

## Link Between 1995 and 2010 HSC Exams Leads To Generalised Wallis Product

by DEREK BUCHANAN

January 6, 2011

In the 1995 HSC exam [1] we saw that  $\frac{\pi}{2} \left( \frac{2}{2+\frac{1}{N}} \right) = \frac{\pi}{2} \left( \frac{2N}{2N+1} \right) < \prod_{n=1}^N \frac{(2n)^2}{(2n)^2-1} < \frac{\pi}{2}$  from which we can deduce the Wallis product by taking the limit as  $N \rightarrow \infty$ ,

$$\frac{\pi}{2} = \lim_{N \rightarrow \infty} \frac{\pi}{2} \left( \frac{2}{2+\frac{1}{N}} \right) \leq \prod_{n>0} \frac{(2n)^2}{(2n)^2-1} \leq \frac{\pi}{2} \text{ \& \therefore } \prod_{n>0} \frac{(2n)^2}{(2n)^2-1} = \frac{\pi}{2}.$$

What if we replace the 2 with a  $k$ ? We don't quite get  $\prod_{n>0} \frac{(kn)^2}{(kn)^2-1} = \frac{\pi}{k}$  (which is only true if  $k = 2$ ) but rather something similar:

**Generalised Wallis Product.**  $\prod_{n>0} \frac{(kn)^2}{(kn)^2-1} = \frac{\pi/k}{\sin(\pi/k)}$  for  $k > 1$ .

To prove this we use something from the 2010 HSC exam [2], namely that  $\sum_{n>0} \frac{1}{n^2} = \frac{\pi^2}{6}$  &  $\therefore$

$$\pi^2 = \sum_{n>0} \frac{6}{n^2}.$$

**Proof.** Suppose  $y = \frac{1}{x} + \sum_{n>0} \frac{2x}{x^2-n^2}$  for  $0 < x < 1$ . Then

$$\begin{aligned} \pi^2 + y' + y^2 &= \sum_{n>0} \frac{6}{n^2} - \frac{1}{x^2} + \sum_{n>0} \frac{(x^2-n^2)(2)-2x(2x)}{(x^2-n^2)^2} + \frac{1}{x^2} + \left( \sum_{n>0} \frac{2x}{x^2-n^2} \right)^2 + \frac{2}{x} \sum_{n>0} \frac{2x}{x^2-n^2} \\ &= \sum_{n>0} \left( \frac{6}{n^2} + \frac{-2x^2-2n^2}{(x^2-n^2)^2} + \frac{4x^2}{(x^2-n^2)^2} + \frac{4}{x^2-n^2} \right) + 4x^2 \sum_{\substack{m,n>0 \\ m \neq n}} \left( \frac{1}{x^2-n^2} \cdot \frac{1}{x^2-m^2} \right) \\ &= \sum_{n>0} \frac{6x^2}{n^2(x^2-n^2)} - 4x^2 \sum_{\substack{m,n>0 \\ m \neq n}} \left( \frac{1}{x^2-n^2} - \frac{1}{x^2-m^2} \right) \cdot \frac{1}{m^2-n^2} \\ &= \sum_{n>0} \left( \frac{6x^2}{n^2(x^2-n^2)} - \frac{8x^2}{x^2-n^2} \sum_{\substack{m>0 \\ m \neq n}} \frac{1}{m^2-n^2} \right) \\ &= \sum_{n>0} \frac{x^2}{x^2-n^2} \left( \frac{6}{n^2} - \frac{4}{n} \sum_{\substack{m>0 \\ m \neq n}} \left( \frac{1}{m-n} - \frac{1}{m+n} \right) \right) \\ &= \sum_{n>0} \frac{x^2}{x^2-n^2} \left( \frac{6}{n^2} - \frac{4}{n} \left( \frac{1}{2n-n} + \frac{1}{3n-n} \right) \right) \\ &= 0 \end{aligned}$$

whereupon  $\frac{dy}{dx} = -y^2 - \pi^2$  &  $\therefore -\pi \int dx = \int \frac{\pi dy}{y^2+\pi^2}$  &  $-\pi x = \tan^{-1} \frac{y}{\pi} + c$  for a constant  $c$ . Letting  $x = \frac{1}{2}$ , then  $y = 2 + \sum_{n>0} \frac{1}{\frac{1}{4}-n^2} = 2 + \sum_{n>0} \left( \frac{-2}{2n-1} + \frac{2}{2n+1} \right) = 2 - 2 = 0$  &  $\therefore c = -\frac{\pi}{2}$  whence  $\frac{\pi}{2} - \pi x = \tan^{-1} \frac{y}{\pi}$  &  $\tan\left(\frac{\pi}{2} - \pi x\right) = \cot \pi x = \frac{y}{\pi}$ . So we now have that  $y = \pi \cot \pi x$ .

Replacing  $x$  with  $\frac{x}{\pi}$  we have that  $\pi \cot x = \frac{\pi}{x} + \sum_{n>0} \frac{2x/\pi}{(x/\pi)^2 - n^2} = \frac{\pi}{x} + \sum_{n>0} \frac{2x\pi}{x^2 - \pi^2 n^2}$  and so  $\cot x = \frac{1}{x} + \sum_{n>0} \frac{2x}{x^2 - \pi^2 n^2}$  for  $0 < x < \pi$ .

So  $\int_0^x (\cot t - \frac{1}{t}) dt = [\ln \frac{\sin t}{t}]_0^x = \ln \frac{\sin x}{x} = \int_0^x \sum_{n>0} \frac{2t}{t^2 - \pi^2 n^2} dt = \sum_{n>0} \ln(1 - \frac{x^2}{\pi^2 n^2})$   
 $= \ln \prod_{n>0} (1 - \frac{x^2}{\pi^2 n^2})$  &  $\therefore \sin x = x \prod_{n>0} (1 - \frac{x^2}{\pi^2 n^2})$  for  $0 < x < \pi$ .<sup>†</sup>

Therefore  $\frac{x}{\sin x} = \prod_{n>0} \frac{\pi^2 n^2}{\pi^2 n^2 - x^2}$ .

Letting  $x = \frac{\pi}{k}$ , then for  $k > 1$ ,  $\prod_{n>0} \frac{\pi^2 n^2}{\pi^2 n^2 - (\pi/k)^2} = \prod_{n>0} \frac{(kn)^2}{(kn)^2 - 1} = \frac{\pi/k}{\sin(\pi/k)}$  □

**Corollary.**  $\lim_{k \rightarrow \infty} \prod_{n>0} \frac{(kn)^2}{(kn)^2 - 1} = 1$ .

**Proof.**  $\lim_{k \rightarrow \infty} \frac{\pi/k}{\sin(\pi/k)} = \lim_{\alpha \rightarrow 0} \frac{\alpha}{\sin \alpha} = 1$  where  $\alpha = \frac{\pi}{k}$ . □

## REFERENCES

- 
1. [http://www.boardofstudies.nsw.edu.au/hsc\\_exams/hsc2000exams/hsc00\\_maths/95MAT4U.PDF](http://www.boardofstudies.nsw.edu.au/hsc_exams/hsc2000exams/hsc00_maths/95MAT4U.PDF)
  2. [http://www.boardofstudies.nsw.edu.au/hsc\\_exams/hsc2010exams/pdf.doc](http://www.boardofstudies.nsw.edu.au/hsc_exams/hsc2010exams/pdf.doc)  
[/2010-hsc-exam-mathematics-extension-2.pdf](#)

---

<sup>†</sup>Actually this product expansion for  $\sin x$  is valid if  $x$  is any complex number. However the proof of this is not necessary for the purposes of this article.