Matrices summary

Using augmented matrices for solving systems of equations in 2 variables.

A matrix is a rectangular array of numbers. Each number in the matrix is called an element of the matrix. We can associate a system

$$\begin{aligned} a_1x_1 + b_1x_2 &= k_1 \\ a_2x_1 + b_2x_2 &= k_2 \end{aligned}$$

with an augmented matrix

$$\left(\begin{array}{cc|c} a_1 & b_1 & k_1 \\ a_2 & b_2 & k_2 \end{array}\right) \begin{array}{l} \leftarrow R_1 (\text{row } 1) \\ \leftarrow R_2 (\text{row } 2) \end{array}$$

Two augmented matrices M_1, M_2 are row-equivalent if they are augmented matrices of equivalent systems and we write $M_1 \sim M_2$. An augmented matrix can be transformed into a row equivalent matrix by using row operations:

- Two rows are interchanged ($R_i \leftrightarrow R_j$)
- A row is multiplied by a non-zero constant ($kR_i \rightarrow R_i$)
- A constant multiple of one row is added to another row ($R_i + kR_j \rightarrow R_i$)

A matrix is in row echelon form if

- All nonzero rows (rows with at least one nonzero element) are above any rows of all zeroes (all zero rows, if any, belong at the bottom of the matrix).
- The leading coefficient (the first nonzero number from the left, also called the pivot) of a nonzero row is always strictly to the right of the leading coefficient of the row above it).
- All entries in a column below a leading entry are zeroes (implied by the first two criteria).

A matrix is in reduced row echelon form (also called row canonical form) if it satisfies the following conditions:

- It is in row echelon form.
- Every leading coefficient is 1 and is the only nonzero entry in its column.

If M_2 is in reduced row echelon form, these are the consequences for the original system of equations:

$$M_2 = \left(\begin{array}{cc|c} 1 & 0 & m \\ 0 & 1 & n \end{array}\right) \Rightarrow \text{consistent \& independent} \Rightarrow \begin{cases} 1 \text{ solution} \\ (x_1 = m, x_2 = n) \end{cases}$$

$$M_2 = \left(\begin{array}{cc|c} 1 & m & n \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \text{consistent \& dependent} \Rightarrow \begin{cases} \text{infinitely many solutions} \\ (x_2 = t, x_1 = n - mt) \end{cases}$$

$$M_2 = \left(\begin{array}{cc|c} 1 & m & n \\ 0 & 0 & 1 \end{array}\right) \Rightarrow \text{inconsistent} \Rightarrow \text{no solutions}$$

This can be generalised for systems of more than 2 variables.

Gauss-Jordan elimination

We can use this method to reduce a matrix to reduced row echelon form.

1. Choose the leftmost nonzero column and use appropriate row operations to get a 1 at the top.
2. Use multiples of the first row to get zeros in all places below the 1 obtained in step 1.
3. Hide the top row and first column of the matrix. Repeat steps 1 and 2 with the submatrix (the matrix that remains after deleting the top row and first column). Continue the process (steps 1-3) until it is not possible to go further.
4. We now consider the whole matrix. Begin with the bottom nonzero row and use appropriate multiples of it to get zeros above the leftmost 1. Continue this process, moving up row by row, until the matrix is in reduced row echelon form.

Addition and multiplication by a number

A matrix is $m \times n$ if it has m rows and n columns. If $m = n$, it is a square matrix. A matrix with 1 column is a column matrix. A matrix with 1 row is a row matrix. Two matrices are equal if they have the same dimensions and their corresponding elements are equal. If two matrices have the same dimensions, their sum is a matrix with elements that are the sum of the corresponding elements of the two given matrices. If matrices A, B, C have the same dimensions, addition is commutative and associative $\therefore A + B = B + A$ and $(A+B)+C = A+(B+C)$. A matrix with elements that are all 0 is called the zero matrix. The negative of matrix A is $-A$ where the elements of $-A$ are the negative of the elements of A. $A - B = A + (-B)$. The product of a number k and A is kA and is the matrix whose elements are the product of k and the elements of A. The principal diagonal of

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix}$$

consists of $a_{1,1}, a_{2,2}, ..., a_{m,m}$ if $m \leq n$ or $a_{1,1}, a_{2,2}, ..., a_{n,n}$ if $m \geq n$.

Matrix multiplication

If A is an $m \times p$ matrix and B is a $p \times n$ matrix then the matrix product, AB, of A and B is an $m \times n$ matrix whose element in the i-th row and j-th column is the real number obtained from the product of the i-th row of A and the j-th column of B. Matrix multiplication is not commutative, i.e., generally, $AB \neq BA$. The zero property of real numbers (i.e., if $a, b \in \mathbb{R}$ and $ab = 0$ then at least one of a or b must be 0) is also not satisfied by matrices, i.e., it is possible that $AB = 0$ but $A \neq 0$ and $B \neq 0$ (where 0 is a zero matrix). If $k \in \mathbb{R}$ and sums and products are defined, then the following properties hold for matrices A, B, C:

- Associative property: $A(BC) = (AB)C$
- Left distributive property: $A(B + C) = AB + AC$
- Right distributive property: $(B + C)A = BA + CA$
- Left multiplication property: If $A = B$ then $CA = CB$
- Right multiplication property: If $A = B$ then $AC = BC$

- $k(AB) = (kA)B = A(kB)$

Inverse of a square matrix

An identity matrix for multiplication is of the form

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

If I and M are square of order n (i.e., $n \times n$) then $IM = MI = M$. If there exists a square matrix M^{-1} such that $M^{-1}M = MM^{-1} = I$, M^{-1} is called the multiplicative inverse of M. If $(M|I)$ can be transformed by row operations into $(I|B)$ then $B = M^{-1}$. If we obtain 0's in one or more rows to the left of the vertical line then M^{-1} does not exist.

Matrix equations

If $AX = B$ where A is $n \times n$ and B and X are $n \times 1$ then $X = A^{-1}B$.

Determinants

The determinant of the 1×1 matrix $(a_{1,1})$ is $\det(a_{1,1}) = |a_{1,1}| = a_{1,1}$. If $n > 1$ and A is the $n \times n$ matrix $\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix}$, $\det(A) = |A| = \sum_{j=1}^{n} (-1)^{i+j} a_{i,j} M_{i,j} = \sum_{i=1}^{n} (-1)^{i+j} a_{i,j} M_{i,j}$ where $M_{i,j}$ is the (i,j)-minor of A = the determinant of the matrix A with the i-th row and j-th column removed.

Cramer's Rule: If $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, $b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$ and $Ax = b$ then $x_i = \frac{|A_i|}{|A|}$ for $i = 1, 2, \ldots, n$ where A_i is A with the i-th column replaced by b.

Proof. Let a_i = i-th column of A, e_i = i-th column of the identity matrix I_n of order n and $X_i = I_n$ with the i-th column replaced by x. Then

$$\begin{aligned} x_i &= \frac{|A| x_i}{|A|} \\ &= \frac{|A| (-1)^{i+i} x_i |I_{n-1}|}{|A|} \\ &= \frac{|A||X_i|}{|A|} \\ &= \frac{|AX_i|}{|A|} \\ &= \frac{|A(e_1, \ldots, e_{i-1}, x, e_{i+1}, \ldots, e_n)|}{|A|} \\ &= \frac{|(Ae_1, \ldots, Ae_{i-1}, Ax, Ae_{i+1}, \ldots, Ae_n)|}{|A|} \\ &= \frac{|(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n)|}{|A|} \\ &= \frac{|A_i|}{|A|} \qquad \square \end{aligned}$$

Inverse using adjugate and determinant

$A_{i,j} = (-1)^{i+j} M_{i,j}$ is called the (i, j)-cofactor of A. The transpose, A^T of A is the matrix obtained from A by reflecting elements of A by the principal diagonal (i.e., by swapping $a_{i,j}$ with $a_{j,i}$). The adjugate of A is $\text{adj}(A) = (A^c)^T$ where A^c is the cofactor matrix consisting of elements in the (i, j)-th position as $A_{i,j}$. Since $A \cdot \text{adj}(A) = |A| \cdot I$ then $A^{-1} = \frac{\text{adj}(A)}{|A|}$. Hence also, from this we can clearly see that A^{-1} exists $\Leftrightarrow |A| \neq 0$.

Inverse using Excel

For higher order matrices or systems of equations, it is not efficient to calculate A^{-1} or $A^{-1}b$ by hand, and so we can use Excel to do it.

To find A^{-1} and solve $Ax = b$ type your matrix A and result vector b as follows:

cell	contents	cell	contents	...	cell	contents	cell	contents
A1	$a_{1,1}$	B1	$a_{1,2}$	...	Λ1	$a_{1,n}$	Φ1	b_1
⋮	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋮
An	$a_{n,1}$	Bn	$a_{n,2}$	...	Λn	$a_{n,n}$	Φn	b_n

(where Λ is the n-th letter and Φ is the $(n + 1)$-th letter).

To find A^{-1} select cells A$(n + 1)$: Λ$(2n)$, type =MINVERSE(A1:Λn)
and press Ctrl+Shift+Enter for PC, or Cmd+Enter for Mac.

Once this is done, to find $x = A^{-1}b$ select cells Φ$(n+1)$: Φ$(2n)$ and type =MMULT(A$(n+1)$:Λ$(2n)$,Φ1 : Φn) and press Ctrl+Shift+Enter for PC, or Cmd+Enter for Mac.

Inverse using wolframalpha

This can be done more easily with the wolframalpha iphone app, or website wolframalpha.com by typing

inverse$\{\{a_{1,1}, ..., a_{1,n}\}, ..., \{a_{n,1}, ..., a_{n,n}\}\}$

and

inverse$\{\{a_{1,1}, ..., a_{1,n}\}, ..., \{a_{n,1}, ..., a_{n,n}\}\} * \{\{b_1\}, ..., \{b_n\}\}$ respectively.

Note also that determinants and adjugate matrices can also be found this way. Type

det$\{\{a_{1,1}, ..., a_{1,n}\}, ..., \{a_{n,1}, ..., a_{n,n}\}\}$

and

adj$\{\{a_{1,1}, ..., a_{1,n}\}, ..., \{a_{n,1}, ..., a_{n,n}\}\}$ respectively.