Link Between 1995 and 2010 HSC Exams Leads To Generalised Wallis Product

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In the 1995 HSC exam [**1**] we saw that $\frac{\pi}{2}(\frac{2}{2+\frac{1}{N}}) = \frac{\pi}{2}(\frac{2N}{2N+1}) < \prod_{n=1}^{N} \frac{(2n)^2}{(2n)^2-1} < \frac{\pi}{2}$ from which we can deduce the Wallis product by taking the limit as $N \to \infty$,
$\frac{\pi}{2} = \lim_{N\to\infty} \frac{\pi}{2}(\frac{2}{2+\frac{1}{N}}) \le \prod_{n>0} \frac{(2n)^2}{(2n)^2-1} \le \frac{\pi}{2}$ & $\therefore \prod_{n>0} \frac{(2n)^2}{(2n)^2-1} = \frac{\pi}{2}$.

What if we replace the 2 with a k? We don't quite get $\prod_{n>0} \frac{(kn)^2}{(kn)^2-1} = \frac{\pi}{k}$ (which is only true if $k = 2$) but rather something similar:

Generalised Wallis Product. $\prod_{n>0} \frac{(kn)^2}{(kn)^2-1} = \frac{\pi/k}{\sin(\pi/k)}$ for $k > 1$.

To prove this we use something from the 2010 HSC exam [**2**], namely that $\sum_{n>0} \frac{1}{n^2} = \frac{\pi^2}{6}$ & $\therefore$ $\pi^2 = \sum_{n>0} \frac{6}{n^2}$.

Proof. Suppose $y = \frac{1}{x} + \sum_{n>0} \frac{2x}{x^2-n^2}$ for $0 < x < 1$. Then

$$\begin{aligned}
\pi^2 + y' + y^2 &= \sum_{n>0} \frac{6}{n^2} - \frac{1}{x^2} + \sum_{n>0} \frac{(x^2-n^2)(2)-2x(2x)}{(x^2-n^2)^2} + \frac{1}{x^2} + \Big(\sum_{n>0} \frac{2x}{x^2-n^2}\Big)^2 + \frac{2}{x}\sum_{n>0} \frac{2x}{x^2-n^2} \\
&= \sum_{n>0} \big(\frac{6}{n^2} + \frac{-2x^2-2n^2}{(x^2-n^2)^2} + \frac{4x^2}{(x^2-n^2)^2} + \frac{4}{x^2-n^2}\big) + 4x^2 \sum_{\substack{m,n>0 \\ m\ne n}} \big(\frac{1}{x^2-n^2} \cdot \frac{1}{x^2-m^2}\big) \\
&= \sum_{n>0} \frac{6x^2}{n^2(x^2-n^2)} - 4x^2 \sum_{\substack{m,n>0 \\ m\ne n}} \big(\frac{1}{x^2-n^2} - \frac{1}{x^2-m^2}\big) \cdot \frac{1}{m^2-n^2} \\
&= \sum_{n>0} \big(\frac{6x^2}{n^2(x^2-n^2)} - \frac{8x^2}{x^2-n^2} \sum_{\substack{m>0 \\ m\ne n}} \frac{1}{m^2-n^2}\big) \\
&= \sum_{n>0} \frac{x^2}{x^2-n^2}\big(\frac{6}{n^2} - \frac{4}{n} \sum_{\substack{m>0 \\ m\ne n}} (\frac{1}{m-n} - \frac{1}{m+n})\big) \\
&= \sum_{n>0} \frac{x^2}{x^2-n^2}\big(\frac{6}{n^2} - \frac{4}{n}(\frac{1}{2n-n} + \frac{1}{3n-n})\big) \\
&= 0
\end{aligned}$$

whereupon $\frac{dy}{dx} = -y^2 - \pi^2$ & $\therefore -\pi \int dx = \int \frac{\pi\, dy}{y^2+\pi^2}$ & $-\pi x = \tan^{-1} \frac{y}{\pi} + c$ for a constant c. Letting $x = \frac{1}{2}$, then $y = 2 + \sum_{n>0} \frac{1}{\frac{1}{4}-n^2} = 2 + \sum_{n>0} (\frac{-2}{2n-1} + \frac{2}{2n+1}) = 2 - 2 = 0$ & $\therefore c = -\frac{\pi}{2}$ whence $\frac{\pi}{2} - \pi x = \tan^{-1} \frac{y}{\pi}$ & $\tan(\frac{\pi}{2} - \pi x) = \cot \pi x = \frac{y}{\pi}$. So we now have that $y = \pi \cot \pi x$.

Replacing x with $\frac{x}{\pi}$ we have that $\pi \cot x = \frac{\pi}{x} + \sum\limits_{n>0} \frac{2x/\pi}{(x/\pi)^2 - n^2} = \frac{\pi}{x} + \sum\limits_{n>0} \frac{2x\pi}{x^2 - \pi^2 n^2}$ and so $\cot x = \frac{1}{x} + \sum\limits_{n>0} \frac{2x}{x^2 - \pi^2 n^2}$ for $0 < x < \pi$.

So $\int_0^x (\cot t - \frac{1}{t})\, dt = \left[\ln \frac{\sin t}{t}\right]_0^x = \ln \frac{\sin x}{x} = \int_0^x \sum\limits_{n>0} \frac{2t}{t^2 - \pi^2 n^2}\, dt = \sum\limits_{n>0} \ln(1 - \frac{x^2}{\pi^2 n^2})$

$= \ln \prod\limits_{n>0} (1 - \frac{x^2}{\pi^2 n^2})$ & $\therefore$ $\sin x = x \prod\limits_{n>0} (1 - \frac{x^2}{\pi^2 n^2})$ for $0 < x < \pi$.[†]

Therefore $\frac{x}{\sin x} = \prod\limits_{n>0} \frac{\pi^2 n^2}{\pi^2 n^2 - x^2}$.

Letting $x = \frac{\pi}{k}$, then for $k > 1$, $\prod\limits_{n>0} \frac{\pi^2 n^2}{\pi^2 n^2 - (\pi/k)^2} = \prod\limits_{n>0} \frac{(kn)^2}{(kn)^2 - 1} = \frac{\pi/k}{\sin(\pi/k)}$ □

Corollary. $\lim\limits_{k\to\infty} \prod\limits_{n>0} \frac{(kn)^2}{(kn)^2 - 1} = 1$.

Proof. $\lim\limits_{k\to\infty} \frac{\pi/k}{\sin(\pi/k)} = \lim\limits_{\alpha\to 0} \frac{\alpha}{\sin \alpha} = 1$ where $\alpha = \frac{\pi}{k}$. □

REFERENCES

1. http://www.boardofstudies.nsw.edu.au/hsc_exams/hsc2000exams/hsc00_maths/95MAT4U.PDF

2. http://www.boardofstudies.nsw.edu.au/hsc_exams/hsc2010exams/pdf_doc
/2010-hsc-exam-mathematics-extension-2.pdf

[†] Actually this product expansion for $\sin x$ is valid if x is any complex number. However the proof of this is not necessary for the purposes of this article.