2011 Extension 2 Trial HSC Solutions

(1)(a) $\int_0^1 xe^{-x^2}\,dx = \left[-\frac{1}{2}e^{-x^2}\right]_0^1$ ✓

$= -\frac{1}{2}(e^{-1}-1)$

$= \frac{1}{2}(1-e^{-1})$ ✓

(b) $\int \frac{1}{\sqrt{x^2-12x+61}}\,dx = \int \frac{1}{\sqrt{(x-6)^2+25}}\,dx$ ✓

$= \ln\left(x-6+\sqrt{x^2-12x+61}\right)+c$ ✓

(c) $\int_0^{\frac{\pi}{4}} \sec^4 x \tan x\,dx$

Let $u = \tan x$

$\therefore du = \sec^2 x\,dx$ ✓

x	0	$\frac{\pi}{4}$
u	0	1

$= \int_0^{\frac{\pi}{4}} \sec^2 x(1+\tan^2 x)\tan x\,dx$ ✓

$$\left.\begin{aligned} &= \int_0^1 (u+u^3)\,du \\ &= \left[\frac{u^2}{2}+\frac{u^4}{4}\right]_0^1 \\ &= \frac{3}{4} \end{aligned}\right\}$$ ✓

(d) $\int_0^1 \frac{x^2}{\sqrt{2-x^2}}\,dx$

Let $x = \sqrt{2}\sin\theta$

$\therefore dx = \sqrt{2}\cos\theta\,d\theta$ ✓

x	0	1
θ	0	$\frac{\pi}{4}$

$$\left.\begin{aligned} &= \int_0^{\frac{\pi}{4}} \frac{2\sin^2\theta}{\sqrt{2(1-\sin^2\theta)}}\cdot\sqrt{2}\cos\theta\,d\theta \\ &= \int_0^{\frac{\pi}{4}} 2\sin^2\theta\,d\theta \end{aligned}\right\}$$ ✓

$= 2\int_0^{\frac{\pi}{4}}\left(\frac{1}{2}-\frac{1}{2}\cos 2\theta\right)d\theta$ ✓

$$\left.\begin{aligned} &= \left[\theta-\frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}} \\ &= \frac{\pi}{4}-\frac{1}{2}\sin\frac{\pi}{2} \\ &= \frac{\pi}{4}-\frac{1}{2} \end{aligned}\right\}$$ ✓

(1) (e) Let $\dfrac{x^2+9x}{(x+3)(x^2+9)} = \dfrac{A}{x+3} + \dfrac{Bx+C}{x^2+9}$. ✓

$\therefore\ x^2+9x = A(x^2+9) + (Bx+C)(x+3)$

Let $x=-3$.

$\therefore\ -18 = 18A$

$\therefore\ A = -1$

Let $x = 0$.

$\therefore\ 0 = -9 + 3C$

$\therefore\ C = 3$

Let $x = 1$.

$\therefore\ 10 = -10 + 4(B+3)$

$\therefore\ B+3 = 5$

$\therefore\ B = 2$

✓

$$\therefore \int \frac{x(x+9)}{(x+3)(x^2+9)}dx = \int \frac{-1}{x+3}dx + \int \frac{2x}{x^2+9}dx + \int \frac{3}{9+x^2}dx$$

$$= -\ln|x+3| + \ln(x^2+9) + \tan^{-1}\left(\frac{x}{3}\right) + c$$

✓✓

(2)(a) $\dfrac{23-14i}{3-4i} \cdot \dfrac{3+4i}{3+4i} = \dfrac{69+56-42i+92i}{9-16i^2}$

$= \dfrac{125+50i}{25}$

$= 5+2i$

(b) Let $-16+30i = (a+ib)^2$.

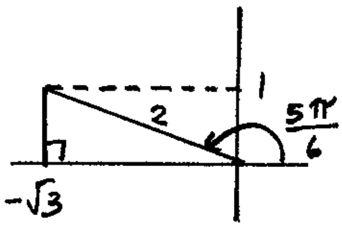
$\therefore\ a^2-b^2=-16$ and $ab=15$

By inspection, $(a,b)=(3,5)$ or $(-3,-5)$.

So the two square roots are $3+5i$ and $-3-5i$.

(c)(i) $w = -\sqrt{3}+i$

$= 2\,\text{cis}\,\dfrac{5\pi}{6}$



(ii) $w^9 = \left(2\,\text{cis}\,\dfrac{5\pi}{6}\right)^9$

$= 2^9\,\text{cis}\,\dfrac{15\pi}{2}$

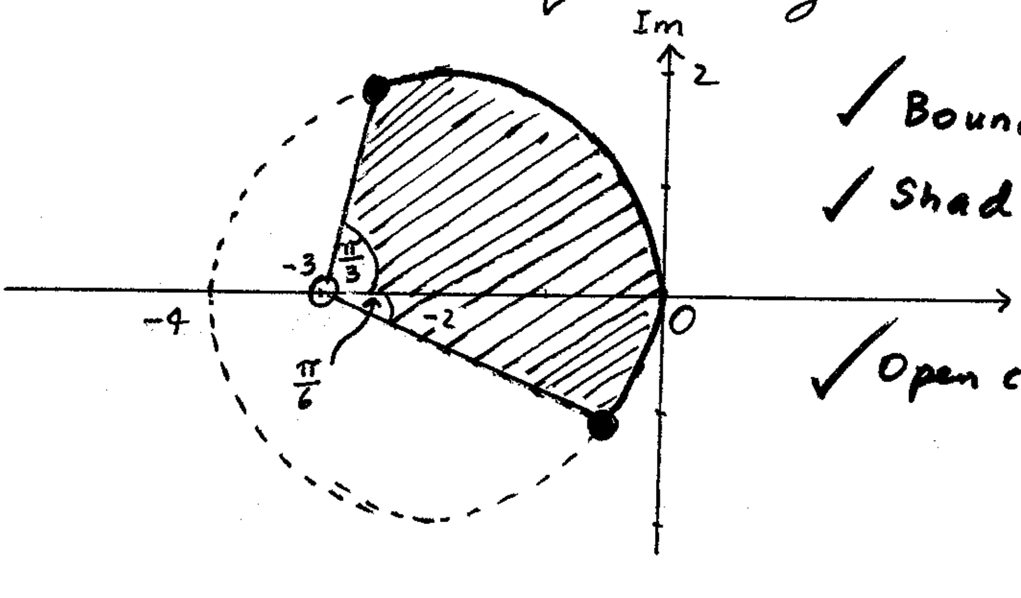
$= 512\,\text{cis}\,\dfrac{3\pi}{2}$

$= 512(0-i)$

$= -512i$, so $w^9+512i=0$.

So w is a root of the equation $z^9+512i=0$.

(d)

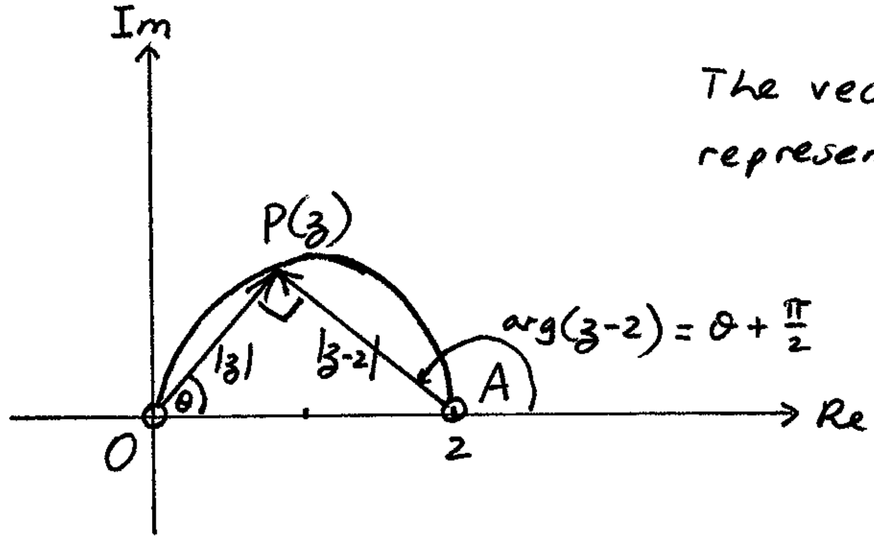


Boundaries

Shading

Open circle

(2)(e)(i)



The vector $\overrightarrow{AP}$ represents $z-2$.

✓

(ii) $\angle APO = \frac{\pi}{2}$ (angle in a semicircle)

So in ΔAPO,

$$\left|\frac{z-2}{z}\right| = \frac{|z-2|}{|z|}$$

$$= \tan\theta .$$

✓

(iii) $\arg\left(\frac{z-2}{z}\right) = \arg(z-2) - \arg z$ ✓

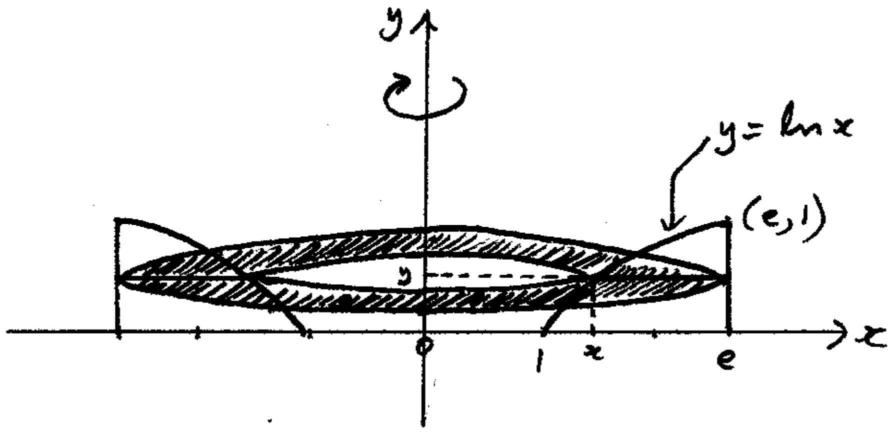
$$= \left(\theta + \frac{\pi}{2}\right) - \theta$$

($\arg(z-2)$ is an exterior angle of ΔAPO)

$$= \frac{\pi}{2}$$

✓

(3)(a)(i)



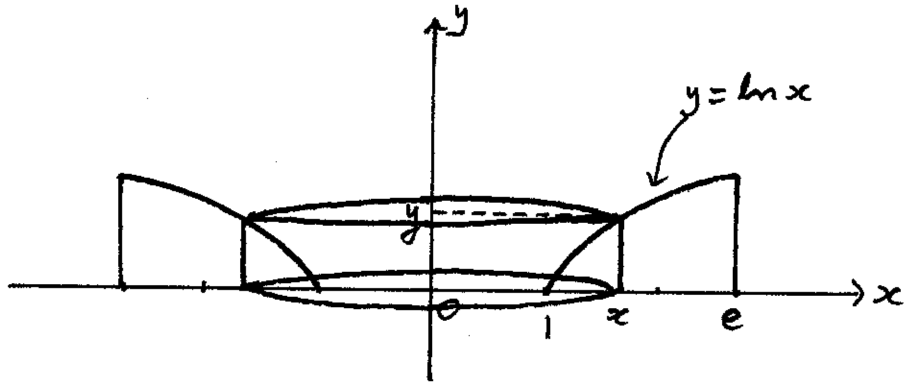
$R = e$
$r = x$

$$A(y) = \pi(R^2 - r^2)$$
$$= \pi(e^2 - x^2), \text{ where } x = e^y$$
$$= \pi(e^2 - e^{2y})$$

So $V = \int_{y=0}^{1} \pi(e^2 - e^{2y})\,dy$

$$= \pi\left[e^2 y - \tfrac{1}{2}e^{2y}\right]_0^1$$
$$= \pi\left(e^2 - \tfrac{1}{2}e^2 - 0 + \tfrac{1}{2}\right)$$
$$= \pi\left(\tfrac{1}{2}e^2 + \tfrac{1}{2}\right)$$
$$= \frac{\pi}{2}(e^2 + 1) \quad u^3$$

(ii)



$$A(x) = 2\pi r h$$
$$= 2\pi . x . y$$
$$= 2\pi x \ln x$$

So $V = \int_{x=1}^{e} 2\pi x \ln x\,dx$

$$= 2\pi\left[\tfrac{1}{2}x^2 \ln x\right]_1^e - 2\pi\int_1^e \tfrac{1}{2}x^2 . \tfrac{1}{x}\,dx$$
$$= \pi e^2 \ln e - \pi\int_1^e x\,dx$$
$$= \pi e^2 - \pi\left(\tfrac{1}{2}e^2 - \tfrac{1}{2}\right)$$
$$= \frac{\pi}{2}(e^2+1) \quad u^3$$

Integration by parts:

Let $u = \ln x$
$\therefore u' = \frac{1}{x}$

Let $v' = x$
$\therefore v = \frac{1}{2}x^2$

(3) (b) (i) $\overline{5+6i} = 5-6i$ is a zero, because all the coefficients of $P(x)$ are <u>real</u>.

Let α be the 3rd zero.

$\therefore (5+6i)+(5-6i)+\alpha = \frac{19}{2}$

$\therefore \alpha = -\frac{1}{2}$

So the zeroes of $P(x)$ are $5+6i$, $5-6i$, $-\frac{1}{2}$.

(ii) $(5+6i)(5-6i)(-\frac{1}{2}) = -\frac{d}{2}$

$\therefore d = 25+36$

$= 61$

(c) Let $u = x^3$, so that $x = u^{\frac{1}{3}}$ (or simply replace x with $x^{\frac{1}{3}}$)

The new equation is

$2u - u^{\frac{2}{3}} + 5 = 0$

$u^{\frac{2}{3}} = 2u+5$

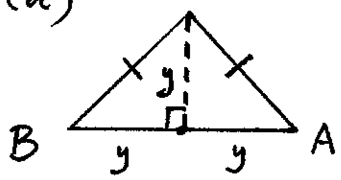
$u^2 = (2u+5)^3$

$$\left.\begin{aligned} u^2 &= 8u^3+60u^2+150u+125 \\ 8u^3&+59u^2+150u+125=0 \end{aligned}\right\}$$

Since u is a dummy variable, the new equation can also be written as

$8x^3+59x^2+150x+125=0$.

(4)(a)



(i) $A(x) = y^2$, where $x + y = 6$

$= (6-x)^2$

(ii) $V = \int_{x=-6}^{6} (6-x)^2 \, dx$

$= 2\left[\frac{(6-x)^3}{-3}\right]_0^6$

$= -\frac{2}{3}(0 - 6^3)$

$= 144 \ u^3$

(b)(i) $m_{PQ} = \dfrac{\frac{3}{p} - \frac{3}{q}}{3p - 3q}$

$= \dfrac{3(q-p)}{3pq(p-q)}$

$= \dfrac{-1}{pq}$

So the chord PQ has equation

$y - \frac{3}{p} = \frac{-1}{pq}(x - 3p)$

$pqy - 3q = -x + 3p$

$x + pqy = 3(p+q)$

(ii) The perpendicular distance from $(0,0)$ to the line $x + pqy - 3(p+q) = 0$ is $\sqrt{5}$ units.

So $\left|\dfrac{1(0) + pq(0) - 3(p+q)}{\sqrt{1^2 + (pq)^2}}\right| = \sqrt{5}$,

so $|-3(p+q)| = \sqrt{5(1+p^2q^2)}$,

so $9(p+q)^2 = 5(1+p^2q^2)$.

(4)(b)(iii) M is the point $\left(\dfrac{3p+3q}{2}, \dfrac{\frac{3}{p}+\frac{3}{q}}{2}\right)$

$$= \left(\frac{3(p+q)}{2}, \frac{3(p+q)}{2pq}\right).$$

So the locus of M has parametric equations

$$x = \frac{3(p+q)}{2} \quad (1) \qquad \text{and} \qquad y = \frac{3(p+q)}{2pq} \quad (2)$$

From (1), $p+q = \dfrac{2x}{3}$.

Substitute into (2): $y = \dfrac{x}{pq}$, so $pq = \dfrac{x}{y}$.

" " part (ii) to get the Cartesian equation:

$$9\left(\frac{2x}{3}\right)^2 = 5\left(1+\frac{x^2}{y^2}\right)$$

$$\frac{4x^2}{5} = 1 + \frac{x^2}{y^2}$$

$$\frac{x^2}{y^2} = \frac{4x^2-5}{5}$$

$$y^2 = \frac{5x^2}{4x^2-5}$$

(c) When $n=2$, LHS $= 2 + H(1)$ and RHS $= 2H(2)$

$= 2+1$ $\qquad$ $= 2\left(1+\frac{1}{2}\right)$

$= 3$ $\qquad$ $= 3$

So the result is true for $n=2$.

Assume that the result is true for the integer $n=k$.

i.e. assume that $k + H(1) + H(2) + \dots + H(k-1) = k\,H(k)$.

Prove that the result is true for $n=k+1$.

i.e. prove that $(k+1) + H(1) + H(2) + \dots + H(k-1) + H(k) = (k+1)H(k+1)$.

LHS $= 1 + (k + H(1) + H(2) + \dots + H(k-1)) + H(k)$

$= 1 + k\,H(k) + H(k)$ (using the assumption)

$= 1 + (k+1)H(k)$

$= \dfrac{k+1}{k+1} + (k+1)\left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{k}\right)$

$= (k+1)\left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{k}+\frac{1}{k+1}\right)$

$= (k+1)H(k+1) =$ RHS.

So, by induction, the result is true for $n = 2, 3, 4, \dots$.

(5)(a) $1 + 2x - x^2 > \dfrac{2}{x}, \quad x \neq 0$

Multiply both sides by x^2:

$$x^2 + 2x^3 - x^4 > 2x$$

$$x^4 - 2x^3 - x^2 + 2x < 0$$

$$x(x^3 - 2x^2 - x + 2) < 0$$

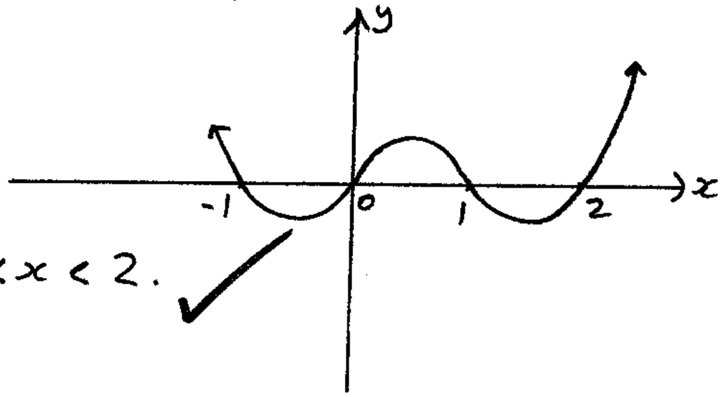
$$x\left(x^2(x-2) - 1(x-2)\right) < 0$$

$$x(x^2-1)(x-2) < 0$$

$$x(x+1)(x-1)(x-2) < 0$$

The solution is

$-1 < x < 0$ or $1 < x < 2$.



(b)(i) At P, $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta}$

$$= \frac{b\cos\theta}{-a\sin\theta} \quad \text{or} \quad -\frac{b\cos\theta}{a\sin\theta}$$

(ii) $m_{SP} \cdot m_{tangent} = \dfrac{b\sin\theta}{a\cos\theta - ae} \cdot \dfrac{+b\cos\theta}{-a\sin\theta}$

$$= \frac{b^2\cos\theta\cancel{\sin\theta}}{a^2\cancel{\sin\theta}(e-\cos\theta)}, \quad \text{where } b^2 = a^2(1-e^2)$$

$$= \frac{\cos\theta(1-e^2)}{e-\cos\theta}$$

(iii) Suppose $m_{SP} \cdot m_{tangent} = -1$.

Then $\cos\theta(1-e^2) = \cos\theta - e$

$$\cancel{\cos\theta} - e^2\cos\theta = \cancel{\cos\theta} - e$$

$$e\cos\theta = 1$$

$$\cos\theta = \frac{1}{e}, \text{ where } 0 < e < 1, \text{ so that } \frac{1}{e} > 1.$$

This is impossible, because $-1 \leq \cos\theta \leq 1$ for all real θ, so, provided $\theta \neq 0$ or π, SP cannot be perpendicular to the tangent.

(c) (i)

$$\cos 3\theta + i\sin 3\theta$$
$$= (\cos\theta + i\sin\theta)^3 \quad \text{(de Moivre)}$$
$$= \cos^3\theta + 3\cos^2\theta . i\sin\theta + 3\cos\theta . i^2\sin^2\theta + i^3\sin^3\theta$$

Equating real and imaginary parts,

$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$ and $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$.

(ii)
$$\tan 3\theta = \frac{\sin 3\theta}{\cos 3\theta}$$
$$= \frac{3\cos^2\theta\sin\theta - \sin^3\theta}{\cos^3\theta - 3\cos\theta\sin^2\theta}$$
$$= \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta},$$ after dividing top and bottom by $\cos^3\theta$.

(iii) Let $\theta = \frac{\pi}{12}$ in (ii).

$$\therefore \tan\frac{\pi}{4} = \frac{3\tan\frac{\pi}{12} - \tan^3\frac{\pi}{12}}{1 - 3\tan^2\frac{\pi}{12}}$$

It follows that $x = \tan\frac{\pi}{12}$ is a root of the equation

$$1 = \frac{3x - x^3}{1 - 3x^2}$$

i.e. $1 - 3x^2 = 3x - x^3$

i.e. $x^3 - 3x^2 - 3x + 1 = 0$

(iv) By inspection, $x = -1$ is a root of the equation. So $(x+1)$ is a factor of the LHS.

$$\begin{array}{r|l} & x^2 - 4x + 1 \\ \hline x+1 & x^3 - 3x^2 - 3x + 1 \\ & x^3 + x^2 \\ \hline & -4x^2 - 3x \\ & -4x^2 - 4x \\ \hline & x + 1 \end{array}$$

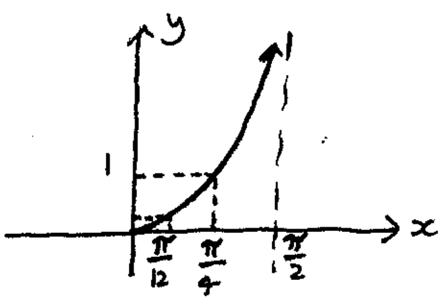
So the equation can be written

$(x+1)(x^2 - 4x + 1) = 0$.

So $\tan\frac{\pi}{12}$ is one of the roots of $x^2 - 4x + 1 = 0$

i.e. $(x-2)^2 = 3$,

from which $x = 2 \pm \sqrt{3}$.



But $\tan\frac{\pi}{12} < \tan\frac{\pi}{4} = 1$,

so $\tan\frac{\pi}{12} = 2 - \sqrt{3}$.

(6)(a)(i)

$$I_n = \int_0^{\frac{\pi}{2}} \sin^{n-1}\theta \sin\theta \, d\theta$$

$$= \left[-\cos\theta \sin^{n-1}\theta\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -\cos\theta.(n-1)\sin^{n-2}\theta\cos\theta \, d\theta$$

Let $u = \sin^{n-1}\theta$

$\therefore u' = (n-1)\sin^{n-2}\theta\cos\theta$

Let $v' = \sin\theta$

$\therefore v = -\cos\theta$

$$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}\theta \cos^2\theta \, d\theta$$

(ii) From (i),

$$I_n = (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}\theta(1-\sin^2\theta)\,d\theta$$

$$I_n = (n-1)\left(\int_0^{\frac{\pi}{2}} \sin^{n-2}\theta\,d\theta - \int_0^{\frac{\pi}{2}} \sin^n\theta\,d\theta\right)$$

$$I_n = (n-1)\left(I_{n-2} - I_n\right)$$

$$(n-1)I_n + I_n = (n-1)I_{n-2}$$

$$nI_n = (n-1)I_{n-2}$$

$$\therefore I_n = \frac{n-1}{n} I_{n-2}$$

(iii)

$I_9 = \frac{8}{9} \times \frac{6}{7} \times \frac{4}{5} \times \frac{2}{3} \times I_1$, where $I_1 = \int_0^{\frac{\pi}{2}} \sin\theta\,d\theta$

$= \left[-\cos\theta\right]_0^{\frac{\pi}{2}}$

$= 1$

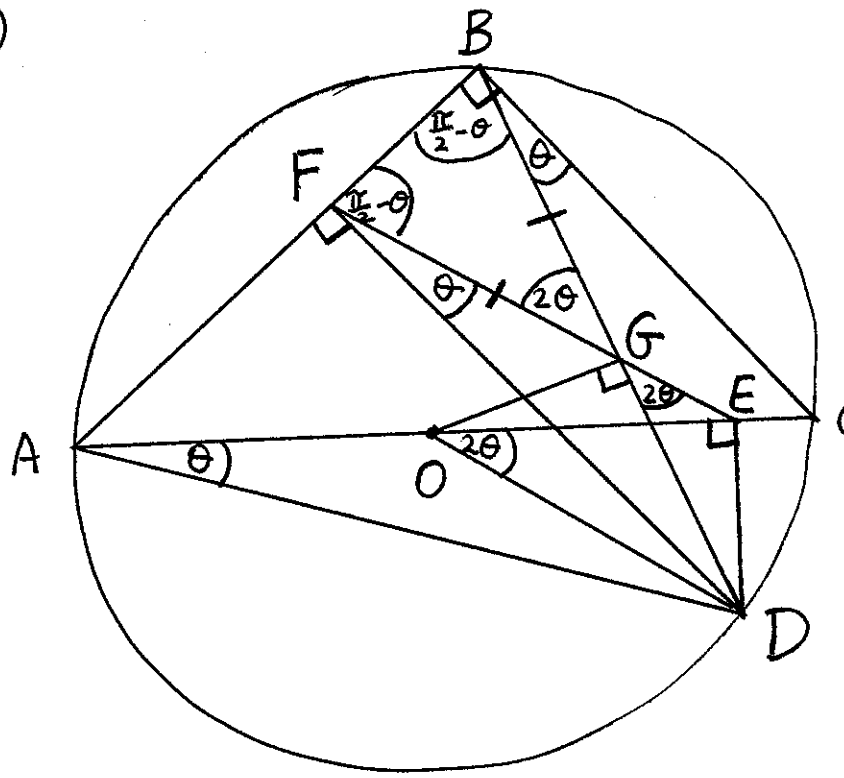
and $I_{10} = \frac{9}{10} \times \frac{7}{8} \times \frac{5}{6} \times \frac{3}{4} \times \frac{1}{2} \times I_0$, where $I_0 = \int_0^{\frac{\pi}{2}} d\theta$

$= \frac{\pi}{2}$.

So $I_9 \times I_{10} = \frac{9!}{10!} \times \frac{\pi}{2}$

$= \frac{\pi}{20}$.

(6)(b)



(i) $\angle AFD = \angle AED = \frac{\pi}{2}$ (given), so A, F, E and D are concyclic (converse of angle in a semicircle).

(ii) Let $\angle DAE = \theta$.
$\therefore \angle DFE = \angle DAE = \theta$ (angles in the same segment of circle ADEF)
and $\angle DBC = \angle DAC = \angle DAE = \theta$ (angles in same segment of circle ADCB).
So $\angle GFB = \angle GBF = \frac{\pi}{2} - \theta$ (complementary angles), so ΔFGB is isosceles (two equal angles).

(iii) $\angle FGB = 2\theta$ (angle sum of ΔFGB), so $\angle DGE = 2\theta$ (vertically opposite)
Also $\angle DOE = 2\theta$ (angle at centre of circle ADCB is twice $\angle DAE$ at the circumference)
So O, D, E and G are concyclic (converse of angles in the same segment)

(iv) $\angle OGD = \angle OED = \frac{\pi}{2}$ (angles in the same segment of circle ODEG)

So $OG \perp BD$.

(6)(c)(i) $P(k) = \dfrac{k}{k+1}$ for $k = 0, 1, 2, \ldots, n$.

So $(k+1)P(k) - k = 0$ for $k = 0, 1, 2, \ldots, n$.

So $x = 0, 1, 2, \ldots, n$ are zeroes of the polynomial $(x+1)P(x) - x$.

(ii) From (i), it follows that

$$(x+1)P(x) - x = A\,x(x-1)(x-2)\ldots(x-n), \quad (*)$$

where "A" is a constant.

leading coefficient

Let $x = -1$.

$\therefore 1 = A(-1)(-2)(-3)\ldots(-1-n)$

$\therefore 1 = A(-1)^{n+1}(n+1)!$

$\therefore A = \dfrac{1}{(n+1)!}$ $\left((-1)^{n+1} = 1, \text{ since } n \text{ is odd} \right)$

(iii) Let $x = n+1$ in $(*)$.

$\therefore (n+2)P(n+1) - (n+1) = \dfrac{1}{(n+1)!}(n+1)(n)(n-1)\ldots 3.2.1$

$\therefore P(n+1) = \dfrac{1 + (n+1)}{n+2}$

$= 1$

(7)(a)(i) By Pythagoras, $DG^2 = DC^2 + CG^2$,

so $DG^2 = 4 + w^2$.

By Pythagoras, $AG^2 = AD^2 + DG^2$

$= 1^2 + (4 + w^2)$

$= 5 + w^2$.

✓

(ii) By Pythagoras, $AH^2 = AD^2 + DH^2 = 1 + w^2$.

Also, $OA^2 = \left(\frac{1}{2}AG\right)^2 = \frac{1}{4}(5 + w^2) = OH^2$. ✓

So in ΔAOH, by the cosine rule,

$$\cos\alpha = \frac{OA^2 + OH^2 - AH^2}{2 \times OA \times OH}$$

$$= \frac{\left|\frac{1}{4}(5+w^2) + \frac{1}{4}(5+w^2) - (1+w^2)\right|}{2 \times \frac{1}{4}(5+w^2)} \cdot \frac{2}{2}$$

$$= \frac{|5 + w^2 - 2 - 2w^2|}{5 + w^2}$$

$$= \frac{|3 - w^2|}{5 + w^2}.$$

✓

(the absolute value is needed because α is acute, so $\cos\alpha > 0$)

(iii) $V = 2w$, $S = 2(2w + 1w + 2)$

$= 6w + 4$

So $r = \frac{V}{S} = \frac{2w}{6w+4}$

$= \frac{1}{3 + \frac{2}{w}}$.

✓

So as $w \to 0^+$, $r \to 0^+$

and as $w \to \infty$, $r \to \left(\frac{1}{3}\right)^-$. ✓ (this is the important part of the solution)

So $0 < r < \frac{1}{3}$ for all values of w.

(4)(a)(iv) If $r \geqslant \frac{1}{4}$,

then $$\frac{w}{3w+2} \geqslant \frac{1}{4}$$

$$4w \geqslant 3w+2$$

$$w \geqslant 2.$$

From (ii), as $w \to \infty$, $\alpha \to \cos^{-1} 1 = 0$.

So the minimum ~~maximum~~ value of $\cos\alpha$ is $\left|\frac{3-2^2}{5+2^2}\right| = \frac{1}{9}$.

So $\alpha \leq \cos^{-1}\frac{1}{9}$. (Note that $\cos\alpha$ is a decreasing function for $0 < \alpha < \frac{\pi}{2}$.)

(b)(i) $F = -mg - 10\%$ of v^2

$$\therefore m\ddot{x} = -mg - \frac{v^2}{10}$$

$$2\ddot{x} = -2\times 10 - \frac{v^2}{10}$$

$$\ddot{x} = -10 - \frac{v^2}{20}$$

$$= -\frac{200+v^2}{20}$$

(ii) $$v\frac{dv}{dx} = -\frac{200+v^2}{20}$$

$$\frac{dx}{dv} = \frac{-20v}{200+v^2}$$

$$x = -10\int \frac{2v}{200+v^2}\,dv$$

$$= -10\ln(200+v^2) + c_1$$

When $x = 0$, $v = u$,

so $c_1 = 10\ln(200+u^2)$.

So $x = 10\ln(200+u^2) - 10\ln(200+v^2)$

$$x = 10\ln\left(\frac{200+u^2}{200+v^2}\right).$$

(7)(b)(iii) $\frac{dv}{dt} = -\frac{200+v^2}{20}$

$$\frac{dt}{dv} = \frac{-20}{200+v^2}$$

$$t = -20\int \frac{1}{200+v^2}\,dv$$

$$= -20 \cdot \frac{1}{10\sqrt{2}} \tan^{-1}\frac{v}{10\sqrt{2}} + c_2$$

When $t=0$, $v=u$, so $c_2 = \frac{2}{\sqrt{2}} \tan^{-1}\frac{u}{10\sqrt{2}}$.

So $t = \sqrt{2}\left(\tan^{-1}\frac{u}{10\sqrt{2}} - \tan^{-1}\frac{v}{10\sqrt{2}}\right)$

(iv) Find the distance AB and the time taken for the first particle.

When $u = 10\sqrt{2}$ and $v=0$,

$$x = 10\ln\left(\frac{200+(10\sqrt{2})^2}{200+0^2}\right)$$

$$= 10\ln 2 \text{ metres.}$$

When $u = 10\sqrt{2}$ and $v=0$,

$$t = \sqrt{2}\left(\tan^{-1}1 - \tan^{-1}0\right)$$

$$= \frac{\pi\sqrt{2}}{4} \text{ seconds } (= 1{\cdot}11\ldots \text{ seconds}).$$

Now consider the second particle, for which $u = 30\sqrt{2}$. Its velocity when it reaches B is given by

$$10\ln 2 = 10\ln\left(\frac{200+(30\sqrt{2})^2}{200+v^2}\right)$$

$$2 = \frac{2000}{200+v^2}$$

$$v^2 + 200 = 1000$$

$$v = 20\sqrt{2} \text{ ms}^{-1} \quad (v>0).$$

Now find the time taken for the second particle to reach B. When $v = 20\sqrt{2}$,

$$t = \sqrt{2}\left(\tan^{-1}3 - \tan^{-1}2\right)$$

$$= 0{\cdot}20067\ldots \text{ seconds.}$$

Now, $\sqrt{2}\left(\tan^{-1}3 - \tan^{-1}2\right) + \frac{3\sqrt{2}}{5} < \frac{\pi\sqrt{2}}{4}$

(i.e. $1{\cdot}0492\ldots < 1{\cdot}11\ldots$).

So the second particle reaches B before the first particle. So the second particle overtakes the first particle while they are both rising.

(8)(a) LHS $= \dfrac{1+\cos\alpha}{\sin\alpha}$

$= \dfrac{2\cos^2\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}$ ✓

$= \cot\frac{\alpha}{2}$

$= \tan\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)$ ✓

$=$ RHS

(b)(i)

$t = \tan\frac{x}{2}$

$\therefore x = 2\tan^{-1}t$

$\therefore dx = \dfrac{2}{1+t^2}dt$

x	0	$\frac{\pi}{2}$
t	0	1

✓

$I = \displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{1+\cos\alpha\sin x}\,dx$

$= \displaystyle\int_0^1 \frac{1}{1+\cos\alpha\cdot\frac{2t}{1+t^2}}\cdot\frac{2}{1+t^2}\,dt$

$= \displaystyle\int_0^1 \frac{2}{1+t^2+2t\cos\alpha}\,dt$ ✓

$= \displaystyle\int_0^1 \frac{2}{(t^2+2t\cos\alpha+\cos^2\alpha)+\sin^2\alpha}\,dt$

$= \displaystyle\int_0^1 \frac{2}{(t+\cos\alpha)^2+\sin^2\alpha}\,dt$ ✓

(8)(b)(ii)

$$I = \int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}-\frac{\alpha}{2}} \frac{2}{\sin^2\alpha\tan^2 u + \sin^2\alpha} . \sin\alpha\sec^2 u\,du$$

$$= \int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}-\frac{\alpha}{2}} \frac{2\sec^2 u}{\sin\alpha(\tan^2 u + 1)}\,du$$

$$= \frac{2}{\sin\alpha}\int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}-\frac{\alpha}{2}} du$$

$$= \frac{2}{\sin\alpha}\left[u\right]_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}-\frac{\alpha}{2}}$$

$$= \frac{2}{\sin\alpha}\left(\frac{\pi}{2} - \frac{\alpha}{2} - \frac{\pi}{2} + \alpha\right)$$

$$= \frac{2}{\sin\alpha} . \frac{\alpha}{2}$$

$$= \frac{\alpha}{\sin\alpha}$$

$t + \cos\alpha = \sin\alpha\tan u$

$\therefore dt = \sin\alpha\sec^2 u\,du$ ✓

When $t=0$,

$\tan u = \cot\alpha$

$\tan u = \tan(\frac{\pi}{2}-\alpha)$

$u = \frac{\pi}{2}-\alpha$ ✓

When $t=1$,

$\tan u = \frac{1+\cos\alpha}{\sin\alpha}$

$= \tan(\frac{\pi}{2}-\frac{\alpha}{2})$

$u = \frac{\pi}{2}-\frac{\alpha}{2}$

(8)(c)(i) $z^{2n+1} = 1$

Let $z = \operatorname{cis}\theta$.

$\therefore \operatorname{cis}(2n+1)\theta = \operatorname{cis}(2k\pi)$, where $k \in \mathbb{Z}$

$\therefore \theta = \dfrac{2k\pi}{2n+1}$ for $k = 0, 1, 2, \dots, 2n$

So the roots are

$$z = \operatorname{cis} 0, \operatorname{cis}\frac{2\pi}{2n+1}, \operatorname{cis}\frac{4\pi}{2n+1}, \dots, \operatorname{cis}\frac{4n\pi}{2n+1}.$$

(ii) $z^{2n+1} - 1 = (z-1)(z^{2n} + z^{2n-1} + \dots + z^2 + z + 1)$

So $(z^{2n} + z^{2n-1} + \dots + z^2 + z + 1)$

$$= \left(z - \operatorname{cis}\frac{2\pi}{2n+1}\right)\left(z - \operatorname{cis}\frac{4n\pi}{2n+1}\right)\left(z - \operatorname{cis}\frac{4\pi}{2n+1}\right)\left(z - \operatorname{cis}\frac{(4n-2)\pi}{2n+1}\right)$$

$$\cdots\left(z - \operatorname{cis}\frac{2n\pi}{2n+1}\right)\left(z - \operatorname{cis}\frac{(2n+2)\pi}{2n+1}\right)$$

$$= \left(z - \operatorname{cis}\frac{2\pi}{2n+1}\right)\left(z - \overline{\operatorname{cis}\frac{2\pi}{2n+1}}\right)\left(z - \operatorname{cis}\frac{4\pi}{2n+1}\right)\left(z - \overline{\operatorname{cis}\frac{4\pi}{2n+1}}\right)$$

$$\cdots\left(z - \operatorname{cis}\frac{2n\pi}{2n+1}\right)\left(z - \overline{\operatorname{cis}\frac{2n\pi}{2n+1}}\right)$$

$$= \left(z^2 - \left(2\cos\frac{2\pi}{2n+1}\right)z + 1\right)\left(z^2 - \left(2\cos\frac{4\pi}{2n+1}\right)z + 1\right)\cdots\left(z^2 - \left(2\cos\frac{2n\pi}{2n+1}\right)z + 1\right)$$

(iii) Let $x = 1$ in the identity in (ii):

$$2n+1 = 2\left(1 - \cos\frac{2\pi}{2n+1}\right).\,2\left(1 - \cos\frac{4\pi}{2n+1}\right)\dots 2\left(1 - \cos\frac{2n\pi}{2n+1}\right)$$

$$2n+1 = 2^n.\,2\sin^2\frac{\pi}{2n+1}\cdot 2\sin^2\frac{2\pi}{2n+1}\dots 2\sin^2\frac{n\pi}{2n+1}$$

(since $1 - \cos 2\theta = 2\sin^2\theta$)

$$\therefore 2^{2n}\sin^2\frac{\pi}{2n+1}\sin^2\frac{2\pi}{2n+1}\dots\sin^2\frac{n\pi}{2n+1} = 2n+1$$

$$\therefore 2^n\sin\frac{\pi}{2n+1}\sin\frac{2\pi}{2n+1}\dots\sin\frac{n\pi}{2n+1} = \sqrt{2n+1}$$