2011 2U Trial HSC Solutions

TOTAL is $10 \times 12 = 120$

(1)(a) $\frac{1}{3+\sqrt{5}} \times \frac{3-\sqrt{5}}{3-\sqrt{5}}$ ✓

$= \frac{3-\sqrt{5}}{4}$ ✓

(b) $\tan\frac{5\pi}{6} = -\tan\frac{\pi}{6}$ ✓

$= -\frac{1}{\sqrt{3}}$ ✓

(c) $y' = 3x^2$ ✓

so $m = 3\left(\frac{1}{2}\right)^2$

$= \frac{3}{4}$ ✓

(d)(i) $\alpha + \beta = \frac{5}{2}$ ✓

(ii) $\alpha\beta = -4$ ✓

(e) $T_n = a + (n-1)d$

so $T_{50} = 119 + (49)(-6)$ ✓

$= -175$ ✓

(f) $f(2x-1) = 13 - 8(2x-1)$ ✓

$= 13 - 16x + 8$

$= 21 - 16x$ ✓

(2)(a)(i) $y' = -4\sin 4x$ ✓

(ii) $y' = -7e^{3-7x}$ ✓

(iii) $y' = \frac{5}{5x+2}$ ✓

(iv) $y = \frac{1}{2}x + 2x^{-1}$ ✓

$\therefore y' = \frac{1}{2} - 2x^{-2}$ ✓

$= \frac{1}{2} - \frac{2}{x^2}$

(b)(i) $3e^{\frac{1}{3}x} + c$ ✓

(ii) $\frac{1}{2}\int \frac{4x+6}{2x^2+6x+3}\,dx$ ✓

$= \frac{1}{2}\ln(2x^2+6x+3) + c$ ✓

(c) $-\frac{1}{2}\left[\cos 2x\right]_0^{\frac{\pi}{2}}$ ✓

$= -\frac{1}{2}(\cos\pi - \cos 0)$

$= -\frac{1}{2}(-1-1)$

$= 1$ ✓

(d) $y = \int (3x^2 - 6x)\,dx$

$= x^3 - 3x^2 + c$ ✓

When $x = 0$, $y = -1$.

So $c = -1$,

so $y = x^3 - 3x^2 - 1$. ✓

(3)(a)(i) When $x = 6$,

$y = 2(6) - 4$

$= 8$.

So $B = (6, 8)$.

(ii) $m_{AB} = 2$

$\therefore m_{BC} = -\frac{1}{2}$

(iii) $y - 8 = -\frac{1}{2}(x - 6)$

$2y - 16 = -x + 6$

$\left.\begin{array}{l} x + 2y - 22 = 0 \\ (\text{i.e. } y = -\frac{1}{2}x + 11) \end{array}\right\}$

(iv) $A = (2, 0)$ and $C = (22, 0)$.

So area $= \frac{1}{2} \times AC \times h$

$= \frac{1}{2} \times (22 - 2) \times 8$

$= 80$ square units.

(b) $x^4 - 8x^2 - 48 = 0$

$(x^2 - 12)(x^2 + 4) = 0$

x^2 cannot be negative,

so $x^2 = 12$

$x = \pm 2\sqrt{3}$.

(c)(i) $1 - \sin^2\theta = \cos^2\theta$

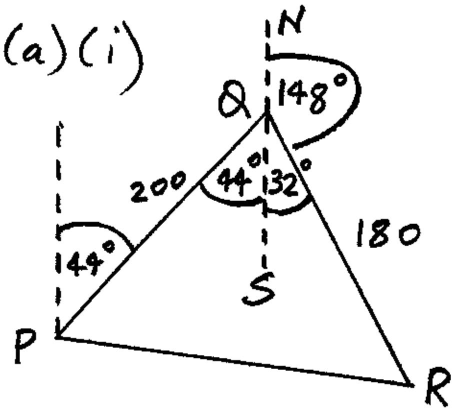
(ii) LHS $= \tan\theta(1 - \sin^2\theta)$

$= \frac{\sin\theta}{\cos\theta} \cdot \cos^2\theta$

$= \sin\theta\cos\theta$

$=$ RHS

(4)(a)(i)



$\angle PQS = 44°$ (alternate $\angle$s on parallel lines)

$\angle SQR = 32°$ (adjacent $\angle$s on a line)

$\therefore \angle PQR = 44° + 32°$

$= 76°$

(ii) $PR^2 = 200^2 + 180^2$

$- 2 \times 200 \times 180 \times \cos 76°$

$= 54981.62\ldots$

$\therefore PR \doteqdot 234.5$ km

(iii) $\frac{\sin P}{180} = \frac{\sin 76°}{234.5}$

$\sin P = \frac{180 \sin 76°}{234.5}$

$P \doteqdot 48°$

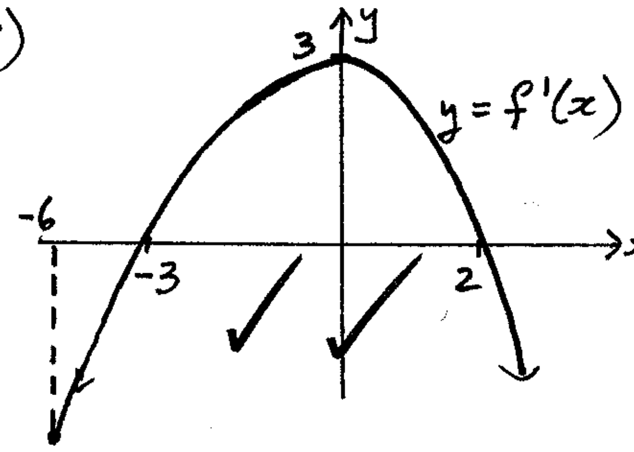
$\left.\begin{array}{l} \text{Now, } 44° + 48° = 92° > 90°, \\ \text{so } R \text{ is south of } P. \end{array}\right\}$

(b)(i) $y \leq 3$

(ii) $-3 < x < 2$

(iii) $-6 < x < 0$ (accept $-6 \leq x < 0$)

(iv)



(5)(a)(i) In Δs ABC and ADC:

$AB = AD$ (given)
$CB = CD$ (given)
AC is common

$\therefore \Delta ABC \equiv \Delta ADC$ (S.S.S.)

$\therefore \angle ACB = \angle ACD$ (matching $\angle$s of congruent Δs)

(ii) In Δs BPC and DPC:

$CB = CD$ (given)
$\angle PCB = \angle PCD$ (proved in (i))
PC is common

$\therefore \Delta BPC \equiv \Delta DPC$ (S.A.S.)

$\therefore \angle BPC = \angle DPC$ (matching $\angle$s of congruent Δs)

(iii) They are both 90°, because they are equal (from (ii)) and supplementary. (Adjacent $\angle$s on a line.)

(b)(i) $T_7 = a + 6d = 49014$ ①
$T_{15} = a + 14d = 77742$ ②

② − ①: $8d = 28728$
$d = 3591$
So her annual pay rise is \$3591.

(ii) Substitute into ①:
$a = 49014 - 6 \times 3591$
$= 27468$
So her first year income was \$27468

(iii) $S_n = \frac{n}{2}(2a + (n-1)d)$
So $S_{18} = 9(54936 + 17 \times 3591)$
$= 1043847$
So she can resign now.

(6)(a)(i) When $t = 0$, $x = 50$

(ii) $v = 4t - 25$
So when $t = 0$, $v = -25\ ms^{-1}$

(iii) When $x = 0$,
$2t^2 - 25t + 50 = 0$
$(2t - 5)(t - 10) = 0$
$t = 2.5$ or 10 seconds

(iv) when $v = 0$,
$4t - 25 = 0$
$t = 6.25$ seconds

(v)

t	2.5	6.25	10
x	0	−28.125	0

So the particle travelled
$2 \times 28.125 = 56.25$ metres

(b)(i) when $\frac{dV}{dt} = 0$,
$20t - 300 = 0$
$t = 15$ days

(ii) $V = \int (20t - 300)\, dt$
$= 10t^2 - 300t + c$
When $t = 15$, $V = 4750$.
So $4750 = 2250 - 4500 + c$,
so $c = 7000$.
So $V = 10t^2 - 300t + 7000$.

(iii) When $t = 0$,
$V = 7000\ L$

(7)(a)(i) When $x=0$, $y=45$, so $OC=45$m.

(ii) $4a = 180$

$a = 45$

So the focus is at O.

(iii) Let $y=0$.

$\therefore x^2 = 180 \times 45$

$= 8100$

$\therefore x = \pm 90$

So the length AB is 180m.

(b)(i) $P = \left(\frac{2\pi}{3}, 0\right)$ Allow $\frac{2\pi}{3}$

(ii) $Q = \left(\frac{\pi}{6}, 4\right)$

(iii) Let $y=2$.

$\therefore \sin 3x = \frac{1}{2}$

$3x = \frac{\pi}{6}$

$x = \frac{\pi}{18}$

So $R = \left(\frac{\pi}{18}, 2\right)$.

$y' = 12\cos 3x$

$\therefore m = 12 \times \cos\frac{\pi}{6}$

$= 12 \times \frac{\sqrt{3}}{2}$

$= 6\sqrt{3}$

The tangent has equation

$y - 2 = 6\sqrt{3}\left(x - \frac{\pi}{18}\right)$

$y = 6\sqrt{3}x + \left(2 - \frac{\pi\sqrt{3}}{3}\right)$

(c) $\log_2(x^2 - x) = 0$

$x^2 - x = 1$

$x^2 - x - 1 = 0$

$\Delta = 5$

So $x = \frac{1 \pm \sqrt{5}}{2}$,

but $x - 1 > 0$, so $x > 1$,

so $x = \frac{1+\sqrt{5}}{2}$.

(8)(a)(i) $\text{LHS} = \frac{(x+1) - 2}{(x+1)^2}$

$= \text{RHS}$

(ii) $\int_0^1 \left(\frac{1}{x+1} - 2(x+1)^{-2}\right)dx$

$= \left[\ln(x+1) - \frac{2(x+1)^{-1}}{-1}\right]_0^1$

$= \left[\ln(x+1) + \frac{2}{x+1}\right]_0^1$

$= \ln 2 + 1 - (\ln 1 + 2)$

$= \ln 2 - 1$

(b)(i) $T_5 = 10000(0\cdot6)^4$

$= 1296$ games

(ii) $S_\infty = \frac{10000}{1 - 0\cdot6}$

$= 25000$ games

(iii) $S_5 = \frac{10000(1 - 0\cdot6^5)}{0\cdot4}$

$= 25000(1 - 0\cdot6^5)$

$= 23056$

So the percentage is

$\left(\frac{23056}{25000} \times \frac{100}{1}\right)\% \doteq 92\cdot2\%$

(c)(i) $a =$ gradient of line

$= \frac{V_1 - V_0}{t_1}$

(ii) Area $= \frac{(V_0 + V_1)t_1}{2}$

(iii) It is the distance travelled by the particle over the time interval $t=0$ to $t=t_1$.

(9)(a)(i)

$$5k^2-20k+24 = 5(k^2-4k)+24$$
$$= 5(k^2-4k+4)+4$$
$$= 5(k-2)^2+4$$
$$\geqslant 4$$
$$> 0 \text{ for all } k$$

(ii) $\Delta = k^2 - 4(k-2)(3-k)$
$$= k^2 + 4(k-2)(k-3)$$
$$= k^2 + 4(k^2-5k+6)$$
$$= 5k^2 - 20k + 24$$
$$> 0 \text{ for all } k \text{ (by (i))}$$

So the equation has 2 distinct roots for all k.

(iii) $k=2$ is the exception, because then the equation becomes $2x+1=0$ which is linear, and only has one root.

(b)(i) Let $f(x) = e^{-x^2}$.
$$\therefore f(-x) = e^{-(-x)^2}$$
$$= e^{-x^2}$$
$$= f(x)$$

$\therefore f(x)$ is even

The graph is symmetrical about the y-axis.

(ii) $y' = -2xe^{-x^2}$

By the product rule,
$$y'' = 4x^2e^{-x^2} - 2e^{-x^2}$$
$$= 2e^{-x^2}(2x^2-1)$$

Let $y''=0$.
$$\therefore x = \pm\frac{1}{\sqrt{2}}$$

So $P = \left(\frac{1}{\sqrt{2}}, e^{-\frac{1}{2}}\right)$ accept this
$$\doteqdot (0.7, 0.6)$$

(iii) LHS $= \frac{d}{dy}(y\ln y - y)$
$$= y\cdot\frac{1}{y} + 1.\ln y - 1$$
$$= \ln y$$
$$= \text{RHS}$$

(iv) $-x^2 = \ln y$, so $x^2 = -\ln y$.
$$V = -\pi\int_{\frac{1}{2}}^{1} \ln y \, dy$$
$$= -\pi\left[y\ln y - y\right]_{\frac{1}{2}}^{1}$$
$$= -\pi\left(1\ln 1 - 1 - \left(\tfrac{1}{2}\ln\tfrac{1}{2} - \tfrac{1}{2}\right)\right)$$
$$= -\pi\left(-\tfrac{1}{2} + \tfrac{1}{2}\ln 2\right)$$
$$= \frac{\pi}{2}(1-\ln 2) \quad u^3$$

(or equivalent)

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(10)(a)(i) $\frac{dQ}{dt} = -k.\, Q_0 e^{-kt}$

$= -kQ$

(ii) When $t = T$, $Q = Q_1$.

$\therefore Q_1 = Q_0 e^{-kT}$

$\frac{Q_1}{Q_0} = e^{-kT}$

$-kT = \ln\left(\frac{Q_1}{Q_0}\right)$

$k = -\frac{1}{T} \ln\left(\frac{Q_1}{Q_0}\right)$

$\left(= \frac{1}{T} \ln\left(\frac{Q_0}{Q_1}\right) \right)$

(iii) When $t = nT$,

$Q = Q_0 e^{-knT}$

$= Q_0 e^{n \ln\left(\frac{Q_1}{Q_0}\right)}$

$= Q_0 e^{\ln\left(\frac{Q_1}{Q_0}\right)^n}$

$= Q_0 . \left(\frac{Q_1}{Q_0}\right)^n$

$= \frac{Q_1^{\,n}}{Q_0^{\,n-1}}$

(b)(i) $I_c = \frac{W}{x^2} + \frac{2W}{(30-x)^2}$

(ii) $\frac{dI_c}{dx} = -\frac{2W}{x^3} + \frac{4W}{(30-x)^3}$

(iii) $\frac{dI_c}{dx} = 0$ when

$\frac{2W}{x^3} = \frac{4W}{(30-x)^3}$

$\frac{1}{x^3} = \frac{2}{(30-x)^3}$

$\left(\frac{30-x}{x}\right)^3 = 2$

$\frac{30}{x} - 1 = \sqrt[3]{2}$

$x = \frac{30}{1+\sqrt[3]{2}}$

$= 13.2748\ldots$

x	13	$\frac{30}{1+\sqrt[3]{2}}$	14
$\frac{dI_c}{dx}$	$-9.6 \times 10^{-5} W$	0	$2.5 \times 10^{-4} W$
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So the distance PL_1 is 13.27m in order to minimise I_c.