QUESTION ONE (12 marks)

a) Evaluate

$$\int_0^3 \frac{dx}{x^2+9} .= \left[\frac{1}{3}\tan^{-1}\frac{x}{3}\right]_0^3 = \frac{\pi}{12}$$

b)

$$m_1 = 2\, and\, m_2 = -\frac{1}{3}$$

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = 7$$

$$\theta = 81^\circ 52'$$

c)

$$\frac{x}{2x-1}\times(2x-1)^2 \le 5(2x-1)^2 \quad x \ne \frac{1}{2}$$

$$2x^2 - x \le 20x^2 - 20x + 5$$

$$0 \le 18x^2 - 19x + 5$$

$$0 \le (9x-5)(2x-1)$$

$$x \ge \frac{5}{9}\ and\ \ x < \frac{1}{2}$$

d)

$$\frac{dy}{dx} = 4\cos^3 x \times -\sin x$$

$$\frac{dy}{dx} = -4\cos^3 x \sin x$$

$$-4\int \cos^3 x \sin x dx = \cos^4 x + c$$

$$\int \cos^3 x \sin x dx = -\frac{1}{4}\cos^4 x + c$$

e)

$$\left(\frac{-6+5}{8}, \frac{-3+25}{8}\right)$$

$$\left(\frac{-1}{8}, 2\frac{3}{4}\right)$$

QUESTION TWO (12 marks)

a)

$$\int_0^{\frac{\pi}{3}} \sin^2(2x)dx = \int_0^{\frac{\pi}{3}} \frac{1-\cos 2(2x)}{2}dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{3}} 1-\cos(4x)dx = \frac{1}{2}\left[x - \frac{1}{4}\sin(4x)\right]_0^{\frac{\pi}{3}}$$

$$= \frac{\pi}{6} + \frac{\sqrt{3}}{16}$$

b)

$$u = \tan x$$

$$\frac{du}{dx} = \sec^2 x \quad du = \sec^2 x\,dx \quad when\, x = 0 \;\; u = \tan 0 = 0$$

$$when\, x = \frac{\pi}{3} \;\; u = \tan\frac{\pi}{3} = \sqrt{3}$$

$$\int_0^{\sqrt{3}} u^2 du = \left[\frac{u^3}{3}\right]_0^{\sqrt{3}} = \frac{3\sqrt{3}}{3} = \sqrt{3}$$

c)

(i) $\,^5_2C(0.4)^2(0.6)^3 = 0.3456$

(ii) $\,^5_3C(0.4)^3(0.6)^2 + \,^5_4C(0.4)^4(0.6)^1 + \,^5_5C(0.4)^5(0.6)^0 = 0.31744$

(iii) The most likely outcome is 3 heads and 2 tails which has 0.3456 chance of occurring.

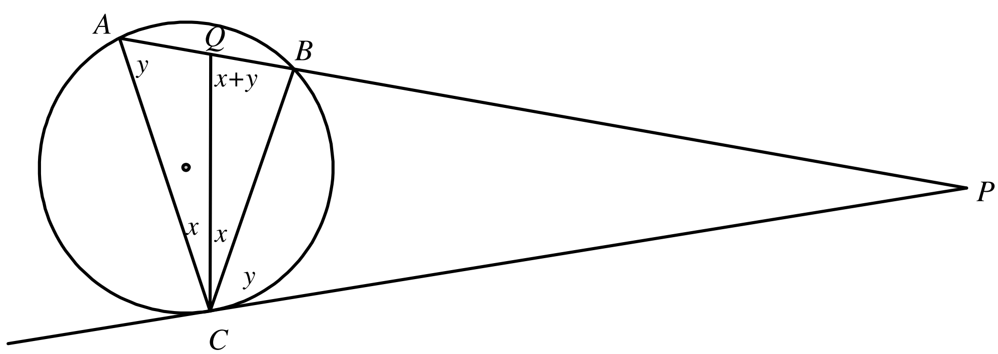
QUESTION THREE (12 marks)

a) $\sin\left(\sin^{-1}\left(\frac{4}{5}\right)-\tan^{-1}\left(\frac{-5}{12}\right)\right).$

$$\sin(A-B) = sinA\cos B - \sin B\cos A$$
$$= \left(\frac{4}{5}\times\frac{-12}{13}\right)-\left(\frac{5}{13}\times\frac{3}{5}\right)$$
$$= \frac{-63}{65}$$

b)

(i)



Let $\angle ACQ = x, \angle QCB = x$ given that QC bisect $\angle ACB$

Let $\angle BCP = y, \angle CAB = y$ (angle between chord and tangent equal angle in the alternate segment)

$\angle CQP = x + y$ (external angle in ΔAQC is equal to the sum of the opposite internal angles)

$\angle QCP = x + y$

ΔPQC is an isosceles Δ. ($\angle CQP = \angle QCP$)

$QP = PC$ (sides opposite equal angles are equal)

(ii)

$PC^2 = PB \times PA$

Let $PB = x, QB = 10 - x$

$10^2 = x \times 12$

$x = 100 \div 12 = 8\frac{1}{3}$

$QB = 10 - 8\frac{1}{3} = 1\frac{2}{3}$

c)

(i)

$$T = 30 + Ae^{-kt}$$
$$Ae^{-kt} = T - 30$$
$$\frac{dT}{dt} = -kAe^{-kt} = -k(T - 30)$$

(ii) Show that the value of A is $220°$ C .

$$T_0 = 250$$
$$250 = 30 + A$$
$$A = 220$$

(iii) Find the value of k to 2 decimal places.

$$150 = 30 + 220e^{-20k}$$
$$120 = 220e^{-20k}$$
$$\frac{120}{220} = e^{-20k}$$
$$k = \frac{1}{20}\ln\frac{11}{6} = 0.03$$

(iv)

$$80 = 30 + 220e^{-0.03t}$$
$$50 = 220e^{-0.03t}$$
$$\frac{50}{220} = e^{-0.03t}$$
$$t = \frac{\ln\frac{5}{22}}{-0.03} = 49\text{ minutes}$$

QUESTION FOUR (12 marks)

a) Prove by mathematical induction that:

$5^n - 4n - 1 \geq 0$, for all $n \geq 1$.

Test for $n = 1$

$5^1 - 4(1) - 1 \geq 0$,

$0 \geq 0$

True for $n = 1$.

Assume true for $n = k$, where $k \geq 1$

assume $5^k - 4k - 1 \geq 0$,

Test for $n = k + 1$

$5^{k+1} - 4(k+1) - 1$

$= 5 \times 5^k - 4k - 4 - 1$

$= 5 \times 5^k - 20k + 16k - 5$

$= 5(5^k - 4k - 1) + 16k$

Since $(5^k - 4k - 1) \geq 0$ and $16k \geq 0$

then $5(5^k - 4k - 1) + 16k \geq 0$, where $k \geq 1$

Since true for $n = 1$, and true for $n = k + 1$,

then by the process of mathematical induction, it is true for all values of n.

b)

(i)

There are 10 years of contributions, then 3 years without contributions, followed by a further 7 years of contribution. The first contribution occurred 20 years before the last.

Contributions occurred quarterly (80 quarters @ 1.5% per quarter)

First contribution= $800(1.015)^{80}$

Second contribution= $800(1.015)^{79}$

Third contribution= $800(1.015)^{78}$

Last contribution before the 3 year break.= $800(1.015)^{41}$

Therefore total contribution make for first 10 years is

$800(1.015)^{80} + 800(1.015)^{79} + \ldots\ldots + 800(1.015)^{41}$.

(ii) Hence, calculate the total amount in the superannuation fund on retirement.

Total contribution = first 10 years contribution +Last 7 years contribution

$= 800(1.015)^{80} + 800(1.015)^{79} + \ldots\ldots + 800(1.015)^{41} + 800(1.015)^{28} + 800(1.015)^{27} + \ldots\ldots + 800(1.015)^{1}$.

$$\frac{800(1.015)(1.015^{28} - 1)}{1.015 - 1} + \frac{800(1.015)^{41}(1.015^{40} - 1)}{1.015 - 1}$$

$\$27998.96 + \$79935.68$

$= \$107934.64$

c)

(i) Show that the value of a is 3 and that b is $\frac{1}{2}$.

$$y = a\cos^{-1} bx$$

$$x = a\cos^{-1} by$$

$$\frac{x}{a} = \cos^{-1} by$$

$$\cos\frac{x}{a} = by$$

$$\frac{1}{b}\cos\frac{x}{a} = y$$

$$\text{amplitude} = 2.\ \text{Therefore}\ \frac{1}{b} = 2, b = \frac{1}{2}$$

$$\text{period} = 6\pi.\text{Therefore}\ \frac{1}{a} = \frac{2\pi}{6\pi}, a = 3$$

(ii)

$$\int_0^{\frac{\pi}{2}} 2\cos\frac{x}{3}\,dx = 2\left[3\sin\frac{x}{3}\right]_0^{\frac{\pi}{2}}$$

$$6\sin\frac{\pi}{6} = 3\,\text{units}^2$$

QUESTION FIVE (12 marks)

a) (i)

$$a = \ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$\frac{1}{2}v^2 = \int 4x\,dx$$

$$\frac{1}{2}v^2 = 2x^2 + c$$

when $x = 1,\ v = -2,$ therefore $c = 0$

$$v^2 = 4x^2$$

$v = \pm\sqrt{4x^2}$ Since velocity is initially negative then

$$v = -\sqrt{4x^2} = -2x$$

(ii) Express *x* as a function of *t*.

$$v = \frac{dx}{dt} = -2x$$

$$\frac{dt}{dx} = \frac{1}{-2x}$$

$$t = \int \frac{1}{-2x}dx = \frac{-1}{2}\ln x + c$$

$$when\ \ t = 0, x = 1, \therefore c = 0$$

$$t = -\frac{1}{2}\ln x$$

$$-2t = \ln x$$

$$x = e^{-2t}$$

(iii) Hence, find the displacement when $t = 2$ seconds to 3 decimal places.

$$when\ t = 2$$

$$x = e^{-4} = 0.018$$

b) (i)

$$(1+x)^{20} + (1-x)^{20}$$

$$= {}^{20}_{0}C + {}^{20}_{1}Cx + {}^{20}_{2}Cx^2 + \ldots\ldots\ {}^{20}_{20}Cx^{20} + {}^{20}_{0}C - {}^{20}_{1}Cx + {}^{20}_{2}Cx^2 - \ldots\ldots + {}^{20}_{20}Cx^{20}$$

$$= 2\left({}^{20}_{0}C + {}^{20}_{2}Cx^2 + {}^{20}_{4}Cx^4 + \ldots\ldots\ {}^{20}_{20}Cx^{20}\right)$$

$$= 2\sum_{k=0}^{10} {}^{20}_{2k}Cx^{2k}$$

(ii)

$$(1+x)^{20}+(1-x)^{20}=2\sum_{k=0}^{10}{}^{20}_{2k}Cx^{2k}$$

$$\sum_{k=0}^{10}{}^{20}_{2k}Cx^{2k}=\frac{1}{2}\left((1+x)^{20}+(1-x)^{20}\right)$$

(iii) Integrate both sides of the equation in part (ii).

$$\frac{1}{2}\left((1+x)^{20}+(1-x)^{20}\right)=\sum_{k=0}^{10}{}^{20}_{2k}Cx^{2k}$$

$$\frac{1}{2}\frac{1}{21}\left((1+x)^{21}-(1-x)^{21}\right)+c=\sum_{k=0}^{10}\frac{1}{2k+1}{}^{20}_{2k}Cx^{2k+1}+c$$

(iv) Hence, or otherwise show:

$$^{20}C_0+\frac{1}{3}\,^{20}C_2+\frac{1}{5}\,^{20}C_4+......+\frac{1}{21}\,^{20}C_{20}=\frac{2^{20}}{21}.$$

Let $x=10\ \therefore\ c=0$

When $x=1$

$$\frac{1}{2}\frac{1}{21}\left((1+1)^{21}-(1-1)^{21}\right)=\sum_{k=0}^{10}\frac{1}{2k+1}{}^{20}_{2k}C1^{2k+1}$$

$$\frac{1}{21}\frac{1}{2}\left(2^{21}\right)=\frac{1}{21}\left(2^{20}\right)=\sum_{k=0}^{10}\frac{1}{2k+1}{}^{20}_{2k}C$$

$$\frac{2^{20}}{21}=\sum_{k=0}^{10}\frac{1}{2k+1}{}^{20}_{2k}C=\left({}^{20}_{0}C+\frac{1}{3}{}^{20}_{2}Cx^{2}+\frac{1}{5}{}^{20}_{4}Cx^{4}+........\frac{1}{21}{}^{20}_{20}Cx^{20}\right)$$

QUESTION SIX (12 marks)

(i)

$$x^2 + y^2 = 100$$
$$x^2 = 100 - y^2$$
$$V = \pi \int_{10-h}^{10} 100 - y^2 dy$$
$$V = \pi \left[100y - \frac{1}{3} y^3 \right]_{10-h}^{10}$$
$$V = \pi \left[\left(100(10) - \frac{1}{3}(10)^3 \right) - \left(100(10-h) - \frac{1}{3}(10-h)^3 \right) \right]$$
$$V = \pi \left[\left(1000 - \frac{1}{3}(1000) \right) - \left(1000 - 100h - \frac{1}{3}(1000 - 300h + 30h^2 - h^3 \right) \right]$$
$$V = \frac{\pi}{3}\left[30h^2 - h^3 \right]$$
$$V = \frac{\pi h^2}{3}\left[30 - h \right]$$

(ii)

$$\frac{dV}{dt} = 2\pi$$
$$V = \frac{\pi h^2}{3}\left[30 - h \right] = 10\pi h^2 - \frac{\pi h^3}{3}$$
$$\frac{dV}{dh} = 20\pi h - \pi h^2$$
$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$
$$\frac{dh}{dt} = \frac{1}{20\pi h - \pi h^2} \times 2\pi$$
when $h = 2$
$$\frac{dh}{dt} = \frac{1}{40\pi - 4\pi} \times 2\pi = \frac{2\pi}{36\pi} = \frac{1}{18} cm \min^{-1}$$

b)

(i) Find an expression for $y = f^{-1}(x)$.

$x = \sqrt{y+3}$

$x^2 = y+3$

$y = x^2 - 3$

$f^{-1}(x) = x^2 - 3$

(ii)

Domain $f^{-1}(x)$ is the range for $f(x)$.

For $f(x) = \sqrt{x+3} \quad x \geq 0$

(iii)

$f^{-1}(f(x)) = f\left(f^{-1}(x)\right) = x$

$y = x$ and $y = \sqrt{x+3}$

$y = y$

$x = \sqrt{x+3}$

$x - \sqrt{x+3} = 0$

(iv) Show that the value of α occurs when $2 < \alpha < 3$.

when $x = 2 \quad 2 - \sqrt{2+3} < 0$

when $x = 3 \quad 3 - \sqrt{3+3} > 0$

$\therefore$ there will be a zero between 2 and 3.

(v) Take $x = 2.5$ as the first approximation of α, use one application of Newton's

method to find a second approximation for α correct to 3 decimal places.

$x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

$x_1 = 2.5 - \dfrac{f(2.5)}{f'(2.5)}$

$x_1 = 2.303$

QUESTION SEVEN (12 marks)

a)

(i) Show that $x = b - a\cos nt$ satisfies $\ddot{x} = -n^2(x-b)$.

$x = b - a\cos nt \quad \therefore \quad a\cos nt = b - x$

$\frac{dx}{dt} = -an\sin nt$

$\frac{d^2x}{dt^2} = an^2\cos nt = n^2(a\cos nt) = n^2(b-x) = -n^2(x-b)$

$\ddot{x} = \frac{d^2x}{dt^2} = -n^2(x-b)$

(ii) Find the values of a, b and n.

$b = 8\frac{5}{6} \quad a = 1\frac{5}{6}$

$\frac{2\pi}{n} = 12\frac{1}{2}\text{ hours}$

$n = \frac{4\pi}{25}$

(iii) Hence, find the earliest time before 3:40pm on this day, a boat may safely enter the harbour if the minimum depth of $9\frac{1}{2}$ metres of water is required.

$x = 8\frac{5}{6} - 1\frac{5}{6}\cos\frac{4\pi t}{25}$

when $x = 9\frac{1}{2}$

$9\frac{1}{2} = 8\frac{5}{6} - 1\frac{5}{6}\cos\frac{4\pi t}{25}$

$\cos\frac{4\pi t}{25} = \frac{-4}{11}$

$\frac{4\pi t}{25} = \cos^{-1}\left(\frac{-4}{11}\right)$

$t = 3\text{hours } 51\text{minutes } 56\text{seconds}$

9:25am + 3hours 51minutes 56seconds = 1:17 pm

b)

(i)

$$\ddot{x} = 0 \qquad \ddot{y} = -g$$

$$\dot{x} = 100 \qquad \dot{y} = -gt + c \qquad c = 0$$

$$x = 100t \qquad y = \frac{-gt^2}{2} + c \qquad \text{when } t = 0 \qquad y = 105 \;\therefore c = 105$$

$$y = \frac{-gt^2}{2} + 105 = \frac{-10t^2}{2} + 105 = -5t^2 + 105$$

(ii) Show that the equation of the line *BL* is $y = -x$.

$m = \tan\theta$

gradient of BL is -1

When $x = 0$, $y = 0$

$y = -x$

(iii) Find the time taken for the bullet to hit the ground at *L*.

$x = 100t \qquad y = -5t^2 + 105$

$$y = -5\left(\frac{x}{100}\right)^2 + 105$$

$$y = \left(\frac{-5x^2}{10000}\right) + 105$$

$$y = -x$$

$$-x = \frac{-5x^2}{10000} + 105$$

$$x^2 - 2000x - 210000 = 0$$

$$(x - 2100)(x + 100) = 0$$

$$x = 2100$$

$$t = \frac{2100}{100} = 21\text{minutes}$$

(iv) Find the distance *BL* to the nearest metre.

$$BL = \sqrt{2100^2 + 2100^2}$$

$$BL = 2970\,\text{cm}$$

BL ≈ 30 m