2011 TRIAL. EXT 1.

Q1

a) $\frac{(n+1)!}{n!} = n+1$ ✓

b) $\int \frac{1}{9+x^2}\,dx = \frac{1}{3}\tan^{-1}\frac{x}{3} + c$ ✓

c) $P(2)=24$

$2^3 + 3\times 2^2 + 2a - 10 = 24$

$8 + 12 + 2a - 10 = 24$ ✓

$2a + 10 = 24$

$2a = 14$ ✓

$a = 7$

d) $y = \sin^{-1} u \qquad u = x^3$

$\frac{dy}{du} = \frac{1}{\sqrt{1-u^2}} \qquad \frac{du}{dx} = 3x^2$ ✓

$\frac{dy}{dx} = \frac{3x^2}{\sqrt{1-x^6}}$ ✓

e) i) $\alpha\beta + \alpha\gamma + \beta\gamma = -4$ ✓

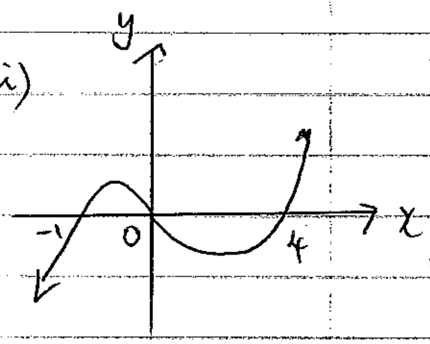
ii) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

$= \frac{\alpha\beta + \alpha\gamma + \beta\gamma}{\alpha\beta\gamma}$

$= \frac{-4}{-12}$ ✓

$= \frac{1}{3}$

f) i)



✓✓

ii) $\frac{x(x+1)}{(x-4)} \geq 0 \qquad x \neq 4$

$[\times (x-4)^2]$

$(x-4)x(x+1) \geq 0$

$-1 \leq x \leq 0$ or $x > 4$ ✓ ✓

Note * $x \neq 4$

Q2

a) $\sin^{-1}\left(\sin\frac{2\pi}{3}\right) = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$= \frac{\pi}{3}$

b) $\lim_{x\to\infty} \frac{3-x}{2x+3}$

$= \lim_{x\to\infty} \frac{\frac{3}{x} - \frac{x}{x}}{\frac{2x}{x} + \frac{3}{x}}$

$= -\frac{1}{2}$

c) $A(2,-4)$ with x_1, y_1 under its coordinates; $B(5,2)$ with x_2, y_2 under its coordinates; $4:-1$ with m, n under the two numbers

$P = \left(\frac{mx_2 - nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$

$= \left(\frac{4\times5 - 1\times2}{4-1}, \frac{4\times2 - 1\times -4}{3}\right)$

$= \left(\frac{20-2}{3}, \frac{8+4}{3}\right)$

$= (6, 4)$

d) $y = \tan^{-1}(\sin x)$

$\frac{dy}{dx} = \frac{\cos x}{1+\sin^2 x}$

At $x = \pi$

$\frac{dy}{dx} = \frac{\cos\pi}{1+\sin^2\pi}$

$= -1$

e) i) $45t - 5t^2 = 0$

$5t(9-t) = 0$

Ball returns at $t = 9s$.

ii) $h = 45t - 5t^2$

$\frac{dh}{dt} = 45 - 10t$

$45 - 10t = 0$

$-10t = -45$

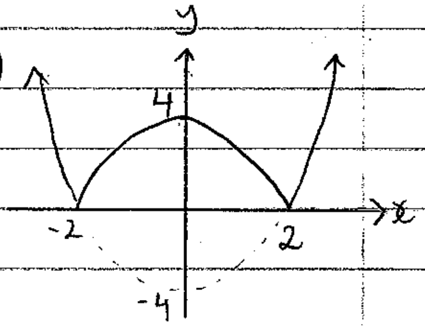
$t = 4.5$

iii) $t = 4.5$

$h = 45\times4.5 - 5\times(4.5)^2$

$= 101.25\,m$

f) i)



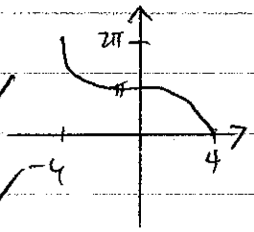
ii) Not differentiable at $(-2,0)$ $(2,0)$.

Q3

a) $f(x) = 2\cos^{-1}\frac{x}{4}$

Stretched horizontally ×4

vertically ×2



Domain $-4 \leq x \leq 4$

Range $0 < y \leq 2\pi$

b) i) $OA = PA = OB = PB = r$

(radii of congruent circles)

$OAPB$ is a rhombus.

ii) $OP = OA = AP$ (radii)

ΔOAP is equilateral

$\angle AOP = \frac{\pi}{3}$

$OP = OB = BP$ (radii)

ΔOBP is equilateral

$\angle POB = \frac{\pi}{3}$

$\angle AOB = \angle AOP + \angle POB$ (adjacent angles)

$= \frac{2\pi}{3}$

iii) Area of segment $= \frac{1}{2}r^2\left(\frac{2\pi}{3} - \sin\frac{2\pi}{3}\right)$

Shaded region $= \pi r^2 - \cancel{2} \times \frac{1}{\cancel{2}} r^2\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$

$= \pi r^2 - r^2\left(\frac{4\pi - 3\sqrt{3}}{6}\right)$

$= \frac{2\pi r^2 + 3\sqrt{3}r^2}{6}$

c) $f(x) = x\log x + x - 1.1$

$f'(x) = (1 + \log x) + 1$

$= 2 + \log x$

$f(1) = \log 1 + 1 - 1.1$

$= -0.1$

$f'(1) = 2$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= 1 - \left(\frac{-0.1}{2}\right)$

$= 1.05$

d) $\left(4x^3 - \frac{1}{x}\right)^{12}$

General term

${}^{12}C_r (4x^3)^{12-r}(-x^{-1})^r$

$= {}^{12}C_r\, 4^{12-r} x^{36-3r} (-1)^r x^{-r}$

For term independant of x

$x^{36-3r} x^{-r} = x^0$

$36 - 4r = 0$

$4r = 36$

$r = 9$

$\therefore$ Term independent of x is

$-{}^{12}C_9\, 4^3$

$= -14080$

Q4

a) $\cos\alpha = \frac{1-t^2}{1+t^2}$ $\quad \cos\alpha = \frac{3}{4}$

$$\frac{1-t^2}{1+t^2} = \frac{3}{4}$$

$$4-4t^2 = 3+3t^2$$ ✓

$$7t^2 = 1$$

$$t = \pm\frac{1}{\sqrt{7}}$$

Since α is acute

$$t = \frac{1}{\sqrt{7}}$$ ✓

$$\tan\frac{\alpha}{2} = \frac{1}{\sqrt{7}}$$

b) $\int_0^1 \frac{4x}{(4x+1)^2}\,dx$ $\quad u = 4x+1$

$\frac{du}{dx} = 4$

$$= \int_1^5 \frac{4\left(\frac{u-1}{4}\right)}{u^2}\,\frac{du}{4}$$

$du = 4dx$

$dx = \frac{du}{4}$

$$= \int_1^5 \frac{u-1}{4u^2}\,du$$

$x = \frac{u-1}{4}$

$$= \int_1^5 \left(\frac{1}{4u} - \frac{1}{4u^2}\right)du$$

$x=0 \quad u=1$

$x=1 \quad u=5$

$$= \frac{1}{4}\left[\log u + \frac{1}{u}\right]_1^5$$

$$= \frac{1}{4}\left[\left(\log 5 + \frac{1}{5}\right) - (\log 1 + 1)\right]$$

$$= \frac{1}{4}\left[\log 5 - \frac{4}{5}\right]$$

c) $y = 2\sin x$ $\quad y^2 = 4\sin^2 x$

$$V = \int_0^{\frac{\pi}{4}} \pi y^2\,dx$$

$$= 4\pi\int_0^{\frac{\pi}{4}} \sin^2 x\,dx$$ ✓

$$= 4\pi\int_0^{\frac{\pi}{4}} \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)dx$$

$$= 4\pi\left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_0^{\frac{\pi}{4}}$$ ✓

$$= 4\pi\left[\left(\frac{\pi}{8} - \frac{1}{4}\sin\frac{\pi}{2}\right) - (0-0)\right]$$

$$= 4\pi\left(\frac{\pi}{8} - \frac{1}{4}\right)$$

$$= \frac{4\pi^2}{8} - \frac{4\pi}{4}$$ ✓

$$= \left(\frac{\pi^2}{2} - \pi\right)u^3$$

d) i) $x = \sqrt{3}\cos 3t - \sin 3t$

$$\dot{x} = -3\sqrt{3}\sin 3t - 3\cos 3t$$ ✓

$$\ddot{x} = -9\sqrt{3}\cos 3t + 9\sin 3t$$

$$= -9\left(\sqrt{3}\cos 3t - \sin 3t\right)$$

$$= -9x$$ ✓

$$= -n^2 x$$

ii) $\sqrt{3}\cos 3t - \sin 3t = 0$

$$\sqrt{3}\cos 3t = \sin 3t$$ ✓

$$\tan 3t = \sqrt{3}$$

$$3t = \frac{\pi}{3}, \frac{4\pi}{3}, \ldots$$

$$t = \frac{\pi}{9}, \frac{4\pi}{9}, \ldots$$

$\therefore$ Particle first passes the origin at $t = \frac{\pi}{9}$ ✓

Q5

a) A. When $n=1$

$RHS = \frac{1}{2\times1+1}$

$= \frac{1}{3}$

$LHS = \frac{1}{(2\times1+1)} \times \frac{1}{(2\times1-1)}$

$= \frac{1}{3} \times \frac{1}{1}$

$= \frac{1}{3}$

$= RHS$ ✓

$\therefore$ Statement true for $n=1$

B. Assume statement true for $n=k$ where k is a positive integer ✓

ie.

** $\frac{1}{3}\times\frac{1}{1} + \frac{1}{5}\times\frac{1}{3} + \dots + \frac{1}{(2k+1)}\times\frac{1}{(2k-1)} = \frac{k}{2k+1}$

Must prove statement for $n=k+1$

ie.

$\frac{1}{3}\times\frac{1}{1} + \frac{1}{5}\times\frac{1}{3} + \dots + \frac{1}{(2(k+1)+1)}\times\frac{1}{(2(k+1)-1)} = \frac{k+1}{2(k+1)+1}$

$= \frac{k+1}{2k+3}$

$LHS =$

$\underbrace{\frac{1}{3}\times\frac{1}{1} + \frac{1}{5}\times\frac{1}{3} + \dots + \frac{1}{(2k+1)}\times\frac{1}{(2k-1)}}_{**} + \frac{1}{(2(k+1)+1)}\times\frac{1}{(2(k+1)-1)}$ ✓

$= \frac{k}{2k+1} + \frac{1}{(2k+3)}\times\frac{1}{(2k+1)}$ By induction hypothesis

$= \frac{k(2k+3)+1}{(2k+3)(2k+1)}$ ✓

$= \frac{2k^2+3k+1}{(2k+3)(2k+1)} = \frac{\cancel{(2k+1)}(k+1)}{(2k+3)\cancel{(2k+1)}} = \frac{k+1}{2k+3} = RHS$

C. It follows from parts A and B by mathematical induction that the statement is true for all positive integers n.

b) i) $\angle PQT = \alpha$ (alternate segment theorem) ✓

ii) let $\angle PRQ = \theta$

$\angle TQS = \theta$ (alternate segment theorem)

In $\triangle PRQ$ ✓

$\angle QPR = 180 - (\alpha+\theta)$ (angle sum of $\triangle$)

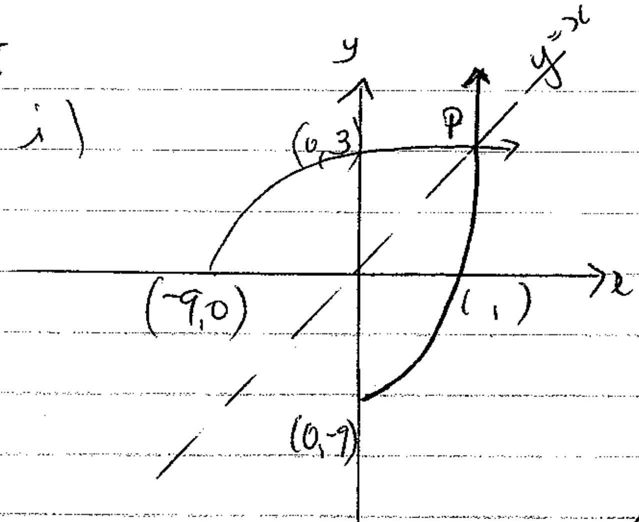
$\angle QSR = \alpha + \theta$ (adjacent $\angle$)

$\angle QPR + \angle QSR = 180$

$\therefore PQSR$ is cyclic quad ✓ as opposite angles in a cyclic quad are supplementary. ✓

Q5

c) i)



ii) Domain of $f^{-1}(x) \geq 0$

iii) $y = \sqrt{x+9}$

interchange x & y

$x = \sqrt{y+9}$

$x^2 = y + 9$

$y = x^2 - 9$ where $x \geq 0$

iv) $x = \sqrt{x+9}$

$x^2 = x + 9$

$x^2 - x - 9 = 0$

$a = 1$ $\quad \Delta = b^2 - 4ac$

$b = -1$ $\quad = 1 - 4 \times 1 \times -9$

$c = -9$ $\quad = 37$

$x = \frac{-1 \pm \sqrt{37}}{2}$ $\quad x \geq 0$

$\therefore x = \frac{1 + \sqrt{37}}{2}$ x-coordinate of P.

Q6

a) i) $v = 600 + Pe^{-kt}$

$\frac{dv}{dt} = \frac{d}{dt}(600 + Pe^{-kt})$

$= -kPe^{-kt}$ ✓

$= -k(v - 600)$

ii) $t = 0 \quad v = 0$

$t = 3 \quad v = 25$

$0 = 600 + Pe^{0}$

$P = -600$ ✓

$25 = 600 - 600e^{-3k}$

$-575 = -600e^{-3k}$

$e^{-3k} = \frac{575}{600}$

$e^{3k} = \frac{24}{23}$

$\ln e^{3k} = \ln\left(\frac{24}{23}\right)$

$3k = \ln\left(\frac{24}{23}\right)$ ✓

$k = \frac{1}{3}\ln\left(\frac{24}{23}\right)$

iii) $t = 10$

$v = 600 - 600e^{-10k}$ ✓

$\doteqdot 79.4\,ms^{-1}$

iv) $v = 600(1 - e^{-kt})$

$t \to \infty \quad e^{-kt} \to 0$

$v = 600$ ✓

b) $T_k = {}^{12}C_k(2x)^{12-k}y^k$

$= {}^{12}C_k 2^{12} x^{12-k} y^k$

$T_{k+1} = {}^{12}C_{k+1}(2x)^{12-(k+1)}y^{k+1}$

$= {}^{12}C_{k+1}(2x)^{11-k}y^{k+1}$

$\frac{T_{k+1}}{T_k} = \frac{{}^{12}C_{k+1}(2x)^{11-k}y^{k+1}}{{}^{12}C_k(2x)^{12-k}y^k}$

$= \dfrac{\frac{12!}{(k+1)!\,(11-k)!}(2x)^{11-k}y^{k+1}}{\frac{12!}{k!\,(12-k)!}(2x)^{12-k}y^k}$ ✓

$= \frac{(12-k)y}{(k+1)2x}$

$\frac{T_{k+1}}{T_k} \geqslant 1$ when $x = 4 \quad y = 5$

$\frac{(12-k)5}{(k+1)8} \geqslant 1$

$60 - 5k \geqslant 8k + 8$

$52 \geqslant 13k$

$k \leqslant 4$ ✓

if $k = 4$ $\quad \frac{T_5}{T_4} = 1$ so $T_4 = T_5$

if $k > 4$ $\quad \frac{T_6}{T_5} < 1$ ✓

if $k < 4$ $\quad \frac{T_4}{T_3} > 1$

Q6

e) $2\cos 3x \sin 4x + 2\cos 3x - \sin 4x - 1 = 0$

i) $2\cos 3x(\sin 4x + 1) - 1(\sin 4x + 1) = 0$ ✓

$(2\cos 3x - 1)(\sin 4x + 1) = 0$

$2\cos 3x = 1$

$\cos 3x = \frac{1}{2}$

$3x = 2n\pi \pm \cos^{-1}(\frac{1}{2})$

$= 2n\pi \pm \frac{\pi}{3}$

$x = \frac{2n\pi}{3} \pm \frac{\pi}{9}$ ✓

for some integer n

$\sin 4x = -1$

$4x = \sin^{-1}(-1) + 2n\pi$

$= -\frac{\pi}{2} + 2n\pi$ for some integer n.

$x = -\frac{\pi}{8} + \frac{n\pi}{2}$

OR ✓

$4x = (\pi - \sin^{-1}(-1)) + 2n\pi$

$= \frac{3\pi}{2} + 2n\pi$

$x = \frac{3\pi}{8} + \frac{n\pi}{2}$

Alternatively.

$4x = m\pi + (-1)^m \sin^{-1}(-1)$

$= m\pi + (-1)^m(\frac{-\pi}{2})$

$x = \frac{m\pi}{4} + (-1)^m(\frac{-\pi}{8})$

for some integer m.

ii) For $0 \le x \le \pi$

$x = \frac{2n\pi}{3} \pm \frac{\pi}{9}$ $\quad n = 0, 1$

$= \frac{+\pi}{9}, \frac{2\pi}{3} \pm \frac{\pi}{9}$

$= \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}$

$x = \frac{m\pi}{4} + (-1)^n(\frac{\pi}{8})$ $(m = 0, 1, 2$

$= \cancel{0 - \frac{\pi}{8}}, \frac{\pi}{4} + \frac{\pi}{8}, \frac{2\pi}{4} - \frac{\pi}{8}$

$\frac{3\pi}{4} + \frac{\pi}{8}, \pi - \frac{\pi}{8}$

$= \frac{3\pi}{8}, \cancel{\frac{3\pi}{8}}, \frac{7\pi}{8}$

✓

Q7

a) $(1+x)^{2n} = (1+x)^n(1+x)^n$

LHS coefficient of x^n is $\binom{2n}{n}$

$$RHS = (1+x)^n(1+x)^n$$

$$= \left[\binom{n}{0}x^0 + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{r}x^r + \dots + \binom{n}{n}x^n\right] \times$$

$$\left[\binom{n}{n}x^n + \binom{n}{n-1}x^{n-1} + \binom{n}{n-2}x^{n-2} + \dots + \binom{n}{n-r}x^{n-r} + \dots + \binom{n}{0}x^0\right]$$

Coefficient of x^n on RHS

$$\binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \dots + \binom{n}{n-r}\binom{n}{r} + \dots + \binom{n}{n}\binom{n}{0}$$

$\binom{n}{0} = \binom{n}{n}$, $\binom{n}{1} = \binom{n}{n-1}$ and $\binom{n}{n-r}\binom{n}{r}$ for all $r \leq n$

$\therefore$ Coefficients of x^n on RHS is $\sum_{k=0}^{n}\binom{n}{k}^2$ ✓

Equating coefficients of x^n on LHS & RHS

$$\binom{2n}{n} = \sum_{k=0}^{n}\binom{n}{k}^2$$

Q7

b) $x = Vt\cos\alpha \qquad y = Vt\sin\alpha - 5t^2$

$t = \dfrac{x}{V\cos\alpha}$

So $y = \dfrac{\cancel{V}x\sin\alpha}{\cancel{V}\cos\alpha} - \dfrac{5x^2}{V^2\cos^2\alpha}$

$= x\tan\alpha - \dfrac{5x^2\sec^2\alpha}{V^2}$ ✓

when $x = p$, $y = h$

$h = p\tan\alpha - \dfrac{5p^2\sec^2\alpha}{V^2}$

$h - p\tan\alpha = \dfrac{-5p^2\sec^2\alpha}{V^2}$ ✓

$V^2 = \dfrac{-5p^2(1+\tan^2\alpha)}{h - p\tan\alpha}$

ii) Also

$V^2 = \dfrac{-5q^2(1+\tan^2\alpha)}{h - q\tan\alpha}$

$\dfrac{\cancel{-5}p^2(1+\tan^2\alpha)}{h - p\tan\alpha} = \dfrac{\cancel{-5}q^2(1+\tan^2\alpha)}{h - q\tan\alpha}$ ✓

$p^2\cancel{(1+\tan^2\alpha)}(h - q\tan\alpha) = q^2\cancel{(1+\tan^2\alpha)}(h - p\tan\alpha)$

$p^2h - p^2q\tan\alpha = q^2h - pq^2\tan\alpha$

$p^2h - q^2h = p^2q\tan\alpha - pq^2\tan\alpha$

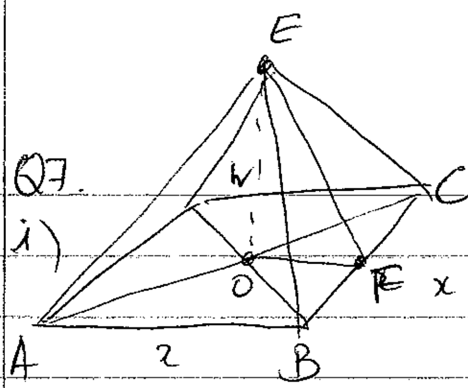
$h(p^2 - q^2) = pq\tan\alpha(p - q)$ ✓

$\tan\alpha = \dfrac{h\cancel{(p-q)}(p+q)}{pq\cancel{(p-q)}}$

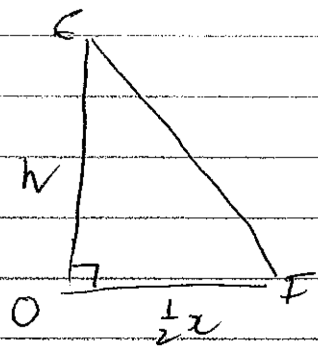
$= \dfrac{h(p+q)}{pq}$ as required.

Q7.

c) i)

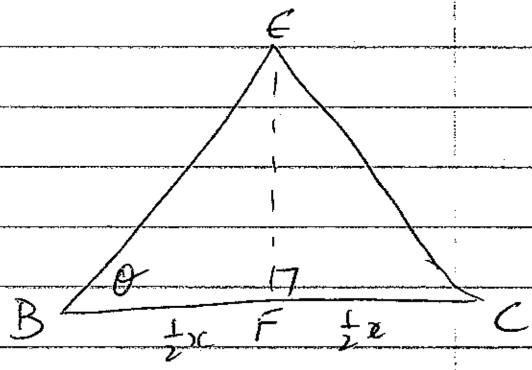


let the height of pyramid be h.



$$V = \frac{1}{3}x^2h$$
$$3V = x^2h$$
$$h = \frac{3V}{x^2}$$

$$EF^2 = h^2 + \frac{1}{4}x^2$$ ✓



Area of $\triangle EBC = \frac{1}{2}x \times EF$

$$\frac{S}{4} = \frac{1}{2}x \times \sqrt{h^2 + \frac{x^2}{4}}$$

$$\frac{S^2}{16} = \frac{x^2}{4} \times \left(h^2 + \frac{x^2}{4}\right)$$

$$S^2 = 4x^2 \times \left[\left(\frac{3V}{x^2}\right)^2 + \frac{x^2}{4}\right]$$

$$= 4x^2 \times \left(\frac{9V^2}{x^4} + \frac{x^2}{4}\right)$$ ✓

$$= \frac{36V^2}{x^2} + x^4 \text{ as required}$$

Q7

c ii) $S^2 = x^4 + 36V^2x^{-2}$

$\frac{d(S^2)}{dx} = 4x^3 - 72V^2x^{-3}$

$\frac{d(S^2)}{dx} = 0$ at stationary point.

$4x^3 - 72V^2x^{-3} = 0$

$x^3 - 18V^2x^{-3} = 0$

$x^6 = 18V^2$ ✓

$x^3 = \sqrt{18}\,V^2$

$= 3\sqrt{2}\,V$

$\frac{d^2S^2}{dx^2} = 12x^2 + 216V^2x^{-4}$

since $x^2 > 0$ $\frac{1}{x^4} > 0$ $V^2 > 0$

$\frac{d^2S^2}{dx^2} > 0$ for all x. ✓

$x^3 = 3\sqrt{2}V$ gives a minimum value for S.

iii) In ΔEBF $\tan\theta = \frac{EF}{\frac{1}{2}x}$

$\tan\theta = \frac{\sqrt{h^2 + \frac{x^2}{4}}}{\frac{1}{2}x}$ $h = \frac{3V}{x^2}$

$= \frac{\sqrt{\frac{x^2}{2} + \frac{x^2}{4}}}{\frac{1}{2}x}$ ✓ and $V = \frac{x^3}{3\sqrt{2}}$

$h = \frac{\cancel{3}}{\cancel{x^2}} \times \frac{x^{\cancel{3}}}{\cancel{3}\sqrt{2}}$

$= \sqrt{\frac{2x^2+x^2}{4}} \div \frac{x}{2}$ $= \frac{x}{\sqrt{2}}$

$= \frac{\sqrt{3}\,\cancel{x}}{\cancel{x}} \times \frac{\cancel{x}}{\cancel{x}}$ ✓

$= \sqrt{3}$

$\theta = 60^\circ$ $\therefore \Delta EBC$ is equilateral.