2011 TRIAL HSC EXT 1 MATHEMATICS - SOLUTIONS

QUESTION 1.

a) $\lim\limits_{x \to 0} \dfrac{\tan 3x}{2x} = \dfrac{3}{2} \lim\limits_{x \to 0} \dfrac{\tan 3x}{3x}$

$= \dfrac{3}{2} \times 1$

$= \dfrac{3}{2}$

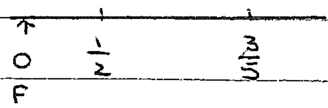
b) $\dfrac{x}{2x-1} \geq 3 \quad , \quad x \neq \dfrac{1}{2}$

$x(2x-1) \geq 3(2x-1)^2$

$3(2x-1)^2 - x(2x-1) \leq 0$

$(2x-1)(6x-3-x) \leq 0$

$(2x-1)(5x-3) \leq 0$



$\therefore \quad \dfrac{1}{2} < x \leq \dfrac{3}{5}$

c) (i) $\alpha\beta + \alpha\gamma + \beta\gamma = \dfrac{c}{a}$

$= -\dfrac{3}{4}$

(ii) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma} = \dfrac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$ $\qquad \alpha\beta\gamma = \dfrac{-d}{a}$

$= -2$

$= -\dfrac{3}{4} \div -2$

$= \dfrac{3}{8}$

d) $\log_5 10 = 2 \cdot 48$

$\log_5 (2 \times 5) = 2 \cdot 48$

$\log_5 2 + \log_5 5 = 2 \cdot 48$

$\log_5 2 = 1 \cdot 48$

$2 \log_5 2 = 2 \cdot 96$

$\therefore \log_5 4 = 2 \cdot 96$

e) $$\int_{2}^{6} \frac{x}{\sqrt{x^2-3}} \, dx$$

$u = x^2 - 3$

$du = 2x \, dx$

$x = 2, \quad u = 1$

$x = 6, \quad u = 33$

$$= \frac{1}{2} \int_{1}^{33} u^{-1/2} \, du$$

$$= \frac{1}{2} \cdot 2 \left[u^{\frac{1}{2}} \right]_{1}^{33}$$

$$= \sqrt{33} - 1$$

QUESTION 2

a) (i) $LHS = \dfrac{1+\cos 2\theta}{\sin 2\theta}$

$$= \frac{1 + 2\cos^2\theta - 1}{2\sin\theta\cos\theta}$$

$$= \frac{\cancel{2}\cos^2\theta}{\cancel{2}\sin\theta\cos\theta}$$

$$= \frac{\cos\theta}{\sin\theta}$$

$$= \cot\theta$$

$$= RHS$$

(ii) $\cot\theta = \left(1 + \cos 2\theta\right) \div \sin 2\theta$

$$\cot\frac{\pi}{12} = \left(1 + \cos\frac{\pi}{6}\right) \div \sin\frac{\pi}{6}$$

$$= \left(1 + \frac{\sqrt{3}}{2}\right) \div \frac{1}{2}$$

$$= 2 + \sqrt{3}$$

b) $P(-4) = 0$ ie $-64 + 16a - 4b + 8 = 0$

$$16a - 4b = 56$$

$$4a - b = 14 \quad \text{——} \quad (1)$$

$P(-1) = 18$ ie $-1 + a - b + 8 = 18$

$$a - b = 11 \quad \text{——} \quad (2)$$

(1) − (2) $\quad 3a = 3$

$$\therefore \quad a = 1, \quad b = -10$$

c) $\displaystyle\int_{\sqrt{2}}^{\sqrt{6}} \frac{dx}{6+3x^2} = \frac{1}{3}\int_{\sqrt{2}}^{\sqrt{6}} \frac{dx}{2+x^2}$

$a^2 = 2$

$a = \sqrt{2}$

$$= \frac{1}{3}\cdot\frac{1}{\sqrt{2}}\left[\tan^{-1}\frac{x}{\sqrt{2}}\right]_{\sqrt{2}}^{\sqrt{6}}$$

$$= \frac{1}{3\sqrt{2}}\left(\tan^{-1}\sqrt{3} - \tan^{-1} 1\right)$$

$$= \frac{1}{3\sqrt{2}}\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$

$$= \frac{\pi}{36\sqrt{2}}$$

d) $\angle COB = 80^\circ$ (angle at the centre is twice the angle at the circumference, subtended by the same arc)

ΔOCB is isosceles as $OC = OB$ (radii)

$\therefore$ $\angle OBC = 50^\circ$ (base angle in isosceles triangle)

QUESTION 3

a) Let $f(x) = 3\sin x - 2x$ $x_1 = 1\cdot 56$

$f'(x) = 3\cos x - 2$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 1\cdot 56 - \left(\frac{3\sin 1\cdot 56 - 2\times 1\cdot 56}{3\cos 1\cdot 56 - 2}\right)$$

$$\doteqdot 1\cdot 4989 \ldots\ldots$$

$\therefore$ a second approx^n to the root is $1\cdot 50$, to 2 dec. pl.

b) $T_{k+1} = {}^nC_k \, a^{n-k} . b^k$

$$= {}^{10}C_k \, (x^2)^{10-k} . (-1)^k . (x^{-3})^k$$

$20 - 2k - 3k = 0$

$k = 4$

$T_{k+1} = {}^{10}C_4 \, (-1)^4$

$\therefore$ the independent term is 210

c) If $f(x)$ is monotonically increasing for all x, then $f'(x) > 0$ for all values of x

$f(x) = \dfrac{e^x}{4 - e^x}$ $u = e^x$ $v = 4 - e^x$

$u' = e^x$ $v' = -e^x$

$$f'(x) = \frac{(4 - e^x)e^x + e^x . e^x}{(4 - e^x)^2}$$

$$= \frac{4e^x}{(4 - e^x)^2}$$

CONT.

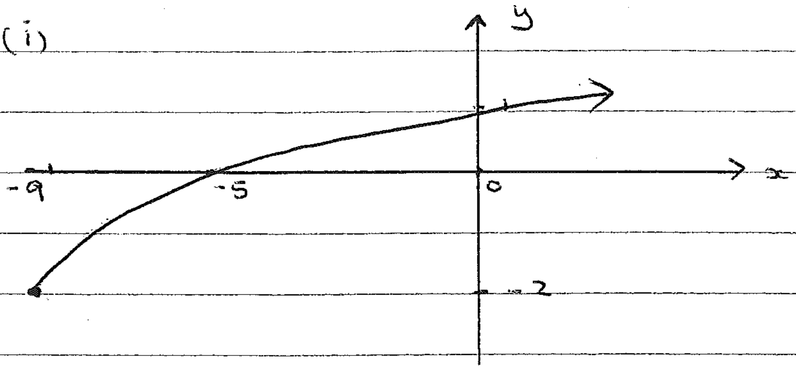
Since $e^{x} > 0$ for all x

and $(4 - e^{x})^{2} > 0$ for all x

then $f'(x) > 0$

and the function is increasing over the domainof x

d) (i)



(ii) $x \geq -9$

(iii) $x = y^2 + 4y - 5$

$x + 5 = y^2 + 4y$

$x + 5 + 4 = y^2 + 4y + 4$

$x + 9 = (y + 2)^2$

$y + 2 = \pm\sqrt{(x + 9)}$

$\therefore$ $y = -2 + \sqrt{x+9}$ is the inverse fn

QUESTION 4

a) $\int \sin\theta \cos^2\theta \, d\theta$

$u = \cos\theta$

$du = -\sin\theta \, d\theta$

$= -\int u^2 \, du$

$= -\frac{1}{3} u^3 + c$

$= -\frac{1}{3} \cos^3\theta + c$

b) $\sin\left[\tan^{-1}(-\sqrt{3})\right] = \sin\left(-\frac{\pi}{3}\right)$

$= -\frac{\sqrt{3}}{2}$

c) $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$

$12 \cdot 6 = 4\pi \times 12^2 \times \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{12 \cdot 6}{4\pi \times 12^2}$

$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$

$A = 4\pi r^2$

$\frac{dA}{dr} = 8\pi r$

$= 8\pi \times 12 \times \frac{12 \cdot 6}{4\pi \times 12^2}$

$\therefore$ surface area is increasing at the rate of $2 \cdot 1 \text{ cm}^2/\text{s}$

d) over page

d) (i) when $x = 0$, $py = 2ap + ap^3$

$y = 2a + ap^2$

$\therefore$ Q is the point $(0, 2a + ap^2)$

(ii) $P(\overset{x_1}{2ap}, \overset{y_1}{ap^2})$ $Q(\overset{x_2}{0}, \overset{y_2}{2a + ap^2})$ $\overset{m}{-1} : \overset{n}{2}$

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right) = \left(\frac{0 + 4ap}{1}, \frac{-2a - ap^2 + 2ap^2}{1}\right)$$

$\therefore$ R is the point $(4ap, ap^2 - 2a)$

(iii) $x = 4ap$ — (1)

$y = ap^2 - 2a$ — (2)

from (1) $p = \dfrac{x}{4a}$

sub in (2) $y = a \cdot \dfrac{x^2}{16a^2} - 2a$

$y + 2a = \dfrac{x^2}{16a}$

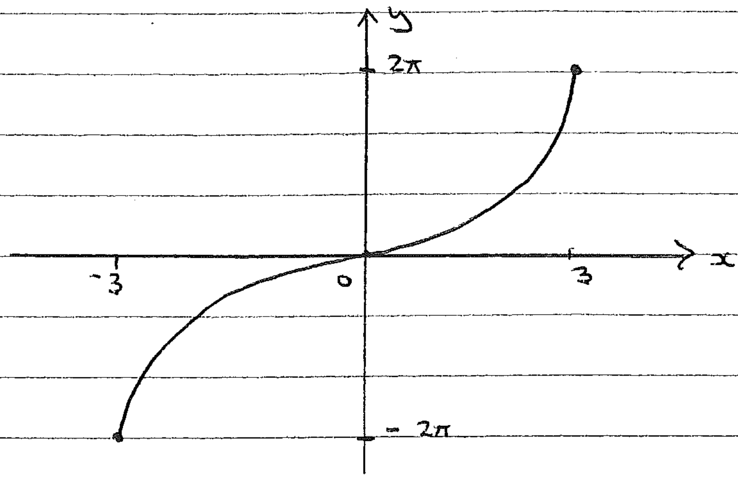
$\therefore$ $x^2 = 16a(y + 2a)$ is the eqn of the locus of R

QUESTION 5

a) (i) domain is $-3 \leq x \leq 3$

range is $-2\pi \leq y \leq 2\pi$

(ii)



(iii) $y = 4\sin^{-1}\left(\frac{x}{3}\right)$

$\frac{dy}{dx} = 4 \cdot \frac{1}{\sqrt{9 - x^2}}$

when $x = \sqrt{5}$, $\frac{dy}{dx} = 4 \cdot \frac{1}{\sqrt{9-5}}$

$\therefore$ gradient of the tangent is 2

b) $V = \pi \int_0^{\pi/4} \sin^2 2x \, dx$

$\cos 2\theta = 1 - 2\sin^2\theta$

$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$

$\sin^2 2x = \frac{1}{2}(1 - \cos 4x)$

$= \frac{\pi}{2} \int_0^{\pi/4} (1 - \cos 4x) \, dx$

$= \frac{\pi}{2} \left[x - \frac{1}{4}\sin 4x \right]_0^{\pi/4}$

$= \frac{\pi}{2} \left[\frac{\pi}{4} - \frac{1}{4}\sin\pi - (0 - 0) \right]$

$\therefore$ the volume is $\frac{\pi^2}{8}$ cubic units

c) (i) $N = 800 + Ae^{kt}$ ie $Ae^{kt} = N - 800$

$\frac{dN}{dt} = k . Ae^{kt}$

ie/ $\frac{dN}{dt} = k(N - 800)$

(ii) $N = 800 + Ae^{kt}$

$t = 0, N = 1000$ so $1000 = 800 + A$

$A = 200$

$N = 800 + 200e^{kt}$

$t = 5, N = 1700$ $1700 = 800 + 200e^{5k}$

$\frac{900}{200} = e^{5k}$

$5k = \log_e 4.5$

$k = \frac{1}{5} \log_e 4.5$

when $t = 10$ $N = 800 + 200e^{2 \log_e 4.5}$

$= 800 + 200(4.5)^2$

$= 4850$

$\therefore$ there are 4850 bacteria after 10 s.

QUESTION 6

a) (i) $\sqrt{3}\sin\theta - \cos\theta = R\sin\theta\cos\alpha - R\cos\theta\sin\alpha$

$R\cos\alpha = \sqrt{3}$ — ①

$R\sin\alpha = 1$ — ②

② ÷ ① $\tan\alpha = 1/\sqrt{3}$

$\alpha = \pi/6$

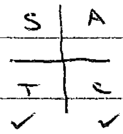
$①^2 + ②^2$ $R^2 = 4$

$R = 2$

$\therefore \sqrt{3}\sin\theta - \cos\theta = 2\sin\left(\theta - \frac{\pi}{6}\right)$

(ii) $2\sin\left(\theta - \frac{\pi}{6}\right) = -1$ for $-\frac{\pi}{6} \leq \theta - \frac{\pi}{6} \leq 2\pi - \frac{\pi}{6}$

$\sin\left(\theta - \frac{\pi}{6}\right) = -\frac{1}{2}$



$\theta - \frac{\pi}{6} = -\frac{\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

$\therefore \theta = 0, \frac{4\pi}{3}, 2\pi$

b) $(3a - 2b)^4$

$= (3a)^4 + 4(3a)^3(-2b) + 6(3a)^2(-2b)^2 + 4(3a)(-2b)^3 + (-2b)^4$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

$= 81a^4 - 216a^3b + 216a^2b^2 - 96ab^3 + 16b^4$

c) $\int_6^{10} \frac{dx}{\sqrt{x^2 - 36}} = \left[\ln\left(x + \sqrt{x^2 - 36}\right)\right]_6^{10}$

$= \ln\left(10 + \sqrt{100 - 36}\right) - \ln 6$

$= \ln 18 - \ln 6$

$= \ln\left(18/6\right)$

$= \ln 3$

d) Step 1 Prove true when $n = 1$

$$LHS = \frac{1}{1 \times 3} = \frac{1}{3} \qquad RHS = \frac{1}{2+1} = \frac{1}{3}$$

$\therefore$ result is true when $n = 1$

Step 2 Assume true when $n = k$

ie/ $S_k = \dfrac{k}{2k+1}$

Prove true when $n = k+1$

ie/ $S_{k+1} = \dfrac{k+1}{2k+3}$

Proof:

$$\begin{aligned} S_{k+1} &= S_k + T_{k+1} \\ &= \frac{k}{(2k+1)} + \frac{1}{(2k+1)(2k+3)} \\ &= \frac{k(2k+3)+1}{(2k+1)(2k+3)} \\ &= \frac{2k^2+3k+1}{(2k+1)(2k+3)} \\ &= \frac{\cancel{(2k+1)}(k+1)}{\cancel{(2k+1)}(2k+3)} \end{aligned}$$

$$\therefore S_{k+1} = \frac{k+1}{2k+3}$$

Step 3 Since the result is true when $n = 1$, it is also true when $n = 1+1$ ie $n = 2$

Since result is true when $n = 2$, it is also true when $n = 2+1$ ie $n = 3$ etc

$\therefore$ result is true for all positive integers.

QUESTION 7

a) $$\ddot{x} = 4x(x^2-1)$$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 4x^3 - 4x$$

$$\frac{1}{2}v^2 = x^4 - 2x^2 + c$$

$x=0,\ v=\sqrt{2}$

$$\frac{1}{2}\cdot 2 = \cancel{4} - 2\cdot 2 + c \quad \Rightarrow c = 1$$

$$\frac{1}{2}v^2 = x^4 - 2x^2 + 1$$

$$\frac{1}{2}v^2 = (x^2-1)^2$$

$$v^2 = 2(x^2-1)^2$$

$$v = \pm\sqrt{2}(x^2-1)$$

when $x=0,\ v=\sqrt{2}$

$$\therefore v = -\sqrt{2}(x^2-1)$$

b) (i) $$\frac{T_{k+1}}{T_k} = \binom{8}{k} x^{8-k}.\ 3^k.\ x^{-2k} \div \left[\binom{8}{k-1} x^{9-k}.\ 3^{k-1}.\ x^{-2k+2}\right]$$

$$= \frac{\cancel{8!}}{k!.(8-k)!} \times \frac{(k-1)!.(9-k)!}{\cancel{8!}} \times 3 \times x^{-3}$$

$$= \frac{9-k}{k}.\frac{3}{x^3}$$

(ii) $$\frac{T_{k+1}}{T_k} > 1$$

$$\frac{3(9-k)}{k} > 1$$

$$27 - 3k > k$$

$$4k < 27$$

$$k < 6\,^2/_4$$

$$\therefore k = 6$$

$$\text{co-efficient} = \binom{8}{6}.3^6$$

$\therefore$ greatest co-efficient is 20 412

c) (i) $v = 0$ at the end points of the motion

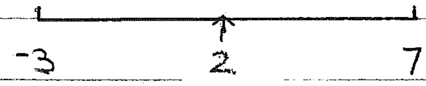
$x^2 - 4x - 21 = 0$

$(x-7)(x+3) = 0$

$x = 7, -3$

$\therefore$ particle is oscillating between $x = -3$ and $x = 7$

(ii) Centre is at $x = 2$



(iii) Amplitude is 5 m

(iv) Max speed occurs at the centre of the motion

when $x = 2$, $v^2 = 21 + 8 - 4$

$v^2 = 25$

$\therefore$ max speed is 5 m/s