

International Mathematical Olympiad 2009

Question 1. Let n be a positive integer and let $a_1, a_2, a_3, \dots, a_k$ ($k \geq 2$) be distinct integers in the set $\{1, 2, \dots, n\}$ such that n divides $a_i(a_{i+1} - 1)$ for $i = 1, 2, \dots, k-1$. Prove that n does not divide $a_k(a_1 - 1)$.

Question 2. Let ABC be a triangle with circumcentre O . The points P and Q are interior points of the sides CA and AB respectively. Let K, L and M be the midpoints of the segments BP, CQ and PQ respectively, and let Γ be the circle passing through K, L and M . Suppose that the line PQ is tangent to the circle Γ . Prove that $OP = OQ$.

Question 3. Suppose that s_1, s_2, s_3, \dots is a strictly increasing sequence of positive integers such that the sub-sequences $s_{s_1}, s_{s_2}, s_{s_3}, \dots$ and $s_{s_1+1}, s_{s_2+1}, s_{s_3+1}, \dots$ are both arithmetic progressions. Prove that the sequence s_1, s_2, s_3, \dots is itself an arithmetic progression.

Question 4. Let ABC be a triangle with $AB = AC$. The angle bisectors of $\angle CAB$ and $\angle ABC$ meet the sides BC and CA at D and E , respectively. Let K be the incentre of triangle ADC . Suppose that $\angle BEK = 45^\circ$. Find all possible values of $\angle CAB$.

Question 5. Determine all functions f from the set of positive integers to the set of positive integers such that, for all positive integers a and b , there exists a non-degenerate triangle with sides of lengths

$$a, f(b) \text{ and } f(b + f(a) - 1).$$

(A triangle is non-degenerate if its vertices are not collinear.)

Question 6. Let a_1, a_2, \dots, a_n be distinct positive integers and let M be a set of $n - 1$ positive integers not containing $s = a_1 + a_2 + \dots + a_n$. A grasshopper is to jump along the real axis, starting at the point 0 and making n jumps to the right with lengths a_1, a_2, \dots, a_n in some order. Prove that the order can be chosen in such a way that the grasshopper never lands on any point in M .