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Evaluating the Probability Integral Using Wallis's Product Formula for π

Paul Levrie and Walter Daems

1. INTRODUCTION. In this note we present a straightforward method for evaluating the probability integral:

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}.$$

By "straightforward" we mean that it can be presented in a standard one-variable calculus course.

This is not the first time an "elementary" proof of this identity has been given. Several other proofs can be found in, e.g., [**1**], [**3**], [**4**], [**5**], [**7**]. Most of them are variations on a theme (double integral, gamma function, . . .). This one is a simplification of one of the most recent ones by Lord [**6**].

The method is based on the following lemma which we will prove later:

Lemma. *The function F with $F(x) = e^{-x} - \left(1 - \frac{x}{n}\right)^n$ satisfies:*

$$\textit{for all } \; x \in [0, n],\; 0 \le F(x) \le \frac{e^{-1}}{n}.$$

2. THE HEART OF THE MATTER. It seems natural to approximate the integrand e^{-x^2} in the probability integral by $(1 - x^2/n)^n$. This yields a simpler integral to evaluate. Is this simplification justified? Yes, it is. Consider the difference between these two expressions, integrated over an interval in which we can apply the lemma (with x replaced by x^2). It satisfies:

$$0 \le \int_0^{\sqrt{n}} \left[e^{-x^2} - \left(1 - \frac{x^2}{n}\right)^n\right] dx \le \int_0^{\sqrt{n}} \frac{e^{-1}}{n}\,dx.$$

This can be rewritten as:

$$0 \le \int_0^{\sqrt{n}} e^{-x^2}\,dx - \int_0^{\sqrt{n}} \left(1 - \frac{x^2}{n}\right)^n dx \le \frac{e^{-1}}{\sqrt{n}}.$$

Hence, taking the limit as $n \to \infty$ we get

$$\int_0^\infty e^{-x^2} dx = \lim_{n\to\infty} \int_0^{\sqrt{n}} \left(1 - \frac{x^2}{n}\right)^n dx.$$

We now evaluate the integral on the right-hand side using the substitution $x = \sqrt{n} \cdot \sin t$, and integration by parts:

$$\int_0^{\sqrt{n}} \left(1 - \frac{x^2}{n}\right)^n dx = \sqrt{n} \cdot \int_0^{\pi/2} \cos^{2n+1} t \, dt$$

$$= \sqrt{n} \cdot \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \cdot \dots \cdot \frac{4}{5} \cdot \frac{2}{3}. \tag{1}$$

To conclude the proof we have to calculate the limit as $n \to \infty$ of this expression. This can be done with the following well-known product formula for π due to John Wallis:

$$\lim_{n\to\infty} \frac{2\cdot 2}{1\cdot 3} \cdot \frac{4\cdot 4}{3\cdot 5} \cdot \frac{6\cdot 6}{5\cdot 7} \cdot \dots \cdot \frac{2n\cdot 2n}{(2n-1)\cdot(2n+1)} = \frac{\pi}{2}.$$

Indeed, as a consequence of this formula we find that

$$\int_0^\infty e^{-x^2} dx = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{2n+1}} \cdot \sqrt{\frac{2\cdot 2}{1\cdot 3} \cdot \frac{4\cdot 4}{3\cdot 5} \cdot \dots \cdot \frac{2n\cdot 2n}{(2n-1)\cdot(2n+1)}} = \frac{1}{\sqrt{2}}\sqrt{\frac{\pi}{2}}$$

which proves the result.

3. TYING UP THE LOOSE ENDS.

A. Proof of the Lemma for $n \geq 2$. First we have to show that $F(x) \geq 0$, or that

$$\left(1 - \frac{x}{n}\right)^n \leq e^{-x} \text{ for } x \in [0, n].$$

If we rewrite this inequality in the form

$$\left(1 - \frac{x}{n}\right)^n \leq \left(e^{-x/n}\right)^n \text{ for } x \in [0, n],$$

then it clearly suffices to prove that

$$0 \leq 1 - \frac{x}{n} \leq e^{-x/n} \text{ for } x \in [0, n],$$

or

$$0 \leq 1 - t \leq e^{-t} \text{ for } t \in [0, 1].$$

This follows immediately from a graph of the exponential function with negative exponent and its tangent at the point $(0, 1)$. A more rigorous proof uses the mean value theorem applied to the function defined by $g(x) = e^{-x}$ on the interval $[0, t]$.

To prove the other part of the lemma we note that $F(0) = 0$, $F(n) = e^{-n}$, and $F'(n) = -e^{-n}$. The function F is continuous on the closed interval $[0, n]$, and hence attains a maximum at some point x_0 of this interval:

$$\text{for all } x \in [0, n], \ F(x) \leq F(x_0). \tag{2}$$

This maximum cannot be at 0 as $F(n) > F(0)$; furthermore, the maximum cannot be at n as $F'(n) < 0$. Since F is differentiable, it follows that $F'(x_0) = 0$, or equivalently:

$$e^{-x_0} - \left(1 - \frac{x_0}{n}\right)^{n-1} = 0.$$

Using this we can rewrite $F(x_0)$ as

$$F(x_0) = e^{-x_0} - e^{-x_0} \cdot \left(1 - \frac{x_0}{n}\right) = \frac{x_0 e^{-x_0}}{n}.$$

Now it is easy to show that the function h defined by $h(x) = xe^{-x}$ reaches its maximum value at $x = 1$. Hence

$$F(x_0) \leq \frac{e^{-1}}{n}.$$

Combined with (2) this proves the lemma. ■

B. Proof of Wallis's Product Formula for π [2, p. 280]. Let us take a closer look at the integration by parts used in (1). In most calculus textbooks the following reduction formula is given as a classic example of the method of integration by parts:

$$\int \cos^m t\, dt = \frac{1}{m} \cos^{m-1} t \cdot \sin t + \frac{m-1}{m} \int \cos^{m-2} t\, dt.$$

If we integrate over the interval $[0, \pi/2]$, the formula reduces to

$$\int_0^{\pi/2} \cos^m t\, dt = \frac{m-1}{m} \int_0^{\pi/2} \cos^{m-2} t\, dt.$$

Iterating for m even and odd we get the well-known Wallis integrals, the second of which is used in (1):

$$C_{2n} = \int_0^{\pi/2} \cos^{2n} t\, dt = \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdot \cdots \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2},$$

$$C_{2n+1} = \int_0^{\pi/2} \cos^{2n+1} t\, dt = \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \cdot \cdots \cdot \frac{4}{5} \cdot \frac{2}{3}.$$

With these we can rewrite Wallis's product formula for π in the following form:

$$\lim_{n\to\infty} \frac{\pi}{2} \cdot \frac{C_{2n+1}}{C_{2n}} = \frac{\pi}{2}.$$

To prove it, we note that from

$$0 \leq \cos x \leq 1 \text{ for } x \in \left[0, \frac{\pi}{2}\right]$$

it follows immediately that

$$0 \leq C_{2n+2} \leq C_{2n+1} \leq C_{2n}.$$

Hence

$$\frac{C_{2n+2}}{C_{2n}} \le \frac{C_{2n+1}}{C_{2n}} \le \frac{C_{2n}}{C_{2n}}.$$

Multiplying by $\pi/2$, noting that $C_{2n+2} = C_{2n} \cdot (2n+1)/(2n+2)$, and taking the limit as $n \to \infty$ proves Wallis's product formula. (For an elementary proof that does not use any calculus, see [**8**].)

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Counting Fields of Complex Numbers

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1. INTRODUCTION. As usual, $\aleph_0 = |\mathbb{N}|$, $c = 2^{\aleph_0} = |\mathcal{P}(\mathbb{N})| = |\mathbb{R}| = |\mathbb{C}|$, and $2^c = |\mathcal{P}(\mathbb{R})| = |\mathcal{P}(\mathbb{C})|$, where $|M|$ denotes the cardinal number of a set M and $\mathcal{P}(M)$ is the power set of M, $\mathcal{P}(M) = \{X \mid X \subset M\}$.

For a field K let $\mathcal{F}(K)$ denote the family of all subfields of K. Further, let $\mathcal{F}(K)^*$ be the quotient set of $\mathcal{F}(K)$ with respect to the field isomorphism relation $\cong$,

$$\mathcal{F}(K)^* := \Big\{\{F \in \mathcal{F}(K) \mid F \cong L\} \,\Big|\, L \in \mathcal{F}(K)\Big\}.$$

Hence $|\mathcal{F}(K)|$ is the *total number* of subfields of K and $|\mathcal{F}(K)^*|$ is the *maximal number* of mutually non-isomorphic subfields of K. Our goal is to determine $|\mathcal{F}(K)|$ and $|\mathcal{F}(K)^*|$ for certain subfields K of $\mathbb{C}$. (The restriction to subfields of $\mathbb{C}$ restricts only the characteristic and cardinality of the field, because every field K of characteristic 0 with $|K| \le c$ can be embedded in $\mathbb{C}$.) Trivially, $\mathcal{F}(K) \subset \mathcal{P}(K)$, and therefore $|\mathcal{F}(K)^*| \le |\mathcal{F}(K)| \le |\mathcal{P}(K)|$ for every field K. Both inequalities can be strict. Table 1 gives an overview of some possible relations among the three cardinal numbers. In the table, $\mathbb{A} \subset \mathbb{C}$ is the field of algebraic numbers and $\mathbb{Q}(\cdots)$ denotes a field extension of $\mathbb{Q}$ as usual. The fields $\Omega^{(p)} \subset \mathbb{A}$ are defined by $\Omega^{(p)} := \mathbb{R} \cap \bigcup_{n=1}^{\infty} \mathbb{Q}(e^{\pi i/p^n}) = \bigcup_{n=1}^{\infty} \mathbb{Q}(\cos(\pi/p^n))$.