Year 12 Ext. 1 TRIAL solutions 2011

① a) $y = (\tan^{-1} x)^2$

$\therefore \frac{dy}{dx} = 2(\tan^{-1}x)^1 \times \frac{d}{dx}(\tan^{-1}x)$

$= 2\tan^{-1}x \times \frac{1}{1+x^2}$

$= \frac{2\tan^{-1}x}{1+x^2}$ (2)

b) $\sum_{n=2}^{5} {}^nC_2 = {}^2C_2 + {}^3C_2 + {}^4C_2 + {}^5C_2$

$= 1 + 3 + 6 + 10$
(using calculator)

$= 20$ (2)

c) $\frac{2x-3}{x-2} \geq 1$ × B.S. by $(\text{denom})^2$

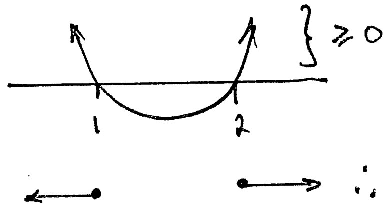
$\therefore \frac{2x-3}{x-2} \times (x-2)^2 \geq 1 \times (x-2)^2$

$\therefore (2x-3)(x-2) \geq x^2 - 4x + 4$

$\therefore 2x^2 - 4x - 3x + 6 \geq x^2 - 4x + 4$

$\therefore x^2 - 3x + 2 \geq 0$

Consider graph of $y = x^2 - 3x + 2$
$= (x-1)(x-2)$



$\therefore x \leq 1$ or $x \geq 2$

but $x \neq 2$ $\therefore$ $x \leq 1$ or $x > 2$ (4)

d) $\lim_{x \to 0} \frac{\sin 3x}{x} = \lim_{x\to 0} 3 \times \frac{\sin 3x}{3x}$

$= 3 \times \lim_{x\to 0} \frac{\sin 3x}{3x}$ $\Big\}$ as $x \to 0$, $3x \to 0$

$= 3 \times 1$

$= 3$ (1)

e) $\int_0^3 \frac{x}{\sqrt{1+x}}\,dx$ $\qquad x = u^2 - 1$

$\therefore dx = 2u\,du$

$= \int_1^2 \frac{u^2-1}{\sqrt{1+u^2-1}} \times 2u\,du$ $\qquad$ at $x = 3$: $u^2 = 4$, $u = 2$ (>0)

$x = 0$: $u^2 = 1$, $u = 1$ (>0)

$= \int_1^2 \frac{u^2-1}{u} \times 2u\,du$

$= 2\int_1^2 u^2 - 1\,du$

$= 2\left[\frac{u^3}{3} - u\right]_1^2$

$= 2\left(\left(\frac{8}{3} - 2\right) - \left(\frac{1}{3} - 1\right)\right)$

$= \frac{8}{3}$ (3)

② a) $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

m_1: $2y = -x - 1$ $\therefore y = -\frac{1}{2}x - \frac{1}{2}$: $m_1 = -\frac{1}{2}$

m_2: $3y = 2x + 6$ $\therefore y = \frac{2}{3}x + 2$: $m_2 = \frac{2}{3}$

$\therefore \tan\theta = \left|\frac{-\frac{1}{2} - \frac{2}{3}}{1 - \frac{1}{2} \times \frac{2}{3}}\right|$

$= \left|\frac{-7}{4}\right|$

$= \frac{7}{4}$

$\therefore \theta = \tan^{-1}(7/4)$

$= 60.25\ldots$

$\theta = 60°$ (nearest deg.) (4)

② b) S.A. = A , Volume = V

$$\therefore \frac{dA}{dt} = \frac{dA}{dV} \times \frac{dV}{dt} \quad \ldots\ldots (1)$$

where $\frac{dV}{dt} = 72$ cm³/s. ..... (2)

Now: $\frac{dA}{dV} = \frac{dA}{dr} \times \frac{dr}{dV}$

where: $A = 4\pi r^2 \quad \therefore \frac{dA}{dr} = 8\pi r \quad \ldots (3)$

$V = \frac{4}{3}\pi r^3 \quad \therefore \frac{dV}{dr} = 4\pi r^2$

or $\frac{dr}{dV} = \frac{1}{4\pi r^2} \quad \ldots (4)$

$\therefore$ (2), (3), (4) into (1) :

$$\frac{dA}{dt} = 8\pi r \times \frac{1}{4\pi r^2} \times 72$$

$$= \frac{144}{r}$$

$\therefore$ when $r = 12$

$$\frac{dA}{dr} = 144/12$$

$$= 12 \text{ cm}^2/\text{s}$$

4

c) (i) $\int \sin^2 x \, dx = \frac{1}{2}\int 1 - \cos 2x \, dx$

$$= \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + c$$

or $= \frac{1}{2}x - \frac{1}{4}\sin 2x + c$ 2

(ii) $\int \frac{dx}{\sqrt{9+4x^2}} = \int \frac{1}{\sqrt{4(9/4 + x^2)}} \, dx$

$$= \frac{1}{2}\int \frac{1}{\sqrt{(\frac{3}{2})^2 + x^2}} \, dx$$

Using Standard Integrals Sheet

$$= \frac{1}{2}\ln\left(x + \sqrt{9/4 + x^2}\right)$$ 2

③ a)(i)

If $N = 5000 + Ae^{kt} \quad \ldots\ldots (1)$

then $dN/dt = k \times Ae^{kt}$

but from (1) : $Ae^{kt} = N - 5000$

$\therefore dN/dt = k(N - 5000)$

$\therefore N = 5000 + Ae^{kt}$ is a solution to $dN/dt = k(N-5000)$ (QED) 1

(ii) At $t = 0$, $N = 15\,000$

$\therefore 15000 = 5000 + Ae^{k \times 0}$

$= 5000 + A$

$\therefore A = 10\,000$

At $t = 2$, $N = 20\,000$

$\therefore 20000 = 5000 + 10\,000 e^{2k}$

$\therefore 10\,000 e^{2k} = 15\,000$

$e^{2k} = 1.5$

$\ln(e^{2k}) = \ln 1.5$

$\therefore 2k = \ln 1.5$

$k = \ln 1.5 \div 2$

$= 0.20273\ldots$

$k = 0.2027$ (4dp) 3

(iii) $\therefore$ At $t = 7$:

$N = 5000 + 10000 e^{0.2027 \times 7}$

$= 46\,325.7\ldots$

$\therefore N = 46\,300$ (nearest 100) 1

b) $P(x) = x^5 + mx^3 + nx$

(i) $\therefore P(a) = a^5 + ma^3 + na$

$P(-a) = (-a)^5 + m(-a)^3 + n(-a)$

$= -a^5 - ma^3 - na$

$= -(a^5 + ma^3 + n)$

$\therefore -P(-a) = -\{-(a^5 + ma^3 + n)\}$

$= a^5 + ma^3 + n$ (1)

i.e. $P(a) = -P(-a)$ $\therefore$ odd (QED)

(ii) When $P(x)$ divided by $x+2$ remainder is $P(-2)$

AND when divided by $x-2$ remainder is $P(2) = 5$ (given)

Now, from (i) $P(a) = -P(-a)$

$\therefore P(-2) = -P(-(-2))$

$= -P(2)$

$= -5$ (2)

c) α, β, γ : roots of $x^3 - 2x^2 + 3x + 7 = 0$

and: $\frac{1 \times \beta\gamma}{\alpha \times \beta\gamma} + \frac{1 \times \alpha\gamma}{\beta \times \alpha\gamma} + \frac{1 \times \alpha\beta}{\gamma \times \alpha\beta}$

$= \frac{\alpha\beta + \alpha\gamma + \beta\gamma}{\alpha\beta\gamma}$

Now: $\alpha\beta + \alpha\gamma + \beta\gamma = c/a$

$= 3/1$

$= 3$

$\alpha\beta\gamma = -d/a$

$= -7/1$

$= -7$ (4)

$\therefore \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 3/-7$ or $-3/7$

(4) a) (i) $\angle MBC = \angle BAC$

$\because$ angle between tangent and chord equals angle in alternate segment (1)

(ii) $\angle BAC + \angle CNM = 180°$ (co-int. $\angle$'s in $\parallel$ lines)

and $\angle BAC = \angle MBC$ (from (i))

$\therefore \angle MBC + \angle CNM = 180°$

$\therefore$ MNCB is cyclic quadrilateral (opp. $\angle$'s add to 180°) (3)

b) Prove: $1 \times 2^0 + 2 \times 2^1 + 3 \times 2^2 + \dots + n \times 2^{n-1} = 1 + (n-1)2^n$ for $n \geq 1$

• Prove true for $n=1$

i.e. $LHS = 1 \times 2^0$ $= 1 \times 1$ $= 1$ $\quad RHS = 1 + (1-1) \times 2^1$ $= 1 + 0$ $= 1$ $(\frac{1}{2})$

$\therefore$ true for $n=1$

• Assume true for $n=k$

i.e. assume: $1 \times 2^0 + 2 \times 2^1 + \dots + k \times 2^{k-1} = 1 + (k-1)2^k$

$\therefore$ prove true for $n=k+1$

i.e. prove: $1 \times 2^0 + 2 \times 2^1 + \dots + k \times 2^{k-1} + (k+1)2^k = 1 + k \times 2^{k+1}$

Now: $LHS = 1 + (k-1)2^k + (k+1)2^k$ {from assumption.

$= 1 + k \times 2^k - 2^k + k \times 2^k + 2^k$

$= 1 + 2k \times 2^k$

$= 1 + k \times 2^1 \times 2^k$

$= 1 + k \times 2^{k+1}$

$= RHS$ (3)

$\therefore LHS = RHS$

• $\therefore$ True for $n=1$ and true for $n=k+1$ when true for $n=k$ $(\frac{1}{2})$

$\therefore$ True for $n = 1, 2, 3, \dots$ (all $n \geq 1$) (4)

(QED)

④ c) (i) Solution to $\frac{\log_e x}{x} + 2 = 0$ corresponds to $y = \frac{\log_e x}{x} + 2$ crossing x-axis $(y=0)$

Now: when $x = 0.4$:

$$\frac{\log_e x}{x} + 2 = \frac{\log_e 0.4}{0.4} + 2$$

$$= -0.29\ldots \text{ (below } x\text{-axis)}$$

when $x = 0.5$:

$$\frac{\log_e x}{x} + 2 = +0.61\ldots \text{ (above } x\text{-axis}$$

$\therefore$ Sign change $\Rightarrow$ solution between $x = 0.4$ and $x = 0.5$ ①

(QED)

(ii) For $f(x) = \frac{\log_e x}{x} + 2$:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

where
$$f'(x) = \frac{x \times \frac{1}{x} - \log_e x \times 1}{x^2}$$

$$= \frac{1 - \log_e x}{x^2}$$

$$\therefore x_2 = 0.4 - \frac{\left(\frac{\log_e 0.4}{0.4} + 2\right)}{\frac{1 - \log_e 0.4}{(0.4)^2}}$$

$$= 0.4242\ldots$$

$\therefore$ $x_2 = 0.424$ (3dp) ③

⑤ a) $A = 5m, 3w$

$B = 4m, 6w$

(i) $P(\text{opposite sexes}) = P(M,W \text{ or } WM)$

$$= P(M,W) + P(W,M)$$

$$= \underset{A}{\frac{5}{8}} \times \underset{B}{\frac{6}{10}} + \underset{A}{\frac{3}{8}} \times \underset{B}{\frac{4}{10}}$$

$$= \frac{21}{40}$$ ②

(ii) $P(\text{man}) = P(\text{Group A, M or Group B, M})$

$$= P(\text{Group A, M}) + P(\text{Group B, M})$$

$$= \frac{1}{2} \times \frac{5}{8} + \frac{1}{2} \times \frac{4}{10}$$

$$= \frac{41}{80}$$ ②

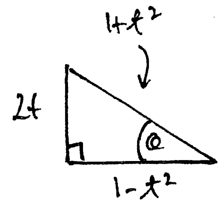
b) $\tan \frac{\theta}{2} = t$

$$\therefore \tan\left(2 \times \theta/2\right) = \tan\theta$$

$$= \frac{2\tan\theta/2}{1 - \tan^2\theta/2}$$

$$= \frac{2t}{1-t^2}$$

$\therefore$



$\Rightarrow$ $\sin\theta = 2t/_{1+t^2}$

$\cos\theta = 1-t^2/_{1+t^2}$

$$\therefore \frac{1-\cos\theta}{\sin\theta} = \frac{1 - \frac{1-t^2}{1+t^2}}{\frac{2t}{1+t^2}} \begin{matrix} \times 1+t^2 \\ \times 1+t^2 \end{matrix}$$

$$= \frac{1+t^2 - 1 + t^2}{2t}$$

$$= \frac{2t^2}{2t}$$

$$= t$$

ie. LHS = RHS (QED) ③

5 c) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = a = 3x^2$

$\therefore \frac{1}{2}v^2 = \int 3x^2\,dx$

$= x^3 + C \qquad (C_1 = 2\times C)$

$\therefore v^2 = 2x^3 + C_1$

and at: $v = -\sqrt{2}$, $x = 1$

$\therefore 2 = 2 + C_1$

$\therefore C_1 = 0$

ie $v^2 = 2x^3$

$\therefore v = \pm\sqrt{2x^3}$

But initially $v = -\sqrt{2}$

$\therefore$ select: $v = -\sqrt{2x^3}$

ie $\frac{dx}{dt} = -\sqrt{2}\,x^{3/2}$

$\therefore \frac{dt}{dx} = -\frac{1}{\sqrt{2}}\,x^{-3/2}$

$\therefore t = \frac{-1}{\sqrt{2}}\int x^{-3/2}\,dx$

$= -\frac{1}{\sqrt{2}} \times \frac{x^{-1/2}}{-\frac{1}{2}} + C_2$

$= \frac{2}{\sqrt{2}}\,x^{-\frac{1}{2}} + C_2$

ie $t = \sqrt{2}\,x^{-\frac{1}{2}} + C_2$

and at: $t = 0$, $x = 1$

$\therefore 0 = \sqrt{2} + C_2$

$\therefore C_2 = -\sqrt{2}$

$\therefore t = \frac{\sqrt{2}}{\sqrt{x}} - \sqrt{2}$

$\therefore \sqrt{x} = \frac{\sqrt{2}}{t + \sqrt{2}}$

$\therefore x = \frac{2}{(t+\sqrt{2})^2}$

5

6 a) (i) Domain:

$y = \sin^{-1}x$: $\quad -1 \le x \le 1$

$\therefore y = \sin^{-1}\left(\frac{x}{2}\right)$: $\quad -1 \le \frac{x}{2} \le 1 \qquad (\times 2)$

$\therefore -2 \le x \le 2$

Range:

$y = \sin^{-1}x$: $\quad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$

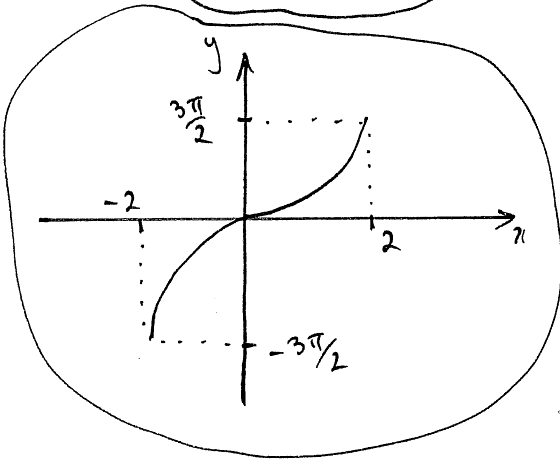
$y = 3\sin^{-1}x$

ie $\frac{y}{3} = \sin^{-1}x$: $\quad -\frac{\pi}{2} \le \frac{y}{3} \le \frac{\pi}{2} \qquad (\times 3)$

$-\frac{3\pi}{2} \le y \le \frac{3\pi}{2}$

2

(ii)



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(iii) $f(x) = 3\sin^{-1}\left(\frac{x}{2}\right)$

$\therefore f'(x) = 3 \times \frac{1}{\sqrt{2^2 - x^2}}$ using Standard Integrals!

$= \frac{3}{\sqrt{4 - x^2}}$

$\therefore$ Slope at $x = 0$ is $f'(0)$

and $f'(0) = \frac{3}{\sqrt{4 - 0^2}}$

$= \frac{3}{2}$

2

6 b) $x = 5\cos(4\pi t) \quad \ldots (1)$

(i) Express as $\ddot{x} = -n^2 x$

from (1): $\dot{x} = 5 \times -\sin(4\pi t) \times 4\pi$

$= -20\pi \sin(4\pi t)$

$\therefore \ddot{x} = -20\pi \times \cos(4\pi t) \times 4\pi$

$= -80\pi^2 \cos(4\pi t)$

$= -16\pi^2 (5\cos(4\pi t))$ 2

ie. $\ddot{x} = -(4\pi)^2 x = -n^2 x$

(QED)

(ii) Period $= 2\pi/n$

$= 2\pi/4\pi$

$= \frac{1}{2}$ (second) 1

(iii) Max Velocity at centre i.e. at $x=0$

$v^2 = n^2(a^2 - x^2)$

where $n = 4\pi$, $a = 5$, $x = 0$

$\therefore v^2 = (4\pi)^2(5^2 - 0)$

$\therefore v = \pm 4\pi \times 5$

$\therefore v_{max} = 20\pi$ m/s 2

(iv) Max. acceleration at amplitude (left)
i.e. at $x = -5$

$\therefore \ddot{x}_{max} = -(4\pi)^2 \times -5$

$= 80\pi^2$ m/s^2 1

(v) from (iii) $v^2 = (4\pi)^2(5^2 - x^2)$

$v^2 = 16\pi^2(25 - x^2)$ 1

7 a)

(i) For 'any' projectile motion; fired from (0,0):

Horizontal	Vertical
$\ddot{x} = 0$	$\ddot{y} = -g$
$\therefore \dot{x} = C_1$	$\therefore \dot{y} = -gt + C_2$
but at $t=0$, $v_x = V\cos\theta$	but at $t=0$, $v_y = V\sin\theta$
$\therefore C_1 = V\cos\theta$	$\therefore C_2 = V\sin\theta$
$\therefore \dot{x} = V\cos\theta \quad (1)$	$\therefore \dot{y} = V\sin\theta - gt \quad (2)$
$\therefore x = Vt\cos\theta + C_3$	$\therefore y = Vt\sin\theta - \frac{1}{2}gt^2 + C_4$
but at $t=0$, $x=0$	but at $t=0$, $y=0$
$\therefore C_3 = 0$	$\therefore C_4 = 0$
$\therefore x = Vt\cos\theta \quad (3)$	$\therefore y = Vt\sin\theta - \frac{1}{2}gt^2 \quad (4)$

$\therefore$ in this question $\theta = 0$ for both projectiles

$\therefore \sin\theta = 0$, $\cos\theta = 1$

- for projectile A, time $= t$

  $\therefore$ for projectile B, time $= t - 10$
  (10 seconds later)

- AND for projectile A:

equation (2) gives: $y = Vt\sin\theta - \frac{1}{2}gt^2 + C_4$

but at $t=0$, $y = 4h$ $\therefore C_4 = 4h$

for projectile B

at $t=0$, $y = h$ $\therefore C_4 = h$

$\therefore$ For A: (3) gives: $x_A = Ut$

'(4)' gives: $y_A = 4h - \frac{1}{2}gt^2$

For B: (3) gives: $x_B = V(t-10)$

$y_B = h - \frac{1}{2}g(t-10)^2$

(QED) 4

7 (a) (ii) Time of flight : when $y=0$

ie. $y(A)=0$

$y(B)=0$

ie. $4h-\frac{1}{2}gt^2=0$

$\therefore h=\frac{1}{8}gt^2$ — (1)

and $h=\frac{1}{2}g(t-10)^2$ — (2)

(1) into (2) : $\frac{1}{8}gt^2=\frac{1}{2}g(t-10)^2$

$\therefore t^2=4(t-10)^2$

$\therefore t=2(t-10)$ or $-t=2(t-10)$

$\therefore t=2t-20$ $\quad \therefore -t=2t-20$

$\therefore t=20$ or $t=20/3$

(t = time of flight for A which must be >10 seconds, for B to be projected 10 seconds after A)

$\therefore t=20$ seconds

and $t-10=10$ seconds

2

$\therefore$ Time of flight : A = 20 sec, B = 10 sec

(iii) When both particles hit ground:

$x_A=x_B$

$\therefore 20u=v(20-10)$

$20u=10v$

$\therefore v=2u$ (QED) 1

b) (i) $x^2=4ay \rightarrow x^2=4y$ $\therefore a=1$

$\therefore$ focus is $(0,1)$, P is $(2t,t^2)$

Divide externally 3:1 $\therefore$ m : n = 3 : −1

$\therefore x=\frac{mx_2+nx_1}{m+n}$

$\therefore x=\frac{3\times 2t-1\times 0}{3-1}$

ie. $x=3t$ — (1)

and $y=\frac{my_2+ny_1}{m+n}$

$\therefore y=\frac{3\times t^2-1\times 1}{3-1}$

$y=\frac{3t^2-1}{2}$ — (2) (QED)

3

(ii) from (1) : $t=x/3$

sub into (2) : $y=\frac{3(x/3)^2-1}{2}$

$=\frac{x^2/3-1}{2}$

$\therefore 2y=x^2/3-1$

$\therefore 6y=x^2-3$ (QED)

Equation of locus $\therefore x^2=6y+3$ ie parabola 1

and $x^2=6(y+\frac{1}{2})$

$=4\times 1\frac{1}{2}(y+\frac{1}{2})$

$\therefore$ focal length of parabola is $1\frac{1}{2}$ 1