International Mathematical Olympiads

1st IMO 1959

A1. Prove that $\frac{21n+4}{14n+3}$ is irreducible for every natural number n.

A2. For what real values of x is $\sqrt{x+\sqrt{2x-1}}+\sqrt{x-\sqrt{2x-1}}=A$ given $A=\sqrt{2}$, (b) $A=1$, (c) $A=2$, where only non-negative real numbers are allowed in square roots and the root always denotes the non-negative root?

A3. Let a, b, c be real numbers. Given the equation for $\cos x$:

$$a\cos^2 x+b\cos x+c=0,$$

form a quadratic equation in $\cos 2x$ whose roots are the same values of x. Compare the equations in $\cos x$ and $\cos 2x$ for $a=4, b=2, c=-1$.

B1. Given the length $|AC|$, construct a triangle ABC with $\angle ABC=90^o$, and the median BM satisfying $BM^2=AB\cdot BC$.

B2. An arbitrary point M is taken in the interior of the segment AB. Squares $AMCD$ and $MBEF$ are constructed on the same side of AB. The circles circumscribed about these squares, with centers P and Q, intersect at M and N.

(a) prove that AF and BC intersect at N;
(b) prove that the lines MN pass through a fixed point S (independent of M);
(c) find the locus of the midpoints of the segments PQ as M varies.

B3. The planes P and Q are not parallel. The point A lies in P but not Q, and the point C lies in Q but not P. Construct points B in P and D in Q such that the quadrilateral $ABCD$ satisfies the following conditions: (1) it lies in a plane, (2) the vertices are in the order A, B, C, D, (3) it is an isosceles trapezoid with AB parallel to CD (meaning that $AD=BC$, but AD is not parallel to BC unless it is a square), and (4) a circle can be inscribed in $ABCD$ touching the sides.

2nd IMO 1960

A1. Determine all three digit numbers N which are divisible by 11 and where $\frac{N}{11}$ is equal to the sum of the squares of the digits of N.

A2. For what real values of x does the following inequality hold:

$$\frac{4x^2}{(1-\sqrt{1+2x})^2}<2x+9?$$

A3. In a given right triangle ABC, the hypoteneuse BC, length a, is divided into n equal parts with n an odd integer. The central part subtends an angle α at A. h is the perpendicular distance from A to BC. Prove that

$$\tan\alpha=\frac{4nh}{an^2-a}.$$

B1. Construct a triangle ABC given the lengths of the altitudes from A and B and the length of the median from A.

B2. The cube $ABCDA'B'C'D'$ has A above A', B above B' and so on. X is any point of the face diagonal AC and Y is any point of $B'D'$.

(a) find the locus of the midpoint of XY;

(b) find the locus of the point Z which lies one-third of the way along XY, so that $ZY = 2XZ$.

B3. A cone of revolution has an inscribed sphere tangent to the base of the cone (and to the sloping surface of the cone). A cylinder is circumscribed about the sphere so that its base lies in the base of the cone. The volume of the cone is V_1 and the volume of the cylinder is V_2.

(a) Prove that $V_1 \neq V_2$;

(b) Find the smallest possible value of $\frac{V_1}{V_2}$. For this case construct the half angle of the cone.

B4. In the isosceles trapezoid $ABCD$ (AB parallel to DC, and $BC = AD$), let $AB = a, CD = c$ and let the perpendicular distance from A to CD be h. Show how to construct all points X on the axis of symmetry such that $\angle BXC = \angle AXD = 90^o$. Find the distance of each such X from AB and from CD. What is the condition for such points to exist?

3rd IMO 1961

A1. Solve the following equations for x, y and z:

$$x + y + z = a; x^2 + y^2 + z^2 = b^2; xy = z^2.$$

What conditions must a and b satisfy for x, y and z to be distinct positive numbers?

A2. Let a, b, c be the sides of a triangle and A its area. Prove that:

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}A$$

When do we have equality?

A3. Solve the equation $\cos^n x - \sin^n x = 1$, where n is a natural number.

B1. P is inside the triangle ABC. PA intersects BC in D, PB intersects AC in E, and PC intersects AB in F. Prove that at least one of $\frac{AP}{PD}, \frac{BP}{PE}, \frac{CP}{PF}$ does not exceed 2, and at least one is not less than 2.

B2. Construct the triangle ABC, given the lengths $AC = b, AB = c$ and the acute $\angle AMB = \alpha$, where M is the midpoint of BC. Prove that the construction is possible iff

$$b \tan \frac{\alpha}{2} \leq c < b.$$

When does equality hold?

B3. Given three non-collinear points A, B, C and a plane p not parallel to ABC and such that A, B, C are all on the same side of p. Take three arbitrary points A', B', C' in p. Let A'', B'', C'' be the midpoints of AA', BB', CC' respectively, and let O be the centroid of A'', B'', C''. What is the locus of O as A', B', C' vary?

4th IMO 1962

A1. Find the smallest natural number with 6 as the last digit, such that if the final 6 is moved to the front of the number it is multiplied by 4.

A2. Find all real x satisfying: $\sqrt{3-x} - \sqrt{x+1} > \frac{1}{2}$.

A3. The cube $ABCDA'B'C'D'$ has upper face $ABCD$ and lower face $A'B'C'D'$ with A directly above A' and so on. The point x moves at a constant speed along the perimeter of $ABCD$, and the point Y moves at the same speed along the perimeter of $B'C'CB$. X leaves A towards B at the same moment as Y leaves B' towards C'. What is the locus of the midpoint of XY?

B1. Find all real solutions to $\cos^2 x + \cos^2 2x + \cos^2 3x = 1$.

B2. Given three distinct points A, B, C on a circle K, construct a point D on K, such that a circle can be inscribed in $ABCD$.

B3. The radius of the circumcircle of an isosceles triangle is R and the radius of its inscribed circle is r. Prove that the distance between the two centers is $\sqrt{R(R-2r)}$.

B4. Prove that a regular tetrahedron has five distinct spheres each tangent to its six extended edges. Conversely, prove that if a tetrahedron has five such spheres then it is regular.

5th IMO 1963

A1. For which real values of p does the equation

$$\sqrt{x^2 - p} + 2\sqrt{x^2 - 1} = x$$

have real roots? What are the roots?

A2. Given a point A and a segment BC, determine the locus of all points P in space for which $\angle APX = 90^o$ for some X on the segment BC.

A3. An n-gon has all angles equal and the lengths of consecutive sides satisfy $a_1 \geq a_2 \geq \ldots \geq a_n$. Prove that all the sides are equal.

B1. Find all solutions $x_1, \ldots, x_5$ to the five equations $x_i + x_{i+2} = yx_{i+1}$ for $i = 1, \ldots, 5$, where subscripts are reduced by 5 if necessary.

B2. Prove that $\cos\frac{\pi}{7} - \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} = \frac{1}{2}$.

B3. Five students A, B, C, D, E were placed 1 to 5 in a contest with no ties. One prediction was that the result would be the order A, B, C, D, E. But no student finished in the position predicted and no two students predicted to finish consecutively did so. For example, the outcome for C and D was not $1, 2$ (respectively), or $2, 3$, or $3, 4$ or $4, 5$. Another prediction was the order D, A, E, C, B. Exactly two students finished in the places predicted and two disjoint pairs predicted to finish consecutively did so. Determine the outcome.

6th IMO 1964

A1. (a) Find all natural numbers n for which 7 divides $2^n - 1$.
(b) Prove that there is no natural number n for which 7 divides $2^n + 1$.

A2. Suppose that a, b, c are the sides of a triangle. Prove that:

$$a^2(b + c - a) + b^2(c + a - b) + c^2(a + b - c) \leq 3abc.$$

A3. Triangle ABC has sides a, b, c. Tangents to the inscribed circle are constructed parallel to the sides. Each tangent forms a triangle with the other two sides of the triangle and a circle is inscribed in each of these triangles. Find the total area of all four inscribed circles.

B1. Each pair from 17 people exchange letters on one of three topics. Prove that there are at least 3 people who write to each other on the same topic. [In other words, if we color the edges of the complete graph K_17 with three colors, then we can find a triangle all the same color.]

B2. 5 points in a plane are situated so that no two of the lines joining a pair of points are coincident, parallel or perpendicular. Through each point lines are drawn perpendicular to each of the lines through two of the other 4 points. Determine the maximum number of intersections these perpendiculars can have.

B3. $ABCD$ is a tetrahedron and D_0 is the centroid of ABC. Lines parallel to DD_0 are drawn through A, B and C and meet the planes BCD, CAD and ABD in A_0, B_0 and C_0 respectively. Prove that the volume of $ABCD$ is one-third of the volume of $A_0B_0C_0D_0$. Is the result true if D_0 is an arbitrary point inside ABC?

7th IMO 1965

A1. Find all x in the interval $[0, 2\pi]$ which satisfy:

$$2\cos x \leq |\sqrt{1 + \sin 2x} - \sqrt{1 - \sin 2x}| \leq \sqrt{2}.$$

A2. The coefficients a_{ij} of the following equations
$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = 0$
$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = 0$
$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = 0$
satisfy the following: (a) a_{11}, a_{22}, a_{33} are positive, (b) other a_{ij} are negative, (c) the sum of the coefficients in each equation is positive. Prove that the only solution is $x_1 = x_2 = x_3 = 0$.

A3. The tetrahedron $ABCD$ is divided into two parts by a plane parallel to AB and CD. The distance of the plane from AB is k times its distance from CD. Find the ratio of the volumes of the two parts.

B1. Find all sets of four real numbers such that the sum of any one and the product of the other three is 2.

B2. The triangle OAB has $\angle O$ acute. M is an arbitrary point on AB. P and Q are the feet of the perpendiculars from M to OA and OB respectively. What is the locus of H, the orthocenter of the triangle OPQ (the point where its altitudes meet)? What is the locus if M is allowed to vary over the interior of OAB?

B3. Given $n > 2$ points in the plane, prove that at most n pairs of points are the maximum distance apart (of any two points in the set).

8th IMO 1966

A1. Problems A, B and C were posed in a mathematical contest. 25 competitors solved at least one of the three. Amongst those who did not solve A, twice as many solved B as C. The number solving only A was one more than the number solving A and at least one other. The number solving just A equalled the number solving just B plus the number solving just C. How many solved just B?

A2. Prove that if $BC + AC = \tan \frac{C}{2}(BC \tan A + AC \tan B)$, then the triangle ABC is isosceles.

A3. Prove that a point in space has the smallest sum of the distances to the vertices of a regular tetrahedron iff it is the center of the tetrahedron.

B1. Prove that $\frac{1}{\sin 2x} + \frac{1}{\sin 4x} + \ldots + \frac{1}{\sin 2^n x} = \cot x - \cot 2^n x$ for any natural number n and any real x (with $\sin 2^n x$ non-zero).

B2. Solve the equations

$$|a_i - a_1|x_1 + |a_i - a_2|x_2 + |a_i - a_3|x_3 + |a_i - a_4|x_4 = 1, i = 1, 2, 3, 4,$$

where a_i are distinct reals.

B3. Take any points K, L, M on the sides BC, CA, AB of the triangle ABC. Prove that at least one of the triangles AML, BKM, CLK has area $\leq \frac{1}{4}$ area ABC.

9th IMO 1967

A1. The parallelogram $ABCD$ has $AB = a, AD = 1, \angle BAD = A$, and the triangle ABD has all angles acute. Prove that circles radius 1 and center A, B, C, D cover the parallelogram iff

$$a \leq \cos A + \sqrt{3} \sin A.$$

A2. Prove that a tetrahedron with just one edge length greater than 1 has volume at most $\frac{1}{8}$.

A3. Let k, m, n be natural numbers such that $m + k + 1$ is a prime greater than $n + 1$. Let $c_s = s(s + 1)$. Prove that

$$(c_{m+1} - c_k)(c_{m+2} - c_k) \ldots (c_{m+n} - c_k)$$

is divisible by the product $c_1 c_2 \ldots c_n$.

B1. $A_0B_0C_0$ and $A_1B_1C_1$ are acute-angled triangles. Construct the triangle ABC with the largest possible area which is circumscribed about $A_0B_0C_0$ (so BC contains B_0, CA contains B_0, and AB contains C_0) and similar to $A_1B_1C_1$.

B2. $a_1, \ldots, a_8$ are reals, not all zero. Let $c_n = a_1^n + a_2^n + \ldots + a_8^n$ for $n = 1, 2, 3, \ldots$. Given that an infinite number of c_n are zero, find all n for which c_n is zero.

B3. In a sports contest a total of m medals were awarded over n days. On the first day one medal and $\frac{1}{7}$ of the remaining medals were awarded. On the second day two medals and $\frac{1}{7}$ of the remaining medals were awarded, and so on. On the last day, the remaining n medals were awarded. How many medals were awarded, and over how many days?

10th IMO 1968

A1. Find all triangles whose side lengths are consecutive integers, and one of whose angles is twice another.

A2. Find all natural numbers n the product of whose decimal digits is $n^2 - 10n - 22$.

A3. a, b, c are real with a non-zero. $x_1, x_2, \ldots, x_n$ satisfy the n equations:
$ax_i^2 + bx_i + c = x_{i+1}$, for $1 \leq i < n$
$ax_n^2 + bx_n + c = x_1$
Prove that the system has zero, 1 or > 1 real solutions according as $(b-1)^2 - 4ac$ is $< 0, = 0$, or > 0.

B1. Prove that every tetrahedron has a vertex whose three edges have the right lengths to form a triangle.

B2. Let f be a real-valued function defined for all real numbers, such that for some $a > 0$ we have
$f(x+a) = \frac{1}{2} + \sqrt{f(x) - f(x)^2}$ for all x.
Prove that f is periodic, and give an example of such a non-constant f for $a = 1$.

B3. For every natural number n evaluate the sum

$$\left[\frac{n+1}{2}\right] + \left[\frac{n+2}{4}\right] + \left[\frac{n+4}{8}\right] + \ldots + \left[\frac{n+2^k}{2^{k+1}}\right] + \ldots,$$

where $[x]$ denotes the greatest integer $\leq x$.

11th IMO 1969

A1. Prove that there are infinitely many positive integers m, such that $n^4 + m$ is not prime for any positive integer n.

A2. Let $f(x) = \cos(a_1 + x) + \frac{1}{2}\cos(a_2 + x) + \frac{1}{4}\cos(a_3 + x) + \ldots + \frac{1}{2^{n-1}}\cos(a_n + x)$, where a_i are real constants and x is a real variable. If $f(x_1) = f(x_2) = 0$, prove that $x_1 - x_2$ is a multiple of π.

A3. For each of $k = 1, 2, 3, 4, 5$ find necessary and sufficient conditions on $a > 0$ such that there exists a tetrahedron with k edges length a and the remainder length 1.

B1. C is a point on the semicircle diameter AB, between A and B. D is the foot of the perpendicular from C to AB. The circle K_1 is the incircle of ABC, the circle K_2 touches CD, DA and the semicircle, the circle K_3 touches CD, DB and the semicircle. Prove that K_1, K_2 and K_3 have another common tangent apart from AB.

B2. Given $n > 4$ points in the plane, no three collinear. Prove that there are at least $\frac{(n-3)(n-4)}{2}$ convex quadrilaterals with vertices amongst the n points.

B3. Given real numbers $x_1, x_2, y_1, y_2, z_1, z_2$ satisfying $x_1 > 0, x_2 > 0, x_1y_1 > z_1^2$, and $x_2y_2 > z_2^2$, prove that:

$$\frac{8}{(x_1+x_2)(y_1+y_2)-(z_1+z_2)^2} \leq \frac{1}{x_1y_1-z_1^2} + \frac{1}{x_2y_2-z_2^2}.$$

Give necessary and sufficient conditions for equality.

12th IMO 1970

A1. M is any point on the side AB of the triangle ABC. r, r_1, r_2 are the radii of the circles inscribed in ABC, AMC, BMC. q is the radius of the circle on the opposite side of AB to C, touching the three sides of AB and the extensions of CA and CB. Similarly, q_1 and q_2. Prove that $r_1r_2q = rq_1q_2$.

A2. We have $0 \leq x_i < b$ for $i = 0, 1, \ldots, n$ and $x_n > 0, x_{n-1} > 0$. If $a > b$, and $x_nx_{n-1}\ldots x_0$ represents the number A base a and B base b, whilst $x_{n-1}x_{n-2}\ldots x_0$ represents the number A' base a and B' base b, prove that $A'B < AB'$.

A3. The real numbers $a_0, a_1, a_2, \ldots$ satisfy $1 = a_0 \leq a_1 \leq a_2 \leq \ldots . b_1, b_2, b_3, \ldots$ are defined by $b_n = \sum_1^n \frac{1-\frac{a_{k-1}}{a_k}}{\sqrt{a_k}}$.

(a) Prove that $0 \leq b_n < 2$.

(b) Given c satisfying $0 \leq c < 2$, prove that we can find a_n so that $b_n > c$ for all sufficiently large n.

B1. Find all positive integers n such that the set $\{n, n+1, n+2, n+3, n+4, n+5\}$ can be partitioned into two subsets so that the product of the numbers in each subset is equal.

B2. In the tetrahedron $ABCD, \angle BDC = 90^o$ and the foot of the perpendicular from D to ABC is the intersection of the altitudes of ABC. Prove that:

$$(AB+BC+CA)^2 \leq 6(AD^2+BD^2+CD^2).$$

When do we have equality?

B3. Given 100 coplanar points, no three collinear, prove that at most 70% of the triangles formed by the points have all angles acute.

13th IMO 1971

A1. Let $E_n = (a_1-a_2)(a_1-a_3)\ldots(a_1-a_n) + (a_2-a_1)(a_2-a_3)\ldots(a_2-a_n) + \ldots + (a_n-a_1)(a_n-a_2)\ldots(a_n-a_{n-1})$. Let S_n be the proposition that $E_n \geq 0$ for all real a_i. Prove that S_n is true for $n = 3$ and 5, but for no other $n > 2$.

A2. Let P_1 be a convex polyhedron with vertices $A_1, A_2, \ldots, A_9$. Let P_i be the polyhedron obtained from P_1 by a translation that moves A_1 to A_i. Prove that at least two of the polyhedra $P_1, P_2, \ldots, P_9$ have an interior point in common.

A3. Prove that we can find an infinite set of positive integers of the from $2^n - 3$ (where n is a positive integer) every pair of which are relatively prime.

B1. All faces of the tetrahedron $ABCD$ are acute-angled. Take a point X in the interior of the segment AB, and similarly Y in BC, Z in CD and T in AD.

(a) If $\angle DAB + \angle BCD \neq \angle CDA + \angle ABC$, then prove none of the closed paths $XYZTX$ has minimal length;

(b) If $\angle DAB + \angle BCD = \angle CDA + \angle ABC$, then there are infinitely many shortest paths $XYZTX$, each with length $2AC \sin k$, where $2k = \angle BAC + \angle CAD + \angle DAB$.

B2. Prove that for every positive integer m we can find a finite set S of points in the plane, such that given any point A of S, there are exactly m points in S at unit distance from A.

B3. Let $A = (a_{ij})$, where $i, j = 1, 2, \ldots, n$, be a square matrix with all a_{ij} non-negative integers. For each i, j such that $a_{ij} = 0$, the sum of the elements in the ith row and the jth column is at least n. Prove that the sum of all the elements in the matrix is at least $\frac{n^2}{2}$.

14th IMO 1972

A1. Given any set of ten distinct numbers in the range $10, 11, \ldots, 99$, prove that we can always find two disjoint subsets with the same sum.

A2. Given $n > 4$, prove that every cyclic quadrilateral can be dissected into n cyclic quadrilaterals.

A3. Prove that $(2m)!(2n)!$ is a multiple of $m!n!(m+n)!$ for any non-negative integers m and n.

B1. Find all positive real solutions to:

$(x_1^2 - x_3x_5)(x_2^2 - x_3x_5) \leq 0$
$(x_2^2 - x_4x_1)(x_3^2 - x_4x_1) \leq 0$
$(x_3^2 - x_5x_2)(x_4^2 - x_5x_2) \leq 0$
$(x_4^2 - x_1x_3)(x_5^2 - x_1x_3) \leq 0$
$(x_5^2 - x_2x_4)(x_1^2 - x_2x_4) \leq 0$

B2. f and g are real-valued functions defined on the real line. For all x and y, $f(x+y)+f(x-y) = 2f(x)g(y)$. f is not identically zero and $|f(x)| \leq 1$ for all x. Prove that $|g(x)| \leq 1$ for all x.

B3. Given four distinct parallel planes, prove that there exists a regular tetrahedron with a vertex on each plane.

15th IMO 1973

A1. $OP_1, OP_2, \ldots, OP_{2n+1}$ are unit vectors in a plane. $P_1, P_2, \ldots, P_{2n+1}$ all lie on the same side of a line through O. Prove that $|OP_1 + \ldots + OP_{2n+1}| \geq 1$.

A2. Can we find a finite set of non-coplanar points, such that given any two points, A and B, there are two others C and D, with the lines AB and CD parallel and distinct?

A3. a and b are real numbers for which the equation $x^4 + ax^3 + bx^2 + ax + 1 = 0$ has at least one real solution. Find the least possible value of $a^2 + b^2$.

B1. A soldier needs to sweep a region with the shape of an equilateral triangle for mines. The detector has an effective radius equal to half the altitude of the triangle. He starts at a vertex of the triangle. What path should he follow in order to travel the least distance and still sweep the whole region?

B2. G is a set of non-constant functions f. Each f is defined on the real line and has the form $f(x) = ax+b$ for some real a, b. If f and g are in G, then so is fg, where fg is defined by $fg(x) = f(g(x))$. If f is in G, then so is the inverse f^{-1}. If $f(x) = ax + b$, then $f^{-1}(x) = \frac{x-b}{a}$. Every f in G has a fixed point (in other words we can find x_f such that $f(x_f) = x_f$. Prove that all the functions in G have a common fixed point.

B3. $a_1, a_2, \ldots, a_n$ are positive reals, and q satisfies $0 < q < 1$. Find $b_1, b_2, \ldots, b_n$, such that:

(a) $a_i < b_i$ for $i = 1, 2, \ldots, n$,

(b) $q < \frac{b_{i+1}}{b_i} < \frac{1}{q}$ for $i = 1, 2, \ldots, n-1$,

(c) $b_1 + b_2 + \ldots + b_n < (a_1 + a_2 + \ldots + a_n)\frac{1+q}{1-q}$.

16th IMO 1974

A1. Three players play the following game. There are three cards each with a different positive integer. In each round, the cards are randomly dealt to the players and each receives the number of counters on his card. After two or more rounds, one player has received 20, another 10 and the third 9 counters. In the last round the player with 10 received the largest number of counters. Who received the middle number on the first round?

A2. Prove that there is a point D on the side AB of the triangle ABC, such that CD is the geometric mean of AD and DB iff $\sin A \sin B \leq \sin^2 \frac{C}{2}$.

A3. Prove that $\sum_0^n \binom{2n+1}{2k+1} 2^{3k}$ is not divisible by 5 for any non-negative integer n.

B1. An 8×8 chessboard is divided into p disjoint rectangles (along the lines between the squares), so that each rectangle has the same number of white squares as black squares, and each rectangle has a different number of squares. Find the maximum possible value of p and all possible sets of rectangle sizes.

B2. Determine all possible values of $\frac{a}{a+b+d} + \frac{b}{a+b+c} + \frac{c}{b+c+d} + \frac{d}{a+c+d}$ for positive reals a, b, c, d.

B3. Let $P(x)$ be a polynomial with integer coefficients of degree $d > 0$. Let n be the number of distinct integer roots to $P(x) = 1$ or -1. Prove that $n \leq d + 2$.

17th IMO 1975

A1. Let $x_1 \geq x_2 \geq \ldots \geq x_n$, and $y_1 \geq y_2 \geq \ldots \geq y_n$ be real numbers. Prove that if z_i is any permutation of the y_i, then:

$$\sum_1^n (x_i - y_i)^2 \leq \sum_1^n (x_i - z_i)^2.$$

A2. Let $a_1 < a_2 < a_3 < \ldots$ be positive integers. Prove that for every $i \geq 1$, there are infinitely many a_n that can be written in the form $a_n = ra_i + sa_j$, with r, s positive integers and $j > i$.

A3. Given any triangle ABC, construct external triangles ABR, BCP, CAQ on the sides, so that $\angle PBC = 45^o, \angle PCB = 30^o, \angle QAC = 45^o, \angle QCA = 30^o, \angle RAB = 15^o, \angle RBA = 15^o$. Prove that $\angle QRP = 90^o$ and $QR = RP$.

B1. Let A be the sum of the decimal digits of 4444^{4444}, and B the sum of the decimal digits of A. Find the sum of the decimal digits of B.

B2. Find 1975 points on the circumference of a unit circle such that the distance between each pair is rational, or prove it impossible.

B3. Find all polynomials $P(x, y)$ in two variables such that:
(1) $P(tx, ty) = t^n P(x, y)$ for some positive integer n and all real t, x, y:
(2) for all real $x, y, z : P(y + z, x) + P(z + x, y) + P(x + y, z) = 0$;
(3) $P(1, 0) = 1$.

18th IMO 1976

A1. A plane convex quadrilateral has area 32, and the sum of two opposite sides and a diagonal is 16. Determine all possible lengths for the other diagonal.

A2. Let $P_1(x) = x^2 - 2$, and $P_{i+1} = P_1(P_i(x))$ for $i = 1, 2, 3, \ldots$. Show that the roots of $P_n(x) = x$ are real and distinct for all n.

A3. A rectangular box can be completely filled with unit cubes. If one places as many cubes as possible, each with volume 2, in the box, with their edges parallel to the edges of the box, one can fill exactly 40% of the box. Determine the possible dimensions of the box.

B1. Determine the largest number which is the product of positive integers with sum 1976.

B2. n is a positive integer and $m = 2n$. $a_{ij} = 0, 1$ or -1 for $1 \leq i \leq n, 1 \leq j \leq m$. The m unknowns $x_1, x_2, \ldots, x_m$ satisfy the n equations:

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{im}x_m = 0,$$

for $i = 1, 2, \ldots, n$. Prove that the system has a solution in integers of absolute value at most m, not all zero.

B3. The sequence $u_0, u_1, u_2, \ldots$ is defined by: $u_0, u_1 = \frac{5}{2}, u_{n+1} = u_n(u_{n-1}^2 - 2) - u_1$ for $n = 1, 2, \ldots$. Prove that $[u_n] = 2^{\frac{2^n - (-1)^n}{3}}$, where $[x]$ denotes the greatest integer less than or equal to x.

19th IMO 1977

A1. Construct equilateral triangles ABK, BCL, CDM, DAN on the inside of a square $ABCD$. Show that the midpoints of KL, LM, MN, NK and the midpoints of $AK, BK, BL, CL, CM, DM, DN, AN$ form a regular dodecahedron.

A2. In a finite sequence of real numbers the sum of any seven successive terms is negative, and the sum of any eleven successive terms is positive. Determine the maximum number of terms in the sequence.

A3. Given an integer $n > 2$, let V_n be the set of integers $1 + kn$ for k a positive integer. A number m in V_n is called indecomposable if it cannot be expressed as the product of two members of V_n. Prove that there is a number in V_n which can be expressed as the product of indecomposable members of V_n in more than one way (decompositions which differ solely in the order of factors are not regarded as different).

B1. Define $f(x) = 1 - a\cos x - b\sin x - A\cos 2x - B\sin 2x$, where a, b, A, B are real constants. Suppose $f(x) \geq)$ for all real x. Prove that $a^2 + b^2 \leq 2$ and $A^2 + B^2 \leq 1$.

B2. Let a and b be positive integers. When $a^2 + b^2$ is divided by $a + b$, the quotient is q and the remainder is r. Find all pairs a, b such that $q^2 + r = 1977$.

B3. The function f is defined on the set of positive integers and its values are positive integers. Given that $f(n+1) > f(f(n))$ for all n, prove that $f(n) = n$ for all n.

20th IMO 1978

A1. m and n are positive integers with $m < n$. The last three decimal digits of 1978^m are the same as the last three decimal digits of 1978^n. Find m and n such that $m + n$ has the least possible value.

A2. P is a point inside a sphere. Three mutually perpendicular rays from P intersect the sphere at points U, V and W. Q denotes the vertex diagonally oppposite P in the parallelepiped determined by PU, PV, PW. Find the locus of Q for all possible sets of such rays from P.

A3. The set of all positive integers is the union of two disjoint subsets $\{f(1), f(2), f(3), \ldots\}$, $\{g(1), g(2), g(3), \ldots\}$, where $f(1) < f(2) < \ldots$, and $g(1) < g(2) < g(3) < \ldots$, and $g(n) = f(f(n)) + 1$ for $n = 1, 2, 3, \ldots$. Determine $f(240)$.

B1. In the triangle ABC, $AB = AC$. A circle is tangent internally to the circumcircle of the triangle and also to AB, AC at P, Q respectively. Prove that the midpoint of PQ is the center of the incircle of the triangle.

B2. $\{a_k\}$ is a sequence of distinct positive integers. Prove that for all positive integers n, $\sum_1^n \frac{a_k}{k^2} \geq \sum_1^n \frac{1}{k}$.

B3. An international society has its members from six different countries. The list of members has 1978 names, numbered $1, 2, \ldots, 1978$. Prove that there is at least one member whose number is the sum of the numbers of two members from his own country, or twice the number of a member from his own country.

21st IMO 1979

A1. Let m and n be positive integers such that

$$\frac{m}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots - \frac{1}{1318} + \frac{1}{1319}.$$

Prove that m is divisible by 1979.

A2. A prism with pentagons $A_1A_2A_3A_4A_5$ and $B_1B_2B_3B_4B_5$ as the top and bottom faces is given. Each side of the two pentagons and each of the 25 segments A_iB_j is colored red or green. Every triangle whose vertices are vertices of the prism and whose sides have all been colored has two sides of a different color. Prove that all 10 sides of the top and bottom faces have the same color.

A3. Two circles in a plane intersect. A is one of the points of intersection. Starting simultaneously from A two points move with constant speed, each traveling along is own circle in the same sense. The two points return to A simultaneously after one revolution. Prove that there is a fixed point P in the plane such that the two points are always equidistant from P.

B1. Given a plane k, a point P in the plane and a point Q not in the plane, find all points R in k such that the ratio $\frac{QP+PR}{QR}$ is a maximum.

B2. Find all real numbers a for which there exist non-negative real numbes x_1, x_2, x_3, x_4, x_5 satisfying:
$x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 = a$,
$x_1 + 2^3x_2 + 3^3x_3 + 4^3x_4 + 5^3x_5 = a^2$,
$x_1 + 2^5x_2 + 3^5x_3 + 4^5x_4 + 5^5x_5 = a^3$.

B3. Let A and E be opposite vertices of an octagon. A frog starts at vertex A. From any vertex except E it jumps to one of the two adjacent vertices. When it reaches E it stops. Let a_n be the number of distinct paths of exactly n jumps ending at E. Prove that: $a_{2n-1} = 0, a_{2n} = \frac{(2+\sqrt{2})^{n-1}}{\sqrt{2}} - \frac{(2-\sqrt{2})^{n-1}}{\sqrt{2}}$.

22nd IMO 1981

A1. P is a point inside the triangle ABC. D, E, F are the feet of the perpendiculars from P to the lines BC, CA, AB respectively. Find all P which minimize:

$$\frac{BC}{PD} + \frac{CA}{PE} + \frac{AB}{PF}.$$

A2. Take r such that $1 \le r \le n$, and consider all subsets of r elements of the set $\{1, 2, \ldots, n\}$. Each subset has a smallest element. Let $F(n, r)$ be the arithmetic mean of these smallest elements. Prove that:

$$F(n, r) = \frac{n+1}{r+1}.$$

A3. Determine the maximum value of $m^2 + n^2$, where m and n are integers in the range $1, 2, \ldots, 1981$ satisfying $(n^2 - mn - m^2)^2 = 1$.

B1. (a) For which $n > 2$ is there a set of n consecutive positive integers such that the largest number in the set is a divisor of the least common multiple of the remaining $n - 1$ numbers?
(b) For which $n > 2$ is there exactly one set having this property?

B2. Three circles of equal radius have a common point O and lie inside a given triangle. Each circle touches a pair of sides of the triangle. Prove that the incenter and the circumcenter of the triangle are collinear with the point O.

B3. The function $f(x, y)$ satisfies: $f(0, y) = y + 1, f(x + 1, 0) = f(x, 1), f(x + 1, y + 1) = f(x, f(x + 1, y))$ for all non-negative integers x, y. Find $f(4, 1981)$.

23rd IMO 1982

A1. The function $f(n)$ is defined on the positive integers and takes non-negative integer values. $f(2) = 0, f(3) > 0, f(9999) = 3333$ and for all m, n: $f(m+n) - f(m) - f(n) = 0$ or 1. Determine $f(1982)$.

A2. A non-isosceles triangle $A_1A_2A_3$ has sides a_1, a_2, a_3 with a_i opposite A_i. M_i is the midpoint of side a_i and T_i is the point where the incircle touches side a_i. Denote by S_i the reflection of T_i in the interior bisector of angle A_i. Prove that the lines M_1S_1, M_2S_2 and M_3S_3 are concurrent.

A3. Consider infinite sequences $\{x_n\}$ of positive reals such that $x_0 = 1$ and $x_0 \geq x_1 \geq x_2 \geq \ldots$.

(a) Prove that for every such sequence there is an $n \geq 1$ such that:

$$\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \ldots + \frac{x_{n-1}^2}{x_n} \geq 3.999.$$

(b) Find such a sequence such that for all n:

$$\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \ldots + \frac{x_{n-1}^2}{x_n} < 4.$$

B1. Prove that if n is a positive integer such that the equation

$$x^3 - 3xy^2 + y^3 = n$$

has a solution in integers x, y, then it has at least three such solutions. Show that the equation has no solutions in integers for $n = 2891$.

B2. The diagonals AC and CE of the regular hexagon $ABCDEF$ are divided by inner points M and N respectively, so that

$$\frac{AM}{AC} = \frac{CN}{CE} = r.$$

Determine r if B, M and N are collinear.

B3. Let S be a square with sides length 100. Let L be a path within S which does not meet itself and which is composed of line segments $A_0A_1, A_1A_2, A_2A_3, \ldots, A_{n-1}A_n$ with $A_0 = A_n$. Suppose that for every point P on the boundary of S there is a point of L at a distance from P no greater than $\frac{1}{2}$. Prove that there are two points X and Y of L such that the distance between X and Y is not greater than 1 and the length of the part of L which lies between X and Y is not smaller than 198.

24th IMO 1983

A1. Find all functions f defined on the set of positive reals which take positive real values and satisfy: $f(xf(y)) = yf(x)$ for all x, y; and $f(x) \to 0$ as $x \to \infty$.

A2. Let A be one of the two distinct points of intersection of two unequal coplanar circles C_1 and C_2 with centers O_1 and O_2 respectively. One of the common tangents to the circles touches C_1 at P_1 and C_2 at P_2, while the other touches C_1 at Q_1 and C_2 at Q_2. Let M_1 be the midpoint of P_1Q_1 and M_2 the midpoint of P_2Q_2. Prove that $\angle O_1AO_2 = \angle M_1AM_2$.

A3. Let a, b and c be positive integers, no two of which have a common divisor greater than 1. Show that $2abc - ab - bc - ca$ is the largest integer which cannot be expressed in the form $xbc + yca + zab$, where x, y, z are non-negative integers.

B1. Let ABC be an equilateral triangle and E the set of all points contained in the three segments AB, BC and CA (including A, B and C). Determine whether, for every partition of E into two disjoint subsets, at least one of the two subsets contains the vertices of a right-angled triangle.

B2. Is it possible to choose 1983 distinct positive integers, all less than or equal to 10^5, no three of which are consecutive terms of an arithmetic progression?

B3. Let a, b and c be the lengths of the sides of a triangle. Prove that

$$a^2b(a-b) + b^2c(b-c) + c^2a(c-a) \geq 0.$$

Determine when equality occurs.

25th IMO 1984

A1. Prove that $0 \leq yz + zx + xy - 2xyz \leq \frac{7}{27}$, where x, y and z are non-negative real numbers satisfying $x + y + z = 1$.

A2. Find one pair of positive integers a, b such that $ab(a+b)$ is not divisible by 7, but $(a+b)^7 - a^7 - b^7$ is divisible by 7^7.

A3. Given points O and A in the plane. Every point in the plane is colored with one of a finite number of colors. Given a point X in the plane, the circle $C(X)$ has center O and radius $OX + \frac{\angle AOX}{OX}$, where $\angle AOX$ is measured in radians in the range $[0, 2\pi)$. Prove that we can find a point X, not on OA, such that its color appears on the circumference of the circle $C(X)$.

B1. Let $ABCD$ be a convex quadrilateral with the line CD tangent to the circle on diameter AB. Prove that the line AB is tangent to the circle on diameter CD iff BC and AD are parallel.

B2. Let d be the sum of the lengths of all the diagonals of a plane convex polygon with $n > 3$ vertices. Let p be its perimeter. Prove that:

$$n - 3 < \frac{2d}{p} < \left[\frac{n}{2}\right]\left[\frac{n+1}{2}\right] - 2,$$

where $[x]$ denotes the greatest integer not exceeding x.

B3. Let a, b, c, d be odd integers such that $0 < a < b < c < d$ and $ad = bc$. Prove that if $a + d = 2^k$ and $b + c = 2^m$ for some integers k and m, then $a = 1$.

26th IMO 1985

A1. A circle has center on the side AB of the cyclic quadrilateral $ABCD$. The other three sides are tangent to the circle. Prove that $AD + BC = AB$.

A2. Let n and k be relatively prime positive integers with $k < n$. Each number in the set $M = \{1, 2, 3, \ldots, n-1\}$ is colored either blue or white. For each i in M, both i and $n-i$ have the same color. For each $i \neq k$ in M both i and $|i-k|$ have the same color. Prove that all numbers in M must have the same color.

A3. For any polynomial $P(x) = a_0 + a_1 x + \ldots + a_k x^k$ with integer coefficients, the number of odd coefficients is denoted by $o(P)$. For $i - 0, 1, 2, \ldots$ let $Q_i(x) = (1+x)^i$. Prove that if $i_1, i_2, \ldots, i_n$ are integers satisfying $0 \leq i_1 < i_2 < \ldots < i_n$, then:

$$o(Q_{i_1} + Q_{i_2} + \ldots + Q_{i_n}) \geq o(Q_{i_1}).$$

B1. Given a set M of 1985 distinct positive integers, none of which has a prime divisor greater than 23, prove that M contains a subset of 4 elements whose product is the 4th power of an integer.

B2. A circle center O passes through the vertices A and C of the triangle ABC and intersects the segments AB and BC again at distinct points K and N respectively. The circumcircles of ABC and KBN intersect at exactly two distinct points B and M. Prove that $\angle OMB$ is a right angle.

B3. For every real number x_1, construct the sequence $x_1, x_2, \ldots$ by setting:

$$x_{n+1} = x_n(x_n + \frac{1}{n}).$$

Prove that there exists exactly one value of x_1 which gives $0 < x_n < x_{n+1} < 1$ for all n.

27th IMO 1986

A1. Let d be any positive integer not equal to $2, 5$ or 13. Show that one can find distinct a, b in the set $\{2, 5, 13, d\}$ such that $ab - 1$ is not a perfect square.

A2. Given a point P_0 in the plane of the triangle $A_1A_2A_3$. Define $A_s = A_{s-3}$ for all $s \geq 4$. Construct a set of points $P_1, P_2, P_3, \ldots$ such that P_{k+1} is the image of P_k under a rotation center A_{k+1} through an angle 120^o clockwise for $k = 0, 1, 2, \ldots$. Prove that if $P_{1986} = P_0$, then the triangle $A_1A_2A_3$ is equilateral.

A3. To each vertex of a regular pentagon an integer is assigned, so that the sum of all five numbers is positive. If three consecutive vertices are assigned the numbers x, y, z respectively, and $y < 0$, then the following operation is allowed: x, y, z are replaced by $x + y, -y, z + y$ respectively. Such an operation is performed repeatedly as long as at least one of the five numbers is negative. Determine whether this procedure necessarily comes to an end after a finite number of steps.

B1. Let A, B be adjacent vertices of a regular n-gon ($n \geq 5$) with center O. A triangle XYZ, which is congruent to and initially coincides with OAB, moves in the plane in such a way that Y and Z each trace out the whole boundary of the polygon, with X remaining inside the polygon. Find the locus of X.

B2. Find all functions f defined on the non-negative reals and taking non-negative real values such that: $f(2) = 0, f(x) \neq 0$ for $0 \leq x < 2$, and $f(xf(y))f(y) = f(x+y)$ for all x, y.

B3. Given a finite set of points in the plane, each with integer coordinates, is it always possible to color the points red or white so that for any straight line L parallel to one of the coordinate axes the difference (in absolute value) between the numbers of white and red points on L is not greater than 1?

28th IMO 1987

A1. Let $p_n(k)$ be the number of permutations of the set $\{1, 2, 3, \ldots, n\}$ which have exactly k fixed points. Prove that $\sum_0^n kp_n(k) = n!$.

A2. In an acute-angled triangle ABC the interior bisector of angle A meets BC at L and meets the circumcircle of ABC again at N. From L perpendiculars are drawn to AB and AC, with feet K and M respectively. Prove that the quadrilateral $AKNM$ and the triangle ABC have equal areas.

A3. Let $x_1, x_2, \ldots, x_n$ be real numbers satisfying $x_1^2 + x_2^2 + \ldots + x_n^2 = 1$. Prove that for every integer $k \geq 2$ there are integers $a_1, a_2, \ldots, a_n$, not all zero, such that $|a_i| \leq k - 1$ for all i, and $|a_1x_1 + a_2x_2 + \ldots + a_nx_n| \leq \frac{(k-1)\sqrt{n}}{k^n-1}$.

B1. Prove that there is no function f from the set of non-negative integers into itself such that $f(f(n)) = n + 1987$ for all n.

B2. Let $n \geq 3$ be an integer. Prove that there is a set of n points in the plane such that the distance between any two points is irrational and each set of three points determines a non-degenerate triangle with rational area.

B3. Let $n \geq 2$ be an integer. Prove that if $k^2 + k + n$ is prime for all integers k such that $0 \leq k \leq \sqrt{\frac{n}{3}}$, then $k^2 + k + n$ is prime for all integers k such that $0 \leq k \leq n - 2$.

29th IMO 1988

A1. Consider two coplanar circles of radii $R > r$ with the same center. Let P be a fixed point on the smaller circle and B a variable point on the larger circle. The line BP meets the larger circle again at C. The perpendicular to BP at P meets the smaller circle again at A (if it is tangent to the circle at P, then $A = P$).

(i) Find the set of values of $AB^2 + BC^2 + CA^2$.

(ii) Find the locus of the midpoint of BC.

A2. Let n be a positive integer and let $A_1, A_2, \ldots, A_{2n+1}$ be subsets of a set B. Suppose that:

(i) Each A_i has exactly $2n$ elements,

(ii) The intersection of every two distinct A_i contains exactly one element, and

(iii) Every element of B belongs to at least two of the A_i.

For which values of n can one assign to every element of B one of the numbers 0 and 1 in such a way that each A_i has 0 assigned to exactly n of its elements?

A3. A function f is defined on the positive integers by: $f(1) = 1, f(3) = 3, f(2n) = f(n), f(4n + 1) = 2f(2n + 1) - f(n)$, and $f(4n + 3) = 3f(2n + 1) - 2f(n)$ for all positive integers n. Determine the number of positive integers $n \leq 1988$ for which $f(n) = n$.

B1. Show that the set of real numbers x which satisfy the inequality:

$$\frac{1}{x-1} + \frac{2}{x-2} + \frac{3}{x-3} + \ldots + \frac{70}{x-70} \geq \frac{5}{4}$$

is a union of disjoint intervals, the sum of whose lengths is 1988.

B2. ABC is a triangle, right-angled at A, and D is the foot of the altitude from A. The straight line joining the incenters of the triangles ABD and ACD intersects the sides AB, AC at K, L respectively. Show that the area of the triangle ABC is at least twice the area of the triangle AKL.

B3. Let a and b be positive integers such that $ab+1$ divides a^2+b^2. Show that $\frac{a^2+b^2}{ab+1}$ is a perfect square.

30th IMO 1989

A1. Prove that the set $\{1, 2, \ldots, 1989\}$ can be expressed as the disjoint union of subsets $A_1, A_2, \ldots, A_{117}$ in such a way that each A_i contains 17 elements and the sum of the elements in each A_i is the same.

A2. In an acute-angled triangle ABC, the internal bisector of angle A meets the circumcircle again at A_1. Points B_1 and C_1 are defined similarly. Let A_0 be the point of intersection of the line AA_1 with the external bisectors of angles B and C. Points B_0 and C_0 are defined similarly. Prove that the area of the triangle $A_0B_0C_0$ is twice the area of the hexagon $AC_1BA_1CB_1$ and at least four times the area of the triangle ABC.

A3. Let n and k be positive integers, and let S be a set of n points in the plane such that no three points of S are collinear, and for any points P of S there are at least k points of S equidistant from P. Prove that $k < \frac{1}{2} + \sqrt{2n}$.

B1. Let $ABCD$ be a convex quadrilateral such that the sides AB, AD, BC satisfy $AB = AD + BC$. There exists a point P inside the quadrilateral at a distance h from the line CD such that $AP = h + AD$ and $BP = h + BC$. Show that:

$$\frac{1}{\sqrt{h}} \geq \frac{1}{\sqrt{AD}} + \frac{1}{\sqrt{BC}}.$$

B2. Prove that for each positive integer n there exist n consecutive positive integers none of which is a prime or a prime power.

B3. A permutation $\{x_1, x_2, \ldots, x_m\}$ of the set $\{1, 2, \ldots, 2n\}$ where n is a positive integer is said to have property P if $|x_i - x_{i+1}| = n$ for at least one i in $\{1, 2, \ldots, 2n-1\}$. Show that for each n there are more permutations with property P than without.

31st IMO 1990

A1. Chords AB and CD of a circle intersect at a point E inside the circle. Let M be an interior point of the segment EB. The tangent at E to the circle through D, E, M intersects the lines BC and AC at F and G respectively. Find $\frac{EF}{EF}$ in terms of $t = \frac{AM}{AB}$.

A2. Take $n \geq 3$ and consider a set E of $2n-1$ distinct points on a circle. Suppose that exactly k of these points are to be colored black. Such a coloring is *good* if there is at least one pair of black points such that the interior of one of the arcs between them contains exactly n points from E. Find the smallest value of k so that every such coloring of k points of E is good.

A3. Determine all integers greater than 1 such that $\frac{2^n+1}{n^2}$ is an integer.

B1. Construct a function from the set of positive rational numbers into itself such that $f(xf(y)) = \frac{f(x)}{y}$ for all x, y.

B2. Given an initial integer $n_0 > 1$, two players A and B choose integers $n_1, n_2, n_3, \ldots$ alternately according to the following rules:

Knowing n_{2k}, A chooses any integer n_{2k+1} such that $n_{2k} \leq n_{2k+1} \leq n_{2k}^2$.

Knowing n_{2k+1}, B chooses any integer n_{2k+2} such that $\frac{n_{2k+1}}{n_{2k+2}} = p^r$ for some prime p and integer $r \geq 1$.

Player A wins the game by choosing the number 1990; player B wins by choosing the number 1. For which n_0 does

(a) A have a winning strategy?

(b) B have a winning strategy?

(c) Neither player have a winning strategy?

B3. Prove that there exists a convex 1990-gon such that all its angles are equal and the lengths of the sides are the numbers $1^2, 2^2, \ldots, 1990^2$ in some order.

32nd IMO 1991

A1. Given a triangle ABC, let I be the incenter. The internal bisectors of angles A, B, C meet the opposite sides in A', B', C' respectively. Prove that

$$\frac{1}{4} < \frac{AI \cdot BI \cdot CI}{AA' \cdot BB' \cdot CC'} \leq \frac{8}{27}.$$

A2. Let $n > 6$ be an integer and let $a_1, a_2, \ldots, a_k$ be all the positive integers less than n and relatively prime to n. If

$$a_2 - a_1 = a_3 - a_2 = \ldots a_k - a_{k-1} > 0,$$

prove that n must be either a prime or a power of 2.

A3. Let $S = \{1, 2, 3, \ldots, 280\}$. Find the smallest integer n such that each n-element subset of S contains five numbers which are pairwise relatively prime.

B1. Suppose G is a connected graph with k edges. Prove that it is possible to label the edges $1, 2, \ldots, k$ in such a way that at each vertex which belongs to two or more edges, the greatest common divisor of the integers labeling those edges is 1.

B2. Let ABC be a triangle and X an interior point of ABC. Show that at least one of the angles XAB, XBC, XCA is less than or equal to 30^o.

B3. Given any real number $a > 1$ construct a bounded infinite sequence $x_0, x_1, x_2, \ldots$ such that $|x_i - x_j||i - j|^a \geq 1$ for every pair of distinct i, j.

33rd IMO 1992

A1. Find all integers a, b, c satisfying $1 < a < b < c$ such that $(a-1)(b-1)(c-1)$ is a divisor of $abc - 1$.

A2. Find all functions f defined on the set of all real numbers with real values, such that $f(x^2 + f(y)) = y + f(x)^2$ for all x, y.

A3. Consider 9 points in space, no 4 coplanar. Each pair of points is joined by a line segment which is colored either blue or red or left uncolored. Find the smallest value of n such that whenever exactly n edges are colored, the set of colored edges necessarily contains a triangle all of whose edges have the same color.

B1. L is a tangent to the circle C and M is a point on L. Find the locus of all points P such that there exist points Q and R on L equidistant from M with C the incircle of the triangle PQR.

B2. Let S be a finite set of points in three-dimensional space. Let S_x, S_y, S_z be the sets consisting of the orthogonal projections of the points of S onto the yz-plane, zx-plane, xy-plane respectively. Prove that:

$$|S|^2 \leq |S_x||S_y||S_z|,$$

where $|A|$ denotes the number of points in the set A. The orthogonal projection of a point onto a plane is the foot of the perpendicular from the point to the plane.

B3. For each positive integer $n, S(n)$ is defined as the greatest integer such that for every positive integer $k \leq S(n), n^2$ can be written as the sum of k positive squares.
(a) Prove that $S(n) \leq n^2 - 14$ for each $n \geq 4$.
(b) Find an integer n such that $S(n) = n^2 - 14$.
(c) Prove that there are infinitely many integers n such that $S(n) = n^2 - 14$.

34th IMO 1993

A1. Let $f(x) = x^n + 5x^{n-1} + 3$, where $n > 1$ is an integer. Prove that $f(x)$ cannot be expressed as the produce of two non-constant polynomials with integer coefficients.

A2. Let D be a point inside the acute-angled triangle ABC such that $\angle ADB = \angle ACB + 90^o$, and $AC \cdot BD = AD \cdot BC$.
(a) Calculate the ratio $AB \cdot CD/(AC \cdot BD)$.
(b) Prove that the tangents at C to the circumcircles of ACD and BCD are perpendicular.

A3. On an infinite chessboard a game is played as follows. At the start n^2 pieces are arranged in an $n \times n$ block of adjoining squares, one piece on each square. A move in the game is a jump in a horizontal or vertical direction over an adjacent occupied square to an unoccupied square immediately beyond. The piece which has been jumped over is removed. Find those values of n for which the game can end with only one piece remaining on the board.

B1. For the points P, Q, R in the plane define $m(PQR)$ as the minimum length of the three altitudes of the triangle PQR (or zero if the points are collinear). Prove that for any points A, B, C, X:

$$m(ABC) \leq m(ABX) + m(AXC) + m(XBC).$$

B2. Does there exist a function f from the positive integers to the positive integers such that $f(1) = 2, f(f(n)) = f(n) + n$ for all n, and $f(n) < f(n+1)$ for all n?

B3. There are $n > 1$ lamps $L_0, L_1, \ldots, L_{n-1}$ in a circle. We use L_{n+k} to mean L_k. A lamp is at all times either on or off. Initially they are all on. Perform steps $s_0, s_1, \ldots$ as follows: at step s_i, if L_{i-1} is lit, then switch L_i from on to off or vice versa, otherwise do nothing. Show that:

(a) There is a positive integer $M(n)$ such that after $M(n)$ steps all the lamps are on again;
(b) If $n = 2^k$, then we can take $M(n) = n^2 - 1$;
(c) If $n = 2^k + 1$, then we can take $M(n) = n^2 - n + 1$.

35th IMO 1994

A1. Let m and n be positive integers. Let $a_1, a_2, \ldots, a_m$ be distinct elements of $\{1, 2, \ldots, n\}$ such that whenever $a_i + a_j \leq n$ for some i, j (possibly the same) we have $a_i + a_j = a_k$ for some k. Prove that:

$$(a_1 + \ldots a_m) \geq \frac{(n+1)}{2}.$$

A2. ABC is an isosceles triangle with $AB = AC$, M is the midpoint of BC and O is the point on the line AM such that OB is perpendicular to AB. Q is an arbitrary point on BC different from B and C. E lies on the line AB and F lies on the line AC such that E, Q, F are distinct and collinear Prove that OQ is perpendicular to EF iff $QE = QF$.

A3. For any positive integer k, let $f(k)$ be the number of elements in the set $\{k+1, k+2, \ldots, 2k\}$ which have exactly three 1s when written in base 2. Prove that for each positive integer m, there is at least one k with $f(k) = m$, and determine all m for which there is exactly one k.

B1. Determine all ordered pairs (m, n) of positive integers for which $\frac{n^3+1}{mn-1}$ is an integer.

B2. Let S be the set of all real numbers greater than -1. Find all functions $f\colon S \to S$ such that $f\big(x + f(y) + xf(y)\big) = y + f(x) + yf(x)$ for all x, y, and $\frac{f(x)}{x}$ is strictly increasing on each of the intervals $-1 < x < 0$ and $0 < x$.

B3. Show that there exists a set A of positive integers with the following property: for any infinite set S of primes, there exist two positive integers $m \in A$ and $n \notin A$, each of which is a product of k distinct elements of S for some $k \geq 2$.

36th IMO 1995

A1. Let A, B, C, D be four distinct points on a line, in that order. The circles with diameter AC and BD intersect at X and Y. The line XY meets BC at Z. Let P be a point on the line XY other than Z. The line CP intersects the circle with diameter AC at C and M, and the line BP intersects the circle with diameter BD at B and N. Prove that the lines AM, DN, XY are concurrent.

A2. Let a, b, c be positive real numbers with $abc = 1$. Prove that:

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

A3. Determine all integers $n > 3$ for which there exist n points $A_1, \ldots, A_n$ in the plane, no three collinear, and real numbers $r_1, \ldots, r_n$ such that for any distinct i, j, k, the area of the triangle $A_iA_jA_k$ is $r_i + r_j + r_k$.

B1. Find the maximum value of x_0 for which there exists a sequence $x_0, x_1, \ldots, x_{1995}$ of positive reals with $x_0 = x_{1995}$ such that for $i = 1, \ldots, 1995$:

$$x_{i-1} + \frac{2}{x_{i-1}} = 2x_i + \frac{1}{x_i}.$$

B2. Let $ABCDEF$ be a convex hexagon with $AB = BC = CD$ and $DE = EF = FA$, such that $\angle BCD = \angle EFA = 60^o$. Suppose that G and H are points in the interior of the hexagon such that $\angle AGB = \angle DHE = 120^o$. Prove that $AG + GB + GH + DH + HE \geq CF$.

B3. Let p be an odd prime number. How many p-element subsets A of $\{1, 2, \ldots, 2p\}$ are there, the sum of whose elements is divisible by p?

37th IMO 1996

A1. We are given a positive integer r and a rectangular board divided into 20×12 unit squares. The following moves are permitted on the board: one can move from one square to another only if the distance between the centers of the two squares is $\sqrt{r}$. The task is to find a sequence of moves leading between two adjacent corners of the board which lie along the long side.
(a) Show that the task cannot be done if r is divisible by 2 or 3.
(b) Prove that the task is possible for $r = 73$.
(c) Can the task be done for $r = 97$?

A2. Let P be a point inside the triangle ABC such that $\angle APB - \angle ACB = \angle APC - \angle ABC$. Let D, E be the incenters of triangles APB, APC respectively. Show that AP, BD, CE meet at a point.

A3. Let S be the set of non-negative integers. Find all functions $f : S \to S$ such that $f(m + f(n)) = f(f(m)) + f(n)$ for all m, n.

B1. The positive integers a, b are such that $15a + 16b$ and $16a - 15b$ are both squares of positive integers. What is the least possible value that can be taken by the smaller of these two squares?

B2. Let $ABCDEF$ be a convex hexagon such that AB is parallel to DE, BC is parallel to EF, and CD is parallel to FA. Let R_A, R_C, R_E denote the circumradii of triangles FAB, BCD, DEF respectively, and let p denote the perimeter of the hexagon. Prove that:

$$R_A + R_C + R_E \geq p/2$$

.

B3. Let p, q, n be three positive integers with $p+q < n$. Let $x_0, x_1, \ldots, x_n$ be integers such that $x_0 = x_n = 0$, and for each $1 \leq i \leq n, x_i - x_{i-1} = p$ or $-q$. Show that there exist indices $i < j$ with (i, j) not $(0, n)$ such that $x_i = x_j$.

38th IMO 1997

A1. In the plane the points with integer coordinates are the vertices of unit squares. The squares are colored alternately black and white as on a chessboard. For any pair of positive integers m and n, consider a right-angled triangle whose vertices have integer coordinates and whose legs, of lengths m and n, lie along the edges of the squares. Let S_1 be the total area of the black part of the triangle, and S_2 be the total area of the white part. Let $f(m,n) = |S_1 - S_2|$.

(a) Calculate $f(m,n)$ for all positive integers m and n which are either both even or both odd.

(b) Prove that $f(m,n) \leq \max(m,n)/2$ for all m, n.

(c) Show that there is no constant C such that $f(m,n) < C$ for all m, n.

A2. $\angle A$ is the smallest angle in the triangle ABC. The points B and C divide the circumcircle of the triangle into two arcs. Let U be an interior point of the arc between B and C which does not contain A. The perpendicular bisectors of AB and AC meet the line AU at V and W, respectively. The lines BV and CW meet at T. Show that $AU = TB + TC$.

A3. Let $x_1, x_2, \ldots, x_n$ be real numbers satisfying $|x_1 + x_2 + \cdots + x_n| = 1$ and $|x_i| \leq (n+1)/2$ for all i. Show that there exists a permutation y_i of x_i such that $|y_1 + 2y_2 + \cdots + ny_n| \leq (n+1)/2$.

B1. An $n \times n$ matrix whose entries come from the set $S = \{1, 2, \ldots, 2n-1\}$ is called a silver matrix if, for each $i = 1, 2, \ldots, n$, the ith row and the ith column together contain all elements of S. Show that:

(a) there is no silver matrix for $n = 1997$;

(b) silver matrices exist for infinitely many values of n.

B2. Find all pairs (a,b) of positive integers that satisfy $a^{b^2} = b^a$.

B3. For each positive integer n, let $f(n)$ denote the number of ways of representing n as a sum of powers of 2 with non-negative integer exponents. Representations which differ only in the ordering of their summands are considered to be the same. For example, $f(4) = 4$, because 4 can be represented as $4, 2+2, 2+1+1$ or $1+1+1+1$. Prove that for any integer $n \geq 3$, $2^{n^2/4} < f(2^n) < 2^{n^2/2}$.

39th IMO 1998

A1. In the convex quadrilateral $ABCD$, the diagonals AC and BD are perpendicular and the opposite sides AB and DC are not parallel. The point P, where the perpendicular bisectors of AB and CD meet, is inside $ABCD$. Prove that $ABCD$ is cyclic iff the triangles ABP and CDP have equal areas.

A2. In a competition there are a contestants and b judges, where $b \geq 3$ is an odd integer. Each judge rates each contestant as either "pass" or "fail". Suppose k is a number such that for any two judges their ratings coincide for at most k contestants. Prove $k/a \geq (b-1)/2b$.

A3. For any positive integer n, let $d(n)$ denote the number of positive divisors of n (including 1 and n). Determine all positive integers k such that $d(n^2) = kd(n)$ for some n.

B1. Determine all pairs (a,b) of positive integers such that $ab^2 + b + 7$ divides $a^2b + a + b$.

B2. Let I be the incenter of the triangle ABC. Let the incircle of ABC touch the sides BC, CA, AB at K, L, M respectively. The line through B parallel to MK meets the lines LM and LK at R and S respectively. Prove that $\angle RIS$ is acute.

B3. Consider all functions $f : N \to N$ on the positive integers satisfying $f(t^2 f(s)) = sf(t)^2$ for all s and t. Determine the least possible value of $f(1998)$.

40th IMO 1999

A1. Find all finite sets S of at least three points in the plane such that for all distinct points A, B in S, the perpendicular bisector of AB is an axis of symmetry for S.

A2. Let $n \geq 2$ be a fixed integer. Find the smallest constant C such that for all non-negative reals $x_1, \ldots, x_n$:

$$\sum_{i<j} x_i x_j (x_i^2 + x_j^2) \leq C(\sum_i x_i)^4.$$

Determine when equality occurs.

A3. Given an $n \times n$ square board, with n even. Two distinct squares of the board are said to be adjacent if they share a common side, but a square is not adjacent to itself. Find the minimum number of squares that can be marked so that every square (marked or not) is adjacent to at least one marked square.

B1. Find all pairs (n, p) of positive integers, such that: p is prime; $n \leq 2p$; and $(p-1)^n + 1$ is divisible by n^{p-1}.

B2. The circles C_1 and C_2 lie inside the circle C, and are tangent to it at M and N, respectively. C_1 passes through the center of C_2. The common chord of C_1 and C_2, when extended, meets C at A and B. The lines MA and MB meet C_1 again at E and F. Prove that the line EF is tangent to C_2.

B3. Determine all functions $f : R \to R$ such that $f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$ for all x, y in R. [R is the reals.]

41st IMO 2000

A1. AB is tangent to the circles $CAMN$ and $NMBD$. M lies between C and D on the line CD, and CD is parallel to AB. The chords NA and CM meet at P; the chords NB and MD meet at Q. The rays CA and DB meet at E. Prove that $PE = QE$.

A2. A, B, C are positive reals with product 1. Prove that $(A - 1 + \frac{1}{B})(B - 1 + \frac{1}{C})(C - 1 + \frac{1}{A}) \leq 1$.

A3. k is a positive real. N is an integer greater than 1. N points are placed on a line, not all coincident. A *move* is carried out as follows. Pick any two points A and B which are not coincident. Suppose that A lies to the right of B. Replace B by another point B' to the right of A such that $AB' = kBA$. For what values of k can we move the points arbitrarily far to the right by repeated moves?

B1. 100 cards are numbered 1 to 100 (each card different) and placed in 3 boxes (at least one card in each box). How many ways can this be done so that if two boxes are selected and a card is taken from each, then the knowledge of their sum alone is always sufficient to identify the third box?

B2. Can we find N divisible by just 2000 different primes, so that N divides $2^N + 1$? [N may be divisible by a prime power.]

B3. $A_1A_2A_3$ is an acute-angled triangle. The foot of the altitude from A_i is K_i and the incircle touches the side opposite A_i at L_i. The line K_1K_2 is reflected in the line L_1L_2. Similarly, the line K_2K_3 is reflected in L_2L_3 and K_3K_1 is reflected in L_3L_1. Show that the three new lines form a triangle with vertices on the incircle.

42nd IMO 2001

A1. ABC is acute-angled. O is its circumcenter. X is the foot of the perpendicular from A to BC. $\angle C \geq \angle B + 30^o$. Prove that $\angle A + \angle COX < 90^o$.

A2. a, b, c are positive reals. Prove that $\frac{a}{\sqrt{a^2+8bc}} + \frac{b}{\sqrt{b^2+8ca}} + \frac{c}{\sqrt{c^2+8ab}} \geq 1$.

A3. Integers are placed in each of the 441 cells of a 21×21 array. Each row and each column has at most 6 different integers in it. Prove that some integer is in at least 3 rows and at least 3 columns.

B1. Let $n_1, n_2, \ldots, n_m$ be integers where m is odd. Let $x = (x_1, \ldots, x_m)$ denote a permutation of the integers $1, 2, \ldots, m$. Let $f(x) = x_1 n_1 + x_2 n_2 + \cdots + x_m n_m$. Show that for some distinct permutations a, b the difference $f(a) - f(b)$ is a multiple of $m!$.

B2. ABC is a triangle. X lies on BC and AX bisects $\angle A$. Y lies on CA and BY bisects $\angle B$. $\angle A = 60^o$. $AB + BX = AY + YB$. Find all possible values for $\angle B$.

B3. $K > L > M > N$ are positive integers such that $KM + LN = (K + L - M + N)(-K + L + M + N)$. Prove that $KL + MN$ is composite.

43rd IMO 2002

A1. S is the set of all (h, k) with h, k non-negative integers such that $h + k < n$. Each element of S is colored red or blue, so that if (h, k) is red and $h' \leq h, k' \leq k$, then (h', k') is also red. A type 1 subset of S has n blue elements with different first member and a type 2 subset of S has n blue elements with different second member. Show that there are the same number of type 1 and type 2 subsets.

A2. BC is a diameter of a circle center O. A is any point on the circle with $\angle AOC > 60^o$. EF is the chord which is the perpendicular bisector of AO. D is the midpoint of the minor arc AB. The line through O parallel to AD meets AC at J. Show that J is the incenter of triangle CEF.

A3. Find all pairs of integers $m > 2, n > 2$ such that there are infinitely many positive integers k for which $k^n + k^2 - 1$ divides $k^m + k - 1$.

B1. The positive divisors of the integer $n > 1$ are $d_1 < d_2 < \ldots < d_k$, so that $d_1 = 1, d_k = n$. Let $d = d_1 d_2 + d_2 d_3 + \cdots + d_{k-1} d_k$. Show that $d < n^2$ and find all n for which d divides n^2.

B2. Find all real-valued functions on the reals such that $(f(x)+f(y))((f(u)+f(v)) = f(xu-yv)+f(xv+yu)$ for all x, y, u, v.

B3. $n > 2$ circles of radius 1 are drawn in the plane so that no line meets more than two of the circles. Their centers are $O_1, O_2, \cdots, O_n$. Show that $\sum_{i<j} 1/O_i O_j \leq (n-1)\pi/4$.

44th IMO 2003

A1. S is the set $\{1, 2, 3, \ldots, 1000000\}$. Show that for any subset A of S with 101 elements we can find 100 distinct elements x_i of S, such that the sets $\{a + x_i | a \in A\}$ are all pairwise disjoint.

A2. Find all pairs (m, n) of positive integers such that $\frac{m^2}{2mn^2 - n^3 + 1}$ is a positive integer.

A3. A convex hexagon has the property that for any pair of opposite sides the distance between their midpoints is $\sqrt{3}/2$ times the sum of their lengths Show that all the hexagon's angles are equal.

B1. $ABCD$ is cyclic. The feet of the perpendicular from D to the lines AB, BC, CA are P, Q, R respectively. Show that the angle bisectors of ABC and CDA meet on the line AC iff $RP = RQ$.

B2. Given $n > 2$ and reals $x_1 \leq x_2 \leq \cdots \leq x_n$, show that $(\sum_{i,j} |x_i - x_j|)^2 \leq \frac{2}{3}(n^2 - 1) \sum_{i,j}(x_i - x_j)^2$. Show that we have equality iff the sequence is an arithmetic progression.

B3. Show that for each prime p, there exists a prime q such that $n^p - p$ is not divisible by q for any positive integer n.

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