathbf{a} \times \mathbf{b} \perp \mathbf{b}$ because $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\mathbf{a} \times \mathbf{b}) = 0$.

Note also that $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$

$\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ is called the scalar triple product of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

Note that although it can also be written as $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$, these parentheses are not actually necessary.

Length of $\mathbf{a} \times \mathbf{b}$

$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| \times |\mathbf{b}| \sin\theta$ where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

Proof.
$$\begin{aligned}|\mathbf{a} \times \mathbf{b}| &= \sqrt{(a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2} \\ &= \sqrt{(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2} \text{ (check this)} \\ &= \sqrt{|\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2} \\ &= \sqrt{|\mathbf{a}|^2|\mathbf{b}|^2 - |\mathbf{a}|^2|\mathbf{b}|^2 \cos^2\theta} \\ &= |\mathbf{a}||\mathbf{b}|\sqrt{1 - \cos^2\theta} \\ &= |\mathbf{a}||\mathbf{b}| \sin\theta \text{ since } 0^\circ \leq \theta \leq 180^\circ. \quad \square\end{aligned}$$

Parallel vectors

For nonzero vectors $\mathbf{a}, \mathbf{b}, \mathbf{a} \parallel \mathbf{b} \Leftrightarrow \mathbf{a} \times \mathbf{b} = \mathbf{0}$ (since $\sin 0^\circ = \sin 180^\circ = 0$)

Area of triangle

The area of a triangle defined by $\mathbf{a}, \mathbf{b}$ is $\frac{1}{2}|\mathbf{a} \times \mathbf{b}|$ (since it is $\frac{1}{2}|\mathbf{a}||\mathbf{b}| \sin\theta$)

Area of parallelogram

The area of a parallelogram defined by $\mathbf{a}, \mathbf{b}$ is $|\mathbf{a} \times \mathbf{b}|$ (since it is $2 \times \frac{1}{2}|\mathbf{a} \times \mathbf{b}|$)

Volume of a parallelepiped

The volume of a parallelepiped defined by $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is $|\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}|$.

Proof. If θ is the angle between $\mathbf{a}$ and the plane defined by $\mathbf{b}, \mathbf{c}$ and ϕ is the angle between $\mathbf{a}$ and $\mathbf{b} \times \mathbf{c}$ then the parallelogramic base defined by $\mathbf{b}, \mathbf{c}$ has area $|\mathbf{b} \times \mathbf{c}|$ and the height is $|\mathbf{a}| \sin\theta$ and hence

$$\begin{aligned}\text{Volume} &= \text{area of base} \times \text{height} \\ &= |\mathbf{b} \times \mathbf{c}||\mathbf{a}| \sin\theta \\ &= |\mathbf{a}||\mathbf{b} \times \mathbf{c}| \cos\phi \\ &= |\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}| \quad \square\end{aligned}$$

Volume of a tetrahedron

The volume of a tetrahedron defined by $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is $\frac{1}{6}|\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}|$

Proof. If θ is the angle between $\mathbf{a}$ and the plane defined by $\mathbf{b}, \mathbf{c}$ and ϕ is the angle between $\mathbf{a}$ and $\mathbf{b} \times \mathbf{c}$ then the triangular base defined by $\mathbf{b}, \mathbf{c}$ has area $\frac{1}{2}|\mathbf{b} \times \mathbf{c}|$ and the height is $|\mathbf{a}| \sin\theta$ and hence

$$\begin{aligned}\text{Volume} &= \tfrac{1}{3} \times \text{area of base} \times \text{height} \\ &= \tfrac{1}{3} \times \tfrac{1}{2}|\mathbf{b} \times \mathbf{c}| \times |\mathbf{a}| \sin\theta \\ &= \tfrac{1}{6}|\mathbf{a}||\mathbf{b} \times \mathbf{c}| \cos\phi \\ &= \tfrac{1}{6}|\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}| \qquad \square\end{aligned}$$

Test for coplanar points

If A, B, C, D have position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$, then

A, B, C, D are coplanar $\Leftrightarrow (\mathbf{b} - \mathbf{a}) \cdot (\mathbf{c} - \mathbf{a}) \times (\mathbf{d} - \mathbf{a}) = 0$

(because the volume of the tetrahedron defined by $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$ would be 0.)

Projection vectors The vector projection of $\mathbf{u}$ in direction $\mathbf{v}$ is $(\mathbf{u} \cdot \hat{\mathbf{v}})\hat{\mathbf{v}}$

Proof.

$$\begin{aligned}\text{Proj}_{\mathbf{v}}\mathbf{u} &= |\mathbf{u}| \cos\theta \tfrac{\mathbf{v}}{|\mathbf{v}|} \text{ where } \theta \text{ is the angle between } \mathbf{u}, \mathbf{v} \\ &= |\mathbf{u}| \tfrac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|} \tfrac{\mathbf{v}}{|\mathbf{v}|} \\ &= \tfrac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2}\mathbf{v} \text{ (actually this is probably the most useful form of this)} \\ &= (\mathbf{u} \cdot \tfrac{\mathbf{v}}{|\mathbf{v}|}) \tfrac{\mathbf{v}}{|\mathbf{v}|} \\ &= (\mathbf{u} \cdot \hat{\mathbf{v}})\hat{\mathbf{v}} \qquad \square\end{aligned}$$

$|\mathbf{u}| \cos\theta = \mathbf{u} \cdot \hat{\mathbf{v}}$ is called the scalar projection of $\mathbf{u}$ in direction $\mathbf{v}$.

The vector projection of $\mathbf{u}$ in a direction orthogonal to $\mathbf{v}$ is $\mathbf{u} - \text{Proj}_{\mathbf{v}}\mathbf{u}$.