2011 Ext 2 Trial – Solution to Question 1

(a)

$$\int_0^1 x\left(5x^2-2\right)^4 dx$$

let $u=5x^2-2, \frac{du}{dx}=10x \Rightarrow \frac{du}{10}=xdx$

when $x=0, u=-2$ & $x=1, u=3$

$$\therefore I=\int_{-2}^{3}\frac{u^4}{10}du=\left[\frac{u^5}{50}\right]_{-2}^{3}=\frac{3^5-(-2)^5}{50}$$

$$=\frac{243+32}{50}=\frac{11}{2} \text{ or } 5.5$$

(b) $\int \cot x dx=\int \frac{\cos x}{\sin x}dx=\ln(\sin x)+C$

(c)

$$\int \frac{1}{x\left(x^2-1\right)}dx$$

$$\frac{1}{x\left(x^2-1\right)}=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{x-1}$$

$$1=A\left(x^2-1\right)+Bx(x-1)+Cx(x+1)$$

$x=1 \Rightarrow 1=2C \qquad C=\frac{1}{2}$

$x=0 \Rightarrow 1=-A \; A=-1$

$x=-1 \Rightarrow 1=2B \qquad B=\frac{1}{2}$

$$I=\int\left(\frac{-1}{x}+\frac{1}{2}\left(\frac{1}{x+1}+\frac{1}{x-1}\right)\right)dx$$

$$\therefore I=-\ln x+\frac{1}{2}\left(\ln(x+1)+\ln(x-1)\right)+C$$

(d)

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\frac{dx}{1-\cos x}$$

$t=\tan\frac{x}{2}, dx=\frac{2dt}{1+t^2}, \cos x=\frac{1-t^2}{1+t^2}$

$$\therefore I=\int_{\tan\left(\frac{\pi}{6}\right)}^{\tan\left(\frac{\pi}{4}\right)}\frac{\frac{2dt}{1+t^2}}{1-\frac{1-t^2}{1+t^2}}=\int_{\frac{1}{\sqrt{3}}}^{1}\frac{2dt}{1+t^2-\left(1-t^2\right)}$$

$$=\int_{\frac{1}{\sqrt{3}}}^{1}\frac{2dt}{2t^2}=\int_{\frac{1}{\sqrt{3}}}^{1}t^{-2}dt=\left[\frac{t^{-1}}{-1}\right]_{\frac{1}{\sqrt{3}}}^{1}$$

$$=\left(-\frac{1}{1}\right)-\left(-\frac{1}{\frac{1}{\sqrt{3}}}\right)=\sqrt{3}-1$$

(e)

$$\int_e^{e^2}\log_e x\,dx$$

$U=\log_e x, V'=1$

$U'=\frac{1}{x}, V=x$

$I=x\log_e x-\int dx$

$=x\log_e x-x$

$$\int_e^{e^2}\log_e x\,dx=\left[x\log_e x-x\right]_e^{e^2}$$

$$=e^2\log_e e^2-e^2-\left(e\log_e e-e\right)$$

$$=2e^2-e^2-e+e=e^2$$

2011 Ext 2 Trial – Solution to Question 2

(a)(i) $z=3+2i \qquad \bar{z}=3-2i$

(a)(ii) $\frac{1}{z}=\frac{\bar{z}}{z\bar{z}}=\frac{3-2i}{9-4i^2}=\frac{3}{13}-\frac{2}{13}i$

(a)(iii) $z^{-2}=\left(\frac{1}{z}\right)^2=\left(\frac{3-2i}{13}\right)^2=\frac{9-12i+4i^2}{169}=\frac{5}{169}-\frac{12}{169}i$

(b)(i) $1-\sqrt{3}i \qquad \theta=-\tan^{-1}\left(\frac{\sqrt{3}}{1}\right)=-\frac{\pi}{3} \qquad r^2=1^2+\left(\sqrt{3}\right)^2 \Rightarrow r=2$

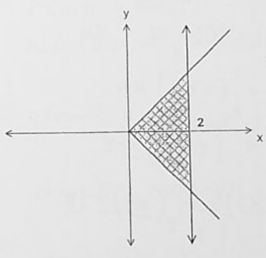
$\therefore 1-\sqrt{3}i=2cis\left(-\frac{\pi}{3}\right)$

(b)(ii) $\left(1-\sqrt{3}i\right)^5=\left[2cis\left(-\frac{\pi}{3}\right)\right]^5=32cis\left(-\frac{5\pi}{3}\right)$

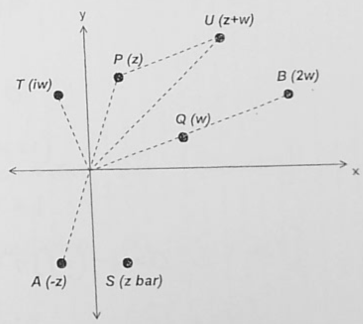
$=32cis\left(\frac{\pi}{3}\right)=32\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=16+16\sqrt{3}i$

(c)

$z\le 2 \quad -\frac{\pi}{4}\le \arg(z)\le\frac{\pi}{4}$

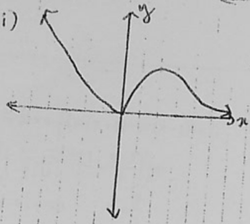


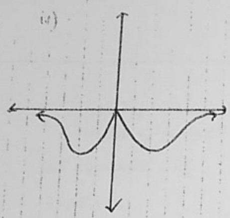
(d)



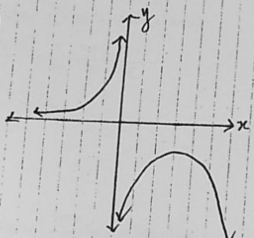
Ext 2 Trial 2018 – Question 3.

a) i)

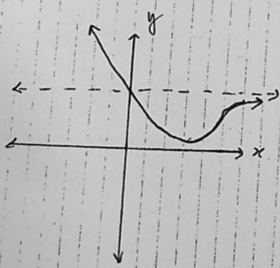




ii)



iii)



b) $x^2 + xy + y^2 = 12$

$2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$

$2x + y + \frac{dy}{dx}(x + 2y) = 0$

$\frac{dy}{dx} = \frac{-2x - y}{x + 2y}$

when tangent is horizontal

$\frac{dy}{dx} = 0$

$0 = -2x - y$

$y = -2x$ ①

$x^2 + xy + y^2 = 12$ ②

sub ① into ②

$x^2 + x(-2x) + (-2x)^2 = 12$

$x^2 = 4$

$x = \pm 2$

when $x = 2$, $y = -4$

$x = -2$, $y = 4$

$\therefore$ tangent horiz at $(2,-4)$ $(-2,4)$

c) i) $2x^3 - 3x^2 + 4x - 1$

$2\left(\frac{x}{2}\right)^3 - 3\left(\frac{x}{2}\right)^2 + 4\left(\frac{x}{2}\right) - 1 = 0$

$x^3 - 3x^2 + 8x - 4 = 0$

ii) $2(\sqrt{x})^3 - 3(\sqrt{x})^2 + 4(\sqrt{x}) - 1 = 0$

$2\sqrt{x}(x+2) = 3x + 1$

$4x(x+2)^2 = (3x+1)^2$

$4x^3 + 7x^2 + 10x - 1 = 0$

Question 4

a) $V_{shell} = \pi(R^2 - r^2)h$

$= \pi\left[(6-x)^2 - (6-(x+\delta x))^2\right]y$

$= \pi(12 - 2x - \delta x)\,\delta x\, y$

$= 2\pi y(6-x)\,\delta x$ (as $\delta x^2 \approx 0$)

$V_{solid} = \int_0^3 2\pi(6-x)(6x + x^2 - x^3)\,dx$

$= 2\pi\int_0^3 x^4 - 7x^3 + 36x\,dx$

$= 2\pi\left[\frac{x^5}{5} - \frac{7x^4}{4} + 18x^2\right]_0^3$

$= \frac{1377\pi}{10}$ units^3

b) i) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\frac{2x}{a^2} + \frac{2y}{b^2} \cdot \frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{b^2 x}{a^2 y}$

at (x_1, y_1)

$\frac{dy}{dx} = \frac{-b^2 x_1}{a^2 y_1}$

$\therefore \quad y - y_1 = \frac{-b^2 x_1}{a^2 y_1}(x - x_1)$

$a^2 y y_1 + b^2 x x_1 = b^2 x_1^2 + a^2 y_1^2$

$\frac{y y_1}{b^2} + \frac{x x_1}{a^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}$

$= 1$ (since (x_1, y_1) lies on $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$)

ii) $\frac{x}{4} + \frac{3\sqrt{3}y}{9 \times 2} = 1$

$\frac{x}{4} + \frac{\sqrt{3}y}{6} = 1$

iii) By symmetry of the ellipce, tangent passes through $(-1, -\frac{3\sqrt{3}}{2})$

$-\frac{x}{4} - \frac{3\sqrt{3}y}{18} = 1$

$-\frac{x}{4} - \frac{\sqrt{3}y}{6} = 1$

iv) Chord passes through the origin :

$m = \frac{\frac{3\sqrt{3}}{2} + \frac{3\sqrt{3}}{2}}{2}$

$= \frac{3\sqrt{3}}{2}$

$y = \frac{3\sqrt{3}}{2} x$

i) Horizontally:

$N \sin\theta - F\cos\theta = \frac{mv^2}{r}$

$N \sin\theta = \frac{mv^2}{r} + F\cos\theta$ ①

Vertically:

$F\sin\theta + N\cos\theta = mg$

$N\cos\theta = mg - F\sin\theta$ ②

ii) $N\sin^2\theta = \frac{mv^2}{r}\sin\theta + F\cos\theta\sin\theta$ ③ [① × sinθ]

$N\cos^2\theta = mg\cos\theta - F\cos\theta\sin\theta$ ④ [② × cosθ]

$N = \frac{mv^2}{r}\sin\theta + mg\cos\theta$ (③ + ④)

$= \frac{mv^2}{r}\sin\theta + \frac{mgr\cot\theta\sin\theta}{r}$

$= \frac{m(v^2 + gr\cot\theta)\sin\theta}{r}$

1(a)(i) $W\hat{B}Z = W\hat{A}D$ (given)

$\therefore WBZA$ is concyclic

(exterior $\angle$ of quadrilateral equals interior opposite $\angle$)

$W\hat{B}A = W\hat{Z}A$ ($\angle$s in same segment)

(ii) $X\hat{C}W = X\hat{B}W$ (given)

$\therefore WXBC$ is concyclic

(WX subtends two equal $\angle$s)

$W\hat{B}C + W\hat{X}C = 180°$ (opposite $\angle$s of a cyclic quadrilateral)

(iii) $W\hat{Z}A = W\hat{X}C$ (exterior $\angle$ of cyclic quadrilateral equals interior opposite $\angle$s)

$\therefore W\hat{B}C + W\hat{Z}A = 180°$

$\therefore W\hat{B}C + W\hat{B}A = 180°$

(b)(i) let $u = x^n$ $\quad v' = e^x$

$u' = nx^{n-1}$ $\quad v = e^x$

$I_n = [x^n e^x]_0^1 - n\int_0^1 x^{n-1} e^x \, dx$

$= e^1 - n I_{n-1}$

(ii) $I_2 = e - 4 I_3$

$I_3 = e - 3I_2$

$I_2 = e - 2I_4$

$I_1 = e - I_0$

$I_0 = \int_0^1 e^x \, dx$

$= [e^x]_0^1$

$= e - 1$

$I_1 = e - e + 1 = 1$

$I_2 = e - 2$

$I_3 = e - 3e + 6 = 6 - 2e$

$I_4 = e - 24 + 8e = 9e - 24$

(c)(i) $\tan 45° = \frac{kL}{x}$

$\therefore x^2 = kL$

$A = x \times 2y$

$= x \times 2\sqrt{16 - x^2}$

(ii) $V = \lim_{\delta x \to 0} \sum_{x=0}^{4} x\sqrt{64 - 4x^2} \, \delta x$

$= \int_0^4 x\sqrt{64 - 4x^2} \, dx$

$= -\frac{1}{8}\int_0^4 y^{\frac{1}{2}} \, dy$ $\quad$ let $y = 64 - 4x^2$

$= -\frac{1}{8}\left[\frac{2y^{\frac{3}{2}}}{3}\right]$ $\quad \frac{dy}{dx} = -8x$

$\quad -\frac{dy}{8x} = dx$

$= \frac{1}{12}\left[y^{\frac{3}{2}}\right]_0^{64}$

$= \frac{1}{12} \times 64^{\frac{3}{2}}$

$= \frac{128}{3}$

6 (a)(i) $w^3 + w + w^2$

$= w(w^2 + 1 + w)$

$w \neq 0$ or 1

$\therefore w^2 + 1 + w = 0$

(ii) $w^4(1 + w^2 + w^4)$

$= w^4(1 + w^2 + w)$

$= 0$

(iii) $p\left(\frac{1}{w^2}\right) = \frac{1}{w^8} + \frac{3}{w^6} + \frac{2}{w^4} + \frac{1}{w^2} - 1$

$= w + 3 + 2w^2 + w - 1$

$= 2 + 2w + 2w^2$

$= 0$

(b)(i) let $u = \cos x$

$\frac{du}{dx} = -\sin x$

$-\frac{du}{\sin x} = dx$

$\int_1^{-1} -\frac{du}{\sqrt{1+u^2}}$

$= \left[\tan^{-1} u\right]_{-1}^{1}$

$= \frac{\pi}{4} - -\frac{\pi}{4}$

$= \frac{\pi}{2}$

(ii) $\frac{du}{dx} = -1$

$-dx = du$

$\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} \, dx$

$= -\int_{\pi}^{0} \frac{(\pi - u)\sin(\pi - u)}{1 + \cos^2(\pi - u)} \, du$

$= \int_0^{\pi} \frac{\pi \sin u - u \sin u}{1 + \cos^2 u} \, du$

$= \int_0^{\pi} \frac{\pi \sin x}{1 + \cos^2 x} \, dx - \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} \, dx$

$\therefore \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1 + \cos^2 x} \, dx$

$= \frac{\pi}{2} \times \frac{\pi}{2}$

$= \frac{\pi^2}{4}$

(c)(i) $x\frac{dy}{dx} + y \times 1 = 0$

$\frac{dy}{dx} = -\frac{y}{x}$

$m_A = -\frac{\frac{c}{t}}{ct}$

$= -\frac{1}{t^2}$

eqn of tangent

$y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$

$t^2 y - ct = -x + ct$

$x + t^2 y = 2ct$

(ii) when $y = 0$, $x = 2ct$

$m_p = \left(\frac{ct + 2ct}{2}, \frac{\frac{c}{t} + 0}{2}\right)$

$= \left(\frac{3ct}{2}, \frac{c}{2t}\right)$

$x = \frac{3ct}{2}$ $\quad y = \frac{c}{2t}$

$xy = \frac{3c^2}{4}$

Question 7:

a) i.

$y=\cot x$

$=\tan\left(\frac{\pi}{2}-x\right)$

$y'=-\sec^2\left(\frac{\pi}{2}-x\right)$

$=-\text{cosec}^2 x$

ii. At $A(x=a)$:

$\sin a=\cot a$

$\sin a=\frac{\cos a}{\sin a}$

$\sin^2 a=\cos a---(1)$

$y=\sin x$	$y=\cot x$
$y'=\cos x$	$y'=-\text{cosec}^2 x$
At $A\ (x=a): m_1=\cos a$	At $A\ (x=a): m_2=-\text{cosec}^2 a$

$m_1\times m_2=\cos a\times-\text{cosec}^2 a$

$=\cos a\times-\frac{1}{\sin^2 a}$

$=\cos a\times-\frac{1}{\cos a}\ \left[\text{from }(1)\right]$

$=-1$

$\therefore$ curves intersect at right angles at A

iii. From (1):

$\sin^2 a=\cos a$

$1-\cos^2 a=\cos a$

$\cos^2 a+\cos a-1=0$

$\cos a=\frac{-1\pm\sqrt{1-4(-1)}}{2}$

$=\frac{-1\pm\sqrt{5}}{2}$

$\sin^2 a=\frac{-1+\sqrt{5}}{2}\ \left[\text{from }(1)\right]$

$\text{cosec}^2 a=\frac{2}{-1+\sqrt{5}}\times\frac{-1-\sqrt{5}}{-1-\sqrt{5}}$

$=\frac{2\left(-1-\sqrt{5}\right)}{1-5}$

$=\frac{-2\left(1+\sqrt{5}\right)}{-4}$

$\text{cosec}^2 a=\frac{1+\sqrt{5}}{2}$

b) i.

$F=\frac{mgR^2}{x^2}$

$\frac{mv^2}{x}=\frac{mgR^2}{x^2}$

$v^2=\frac{gR^2}{x}$

$=\frac{10\times\left(6.4\times10^6\right)^2}{6.403\times10^6}$

$v\cong8000\text{ ms}^{-1}$

ii.

$v=rw$

$v=xw$

$8000=\left(6.403\times10^6\right)w$

$w=1.249\times10^{-3}\text{ rad/s}$

$T=\frac{2\pi}{w}$

$=5030\text{ s}$

$=1\text{ h }24\text{ min}$

iii.

$F=\frac{mgR^2}{x^2}$

$=\frac{300\times10\times\left(6.4\times10^6\right)^2}{\left(6.403\times10^6\right)^2}$

$\cong2997N$

c) i.

$$y=\sin^{-1}x-\sqrt{1-x^2}$$
$$=\sin^{-1}x-\left(1-x^2\right)^{\frac{1}{2}}$$
$$y'=\frac{1}{\sqrt{1-x^2}}-\frac{1}{2}\left(1-x^2\right)^{-\frac{1}{2}}\cdot -2x$$
$$=\frac{1}{\sqrt{1-x^2}}+\frac{x}{\sqrt{1-x^2}}$$
$$=\frac{1+x}{\sqrt{1-x^2}}$$

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ii.

$$\frac{1+x}{\sqrt{1-x^2}}=\frac{1+x}{\sqrt{(1+x)(1-x)}}$$
$$=\sqrt{\frac{1+x}{1-x}}$$
$$\int_0^a\sqrt{\frac{1+x}{1-x}}dx=\left[\sin^{-1}x-\sqrt{1-x^2}\right]_0^a$$
$$=\left(\sin^{-1}a-\sqrt{1-a^2}\right)-\left(\sin^{-1}0-\sqrt{1-0^2}\right)$$
$$=\sin^{-1}a+1-\sqrt{1-a^2}$$

Question 8:

a) i. $\sin(A+B)-\sin(A-B)=\sin A\cos B+\cos A\sin B-(\sin A\cos B-\cos A\sin B)$

$$=2\cos A\sin B$$

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ii.

$$LHS=\sin\frac{\theta}{2}(1+2\cos\theta+2\cos2\theta+2\cos3\theta)$$
$$=\sin\frac{\theta}{2}+2\cos\theta\sin\frac{\theta}{2}+2\cos2\theta\sin\frac{\theta}{2}+2\cos3\theta\sin\frac{\theta}{2}$$
$$=\sin\frac{\theta}{2}+\left\{\sin\left(\theta+\frac{\theta}{2}\right)-\sin\left(\theta-\frac{\theta}{2}\right)\right\}+\left\{\sin\left(2\theta+\frac{\theta}{2}\right)-\sin\left(2\theta-\frac{\theta}{2}\right)\right\}$$
$$+\left\{\sin\left(3\theta+\frac{\theta}{2}\right)-\sin\left(3\theta-\frac{\theta}{2}\right)\right\}$$
$$=\cancel{\sin\frac{\theta}{2}}+\cancel{\sin\frac{3\theta}{2}}-\cancel{\sin\frac{\theta}{2}}+\cancel{\sin\frac{5\theta}{2}}-\cancel{\sin\frac{3\theta}{2}}+\sin\frac{7\theta}{2}-\cancel{\sin\frac{5\theta}{2}}$$
$$=\sin\frac{7\theta}{2}$$
$$=RHS$$

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iii. $\sin\frac{\theta}{2}(1+2\cos\theta+2\cos2\theta+2\cos3\theta)=\sin\frac{7\theta}{2}$

When $\theta=\frac{2\pi}{7}$:

$$RHS=\sin\frac{7\left(\frac{2\pi}{7}\right)}{2}$$
$$=\sin\pi$$
$$=0$$

$$\sin\frac{\theta}{2}(1+2\cos\theta+2\cos2\theta+2\cos3\theta)=0$$

$$\sin\frac{\theta}{2}=0$$

But when $\theta=\frac{2\pi}{7}$:

$$\sin\frac{\pi}{7}\neq0$$
$$\therefore 1+2\cos\theta+2\cos2\theta+2\cos3\theta=0$$

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b) i.

$$f(x)=e^x\left(1-\frac{x}{8}\right)^8$$

$$u=e^x$$
$$u'=e^x$$

$$v=\left(1-\frac{x}{8}\right)^8$$
$$v'=8\left(1-\frac{x}{8}\right)^7\cdot\left(-\frac{1}{8}\right)$$
$$=-\left(1-\frac{x}{8}\right)^7$$

$$f'(x)=-e^x\left(1-\frac{x}{8}\right)^7+e^x\left(1-\frac{x}{8}\right)^8$$
$$=-e^x\left(1-\frac{x}{8}\right)^7\left[1-\left(1-\frac{x}{8}\right)\right]$$
$$=-e^x\left(1-\frac{x}{8}\right)^7\left(\frac{x}{8}\right)$$
$$=\frac{-xe^x}{8}\left(1-\frac{x}{8}\right)^7$$

Stat. points occur when $f'(x)=0$:

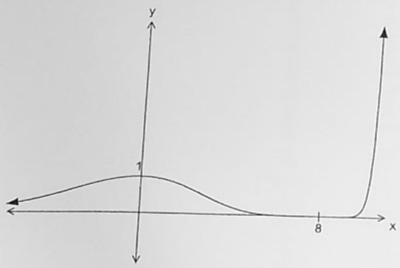
$$\frac{-xe^x}{8}=0 \qquad \left(1-\frac{x}{8}\right)^7=0$$

$$\left.\begin{array}{l}x=0\\y=1\end{array}\right\} \qquad 1-\frac{x}{8}=0$$

$$\left.\begin{array}{l}x=8\\y=0\end{array}\right\}$$

Stat. points at $(0,1)$ and $(8,0)$

ii.



iii. When $x<8$:

$$e^x\left(1-\frac{x}{8}\right)^8 \le 1 \text{ from graph}$$

$$e^x \le \frac{1}{\left(1-\frac{x}{8}\right)^8} \quad \text{Note: } \left(1-\frac{x}{8}\right)^8 > 0 \text{ when } x<8$$

$$e^x \le \left(1-\frac{x}{8}\right)^{-8}$$

iv. When $x=1$:

$$e \le \left(1-\frac{1}{8}\right)^{-8}$$

$$e \le \left(\frac{7}{8}\right)^{-8}$$

$$e \le \left(\frac{8}{7}\right)^8$$

When $x=-1$:

$$e^{-1} \le \left(1+\frac{1}{8}\right)^{-8}$$

$$\frac{1}{e} \le \left(\frac{9}{8}\right)^{-8}$$

$$\frac{1}{e} \le \left(\frac{8}{9}\right)^8$$

$$e \ge \left(\frac{9}{8}\right)^8$$

$$\therefore \left(\frac{9}{8}\right)^8 \le e \le \left(\frac{8}{7}\right)^8$$