SYDNEY GRAMMAR SCHOOL

2011 Trial Examination

FORM VI
MATHEMATICS 2 UNIT

Tuesday 2nd August 2011

General Instructions

- Reading time — 5 minutes
- Writing time — 3 hours
- Write using black or blue pen.
- Board-approved calculators and templates may be used.
- A list of standard integrals is provided at the end of the examination paper.
- All necessary working should be shown in every question.
- Start each question in a new booklet.

Structure of the paper

- Total marks — 120
- All ten questions may be attempted.
- All ten questions are of equal value.

Collection

- Write your candidate number clearly on each booklet.
- Hand in the ten questions in a single well-ordered pile.
- Hand in a booklet for each question, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Place the question paper inside your answer booklet for Question One.

Checklist

- SGS booklets — 10 per boy
- Candidature — 85 boys

Examiner
DS

<u>QUESTION ONE</u> (12 marks) Use a separate writing booklet. **Marks**

(a) Write $\dfrac{1}{3+\sqrt{5}}$ with a rational denominator. $\boxed{2}$

(b) Find the exact value of $\tan \frac{5\pi}{6}$. $\boxed{2}$

(c) Find the gradient of the tangent to the curve $y = x^3$ at the point $\left(\frac{1}{2}, \frac{1}{8}\right)$. $\boxed{2}$

(d) The equation $2x^2 - 5x - 8 = 0$ has roots α and β. Without solving the equation, find the value of:

(i) $\alpha + \beta$ $\boxed{1}$

(ii) $\alpha\beta$ $\boxed{1}$

(e) Find the 50th term of the arithmetic sequence 119, 113, 107, ... $\boxed{2}$

(f) Given that $f(x) = 13 - 8x$, find $f(2x - 1)$ in simplest form. $\boxed{2}$

Exam continues next page ...

QUESTION TWO (12 marks) Use a separate writing booklet. **Marks**

(a) Differentiate:

(i) $y = \cos 4x$ **1**

(ii) $y = e^{3-7x}$ **1**

(iii) $y = \ln(5x + 2)$ **1**

(iv) $y = \dfrac{x}{2} + \dfrac{2}{x}$ **2**

(b) Find:

(i) $\displaystyle\int e^{\frac{1}{3}x}\,dx$ **1**

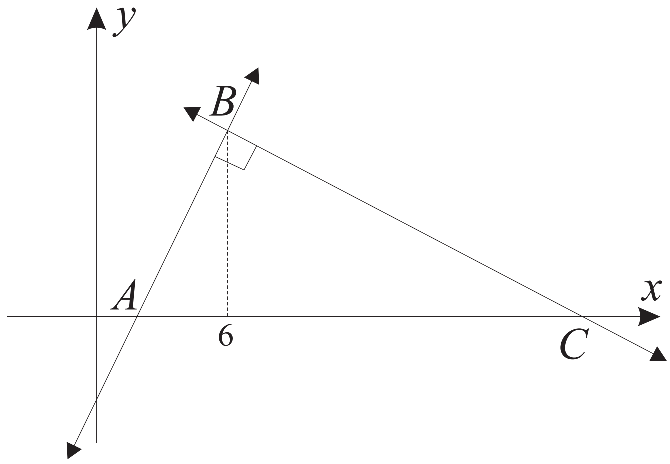
(ii) $\displaystyle\int \frac{2x+3}{2x^2+6x+3}\,dx$ **2**

(c) Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \sin 2x\,dx$. **2**

(d) The curve $y = f(x)$ has gradient function $\dfrac{dy}{dx} = 3x^2 - 6x$. Find the equation of the curve given that its y-intercept is -1. **2**

QUESTION THREE (12 marks) Use a separate writing booklet. **Marks**

(a)



The diagram above shows a right-angled triangle ABC in the number plane. The point B has $x-$coordinate 6, and the side AB has equation $2x - y - 4 = 0$.

(i) Find the $y-$coordinate of B. **1**

(ii) Find the gradient of BC. **2**

(iii) Find the equation of BC. **2**

(iv) Find the area of triangle ABC. **2**

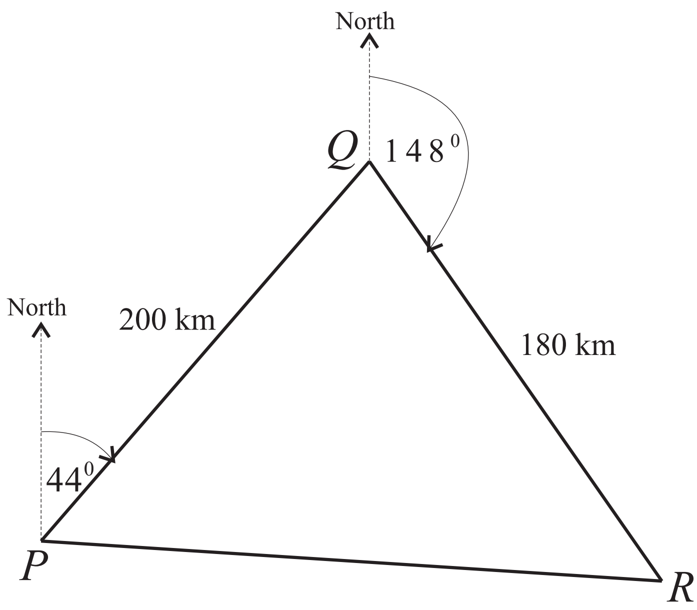
(b) Solve the equation $x^4 - 8x^2 - 48 = 0$. **2**

(c) (i) Simplify $1 - \sin^2\theta$. **1**

(ii) Prove the identity $\tan\theta\left(1 - \sin^2\theta\right) = \sin\theta\cos\theta$. **2**

QUESTION FOUR (12 marks) Use a separate writing booklet. **Marks**

(a)



A boat sails 200 km from P to Q on a bearing of 044°. It then sails 180 km from Q to R on a bearing of 148°.

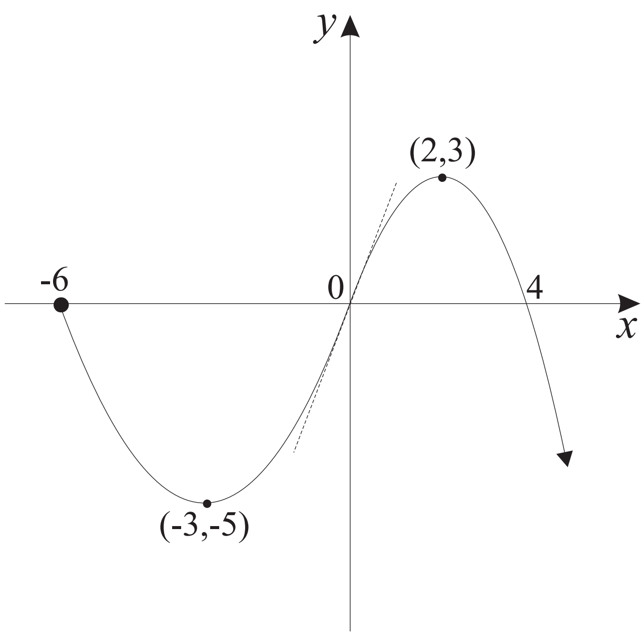
(i) Explain why $\angle PQR = 76°$. **2**

(ii) Use the cosine rule to find the distance of R from P in kilometres correct to one decimal place. **2**

(iii) Use the sine rule to find $\angle RPQ$, and hence determine whether R is north of P or, as the diagram suggests, south of P. **3**

QUESTION FOUR (Continued)

(b)



The function $y = f(x)$, with domain $x \geq -6$, is graphed in the diagram above. The points $(-3, -5)$ and $(2, 3)$ are stationary points, and the origin is a point of inflexion.

(i) What is the range of $f(x)$? **1**

(ii) For what values of x is $f'(x) > 0$? **1**

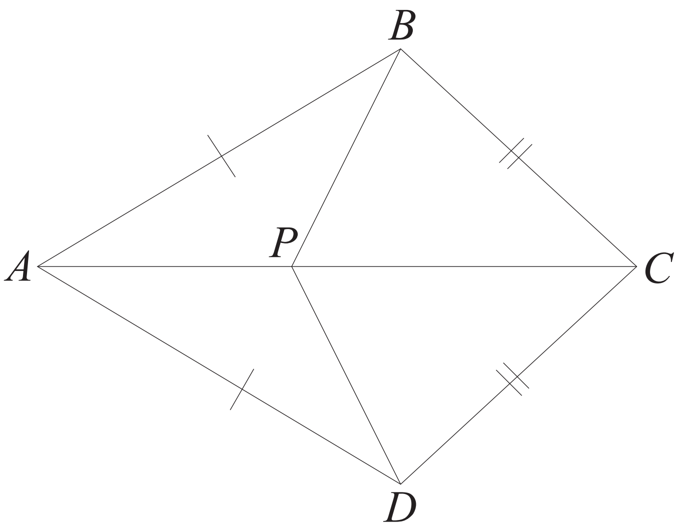
(iii) For what values of x is $f''(x) > 0$? **1**

(iv) Given that the tangent at the origin has gradient 3, sketch the curve $y = f'(x)$. **2**

Exam continues next page ...

QUESTION FIVE (12 marks) Use a separate writing booklet. **Marks**

(a)



The diagram above shows a quadrilateral $ABCD$ in which $AB = AD$ and $CB = CD$. The point P lies on the diagonal AC.

(i) Use congruent triangles to prove that $\angle ACB = \angle ACD$. **3**

(ii) Use congruent triangles to prove that $\angle BPC = \angle DPC$. **3**

(iii) What can be said about the angles BPC and DPC in the special case where the points B, P and D are collinear? Give reasons for your answer. **1**

(b) Maria has worked for Joe King Motors for the past 18 years. Her annual income has increased each year by the same fixed amount. In her 7th year her income was \$49 014, and in her 15th year her income was \$77 742.

(i) Find her annual pay rise. **2**

(ii) Find her income in her first year. **1**

(iii) Maria recently decided that she would resign from her job when her total earnings have exceeded one million dollars. Can she resign yet? **2**

QUESTION SIX (12 marks) Use a separate writing booklet. **Marks**

(a) A particle is moving on the x-axis with displacement x metres after t seconds given by the function

$$x = 2t^2 - 25t + 50.$$

(i) What was the initial position of the particle? **1**

(ii) What was the inital velocity of the particle? **1**

(iii) At what times was the particle at the origin? **2**

(iv) At what time was the particle instantaneously at rest? **1**

(v) How far did the particle travel in between its visits to the origin? **2**

(b) Water started leaking out of a tank. The rate of change of V, the volume of water in the tank t days after the leak started, is given by $\dfrac{dV}{dt} = 20t - 300$ litres per day. When the tank stopped leaking, it still had 4750 L of water in it.

(i) For how many days was the tank leaking? **1**

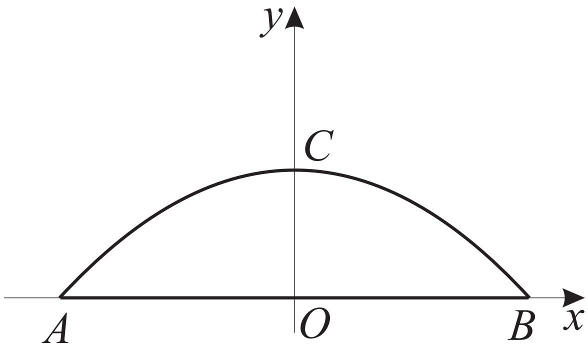
(ii) Find a formula for V. **3**

(iii) How much water was in the tank when it started leaking? **1**

Exam continues next page ...

QUESTION SEVEN (12 marks) Use a separate writing booklet. **Marks**

(a)



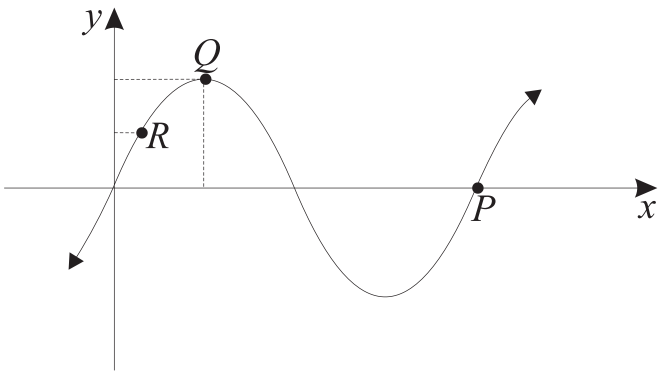
The diagram above shows the side view of a bridge. The level roadway AB lies on the x-axis, and the steel arch ACB is a parabolic arc with equation $x^2 = -180(y - 45)$. On both the x and y axes, one unit represents one metre.

(i) Find the height OC of the highest point C on the steel arch above the roadway. **1**

(ii) Show that the focus of the parabolic steel arch lies on the roadway. **1**

(iii) Find the length of the roadway AB. **2**

(b)



The diagram above shows the graph of the function $y = 4\sin 3x$, where x is measured in radians.

(i) Find the coordinates of the point P. **1**

(ii) Find the coordinates of the point Q. **1**

(iii) Find the equation of the tangent to the curve at the point R, which is half as high as Q is above the x-axis. **3**

(c) Solve the equation $\log_2 x + \log_2(x - 1) = 0$. **3**

Exam continues overleaf ...

QUESTION EIGHT (12 marks) Use a separate writing booklet. **Marks**

(a) (i) Show that $\dfrac{1}{x+1} - \dfrac{2}{(x+1)^2} = \dfrac{x-1}{(x+1)^2}$. **1**

(ii) Hence evaluate $\displaystyle\int_0^1 \frac{x-1}{(x+1)^2}\,dx$. **3**

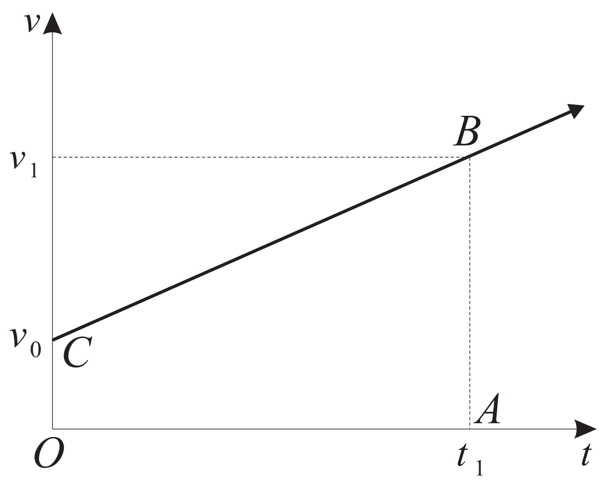
(b) Michael has been selling video games for a while now. He sold 10 000 games in the first month of business, but has found that sales in each subsequent month are only 60% of the sales in the previous month.

(i) How many games did Michael sell in the 5th month? **1**

(ii) Show that Michael will never sell more that 25 000 games. **1**

(iii) What percentage of his eventual sales did Michael make in the first five months? Give your answer correct to the nearest tenth of a percent. **3**

(c)



A particle is initially at the origin and moves in a straight line with constant acceleration. Its velocity-time graph is shown above.

(i) Find the acceleration of the particle in terms of v_0, v_1 and t_1. **1**

(ii) Find the area of the trapezium $OABC$ in terms of v_0, v_1 and t_1. **1**

(iii) What is the physical significance of the area found in part (ii)? **1**

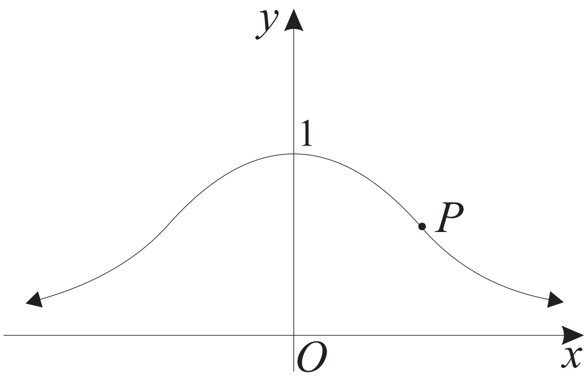
QUESTION NINE (12 marks) Use a separate writing booklet. **Marks**

(a) (i) By completing the square, or otherwise, show that the expression $5k^2 - 20k + 24$ is positive for all values of k. **2**

(ii) Hence show that the equation $(k-2)x^2 + kx + (3-k) = 0$ has two distinct solutions for all values of k. **2**

(iii) In actual fact there is a value of k for which the equation in part (ii) only has one solution. What is this value, and why is it an exception? **1**

(b)



The diagram above shows the graph of the function $y = e^{-x^2}$.

(i) Show that the function $y = e^{-x^2}$ is even. What symmetry is exhibited by the graph of an even function? **1**

(ii) Find, correct to one decimal place, the coordinates of the point of inflexion P in the first quadrant. **3**

(iii) Show that $\dfrac{d}{dy}(y \ln y - y) = \ln y$. **1**

(iv) Find the exact volume of the solid formed when the region in the first quadrant bounded by the curve $y = e^{-x^2}$, the y-axis and the horizontal line $y = \frac{1}{2}$ is rotated about the y-axis. **2**

Exam continues overleaf ...

QUESTION TEN (12 marks) Use a separate writing booklet. **Marks**

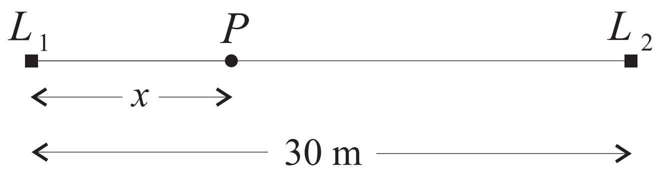
(a) A quantity Q of a radioactive substance decays at a rate that at any time t is proportional to the amount remaining at that time. That is, $\dfrac{dQ}{dt} = -kQ$, where k is a positive constant. Suppose that Q_0 is the initial amount of the substance, and that at time $t = T$, the amount is Q_1.

(i) Show that $Q = Q_0 e^{-kt}$ satisfies the differential equation $\dfrac{dQ}{dt} = -kQ$. **1**

(ii) Find the constant k in terms of T, Q_0 and Q_1. **2**

(iii) Hence show that the amount of the substance at time nT, where n is positive, is $\dfrac{{Q_1}^n}{{Q_0}^{n-1}}$. **2**

(b)



The intensity I produced by a light of power W at a distance x metres from the light is given by $I = \dfrac{W}{x^2}$. Two lights L_1 and L_2, of power W and $2W$ respectively, are positioned 30 metres apart.

(i) Write down an expression for the combined intensity I_c of L_1 and L_2 at a point P which is x metres from L_1, as shown in the diagram. **1**

(ii) Find $\dfrac{dI_c}{dx}$. **2**

(iii) Find the distance PL_1, correct to the nearest centimetre, so that the combined intensity of L_1 and L_2 is at its minimum. **4**

END OF EXAMINATION

B L A N K   P A G E

The following list of standard integrals may be used:

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \;\; n \neq -1; \;\; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \;\; x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \;\; a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \;\; a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \;\; a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \;\; a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \;\; a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \;\; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \;\; a > 0, \;\; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln \left(x + \sqrt{x^2 - a^2} \right), \;\; x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln \left(x + \sqrt{x^2 + a^2} \right)$$

$$\text{NOTE}: \;\; \ln x = \log_e x, \;\; x > 0$$