Question 1

(a) $\int_0^1 \frac{e^x}{(1+e^x)^2}\,dx$

let $u = 1+e^x$
$du = e^x\,dx$
$x=1,\ u=1+e$
$x=0,\ u=2$

$= \int_2^{1+e} \frac{du}{u^2}$

$= \left[-\frac{1}{u}\right]_2^{1+e}$

$= -\frac{1}{1+e} + \frac{1}{2}$

(b) $\int x^3 \log x\,dx$

log is only practical to differentiate
let $u = \log x$ and $\frac{dv}{dx} = x^3$
$\frac{du}{dx} = \frac{1}{x}$ $\quad v = \frac{x^4}{4}$

$= \left[\frac{x^4}{4}\log x\right] - \int x^2\,dx$

$= \frac{x^4}{4}\log x - \frac{x^3}{3} + C$

(c) $\int \frac{dx}{\sqrt{x^2-2x+5}}$

x^2-2x+5
$= x^2-2x+1^2+4$
$= (x-1)^2+4$
letting $u = x-1$ makes it look like $\int \frac{1}{\sqrt{x^2+a^2}}\,dx$
$\frac{d}{dx}(x-1) = 1$ so there's really no effect.

$= \int \frac{dx}{\sqrt{(x-1)^2+4}}$

$= \ln\left((x-1)+\sqrt{(x-1)^2+4}\right)+C$

(d)(i) $\times (x-1)(x^2+4)$:

$5x^2-3x+13 \equiv a(x^2+4)+(bx-1)(x-1)$
$\equiv ax^2+4a+bx^2-bx-x+1$

$a+b=5$
$-b-1=-3 \Rightarrow b=2$
$\Big\}\ b=2,\ a=3$

(ii) $\int\left(\frac{3}{x-1}+\frac{2x-1}{x^2+4}\right)dx = \int\left(\frac{3}{x-1}+\frac{2x}{x^2+4}-\frac{1}{x^2+4}\right)dx$

$= 3\ln(x-1)+\ln(x^2+4)-\frac{1}{2}\tan^{-1}\frac{x}{2}+C$

(e) $\int_0^{3/\sqrt{2}} \frac{dx}{(9-x^2)^{3/2}}$

let $x = 3\sin\theta$.
$9-x^2 = 9(1-\sin^2\theta) = 9\cos^2\theta$.
$dx = 3\cos\theta\,d\theta$
$x = 3/\sqrt{2},\ \theta = \sin^{-1}\frac{x}{3} = \frac{\pi}{4}$
$x=0,\ \theta = \sin^{-1}0 = 0$

$= \int_0^{\pi/4} \frac{3\cos\theta\,d\theta}{3^3\cos^3\theta}$

$= \int_0^{\pi/4} \frac{1}{9}\sec^2\theta\,d\theta$

$= \frac{1}{9}\left[\tan\theta\right]_0^{\pi/4} = \frac{1}{9}$

Question 2

(a) $z = 2+i \quad w = 1-i$

(i) $z\bar{w} = (2+i)(1+i)$
$= 2+2i+i-1$
$= 1+3i$

(ii) $\dfrac{4}{z} = \dfrac{4}{2+i} \times \dfrac{2-i}{2-i} = \dfrac{4(2-i)}{4+1} = \dfrac{8}{5} - \dfrac{4}{5}i$

(b) (i) $\alpha = -1+i$

$|\alpha| = \sqrt{(-1)^2+1^2}$

$\arg\alpha = \pi - \tan^{-1}\left|\frac{1}{-1}\right| = \frac{3\pi}{4}$ $\Big\}$ $\alpha = \sqrt{2}\ \text{cis}\ \frac{3\pi}{4}$

(ii) put $z = \alpha$: $\alpha^4 + 4 = (\sqrt{2})^4 \text{cis}\, 4\left(\frac{3\pi}{4}\right) + 4$
$= 4(\cos 3\pi + i\sin 3\pi) + 4$
$= 0$

(iii) since coefficients of z^4+4 are real, non-real zeroes occur in conjugate pairs.

$(z-(-1+i))(z-(-1-i))$
$= ((z+1)-i)((z+1)+i)$
$= (z+1)^2 - i^2 = z^2+2z+2$ is a factor

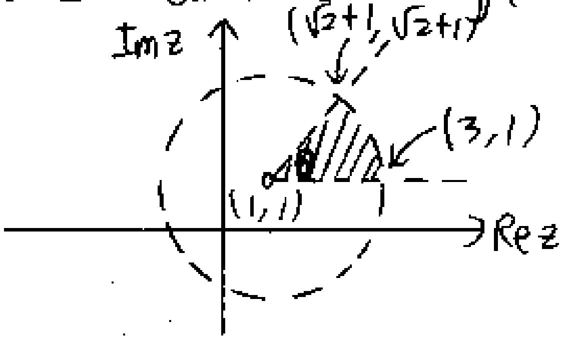
$\therefore z^4+4 = (z^2+2z+2)(z^2+bz+c)$, $c=2$ by inspection.

coeff of z : $2c+2b = 0$
$b = -2$

$\therefore z^4+4 = (z^2+2z+2)(z^2-2z+2)$

but we only need one factor so just choose one of them.

(c) $|z-(1+i)| < 2$ and $0 < \arg(z-(1+i)) < \pi/4$



$\theta = \pi/4$

$x^2+y^2 = 2^2$ and $y = x$
$x^2+x^2 = 4$
$x = \sqrt{2}$

Question 2

(d) $(\cos\theta + i\sin\theta)^5$
$= \cos 5\theta + i\sin 5\theta$
(De Moivre's theorem)

also $(\cos\theta + i\sin\theta)^5$
$= \cos^5\theta + 5\cos^4\theta\, i\sin\theta + 10\cos^3\theta(-1)\sin^2\theta$
$+ 10\cos^2\theta(-i)\sin^3\theta + 5\cos\theta\sin^4\theta$
$+ i\sin^5\theta$
(binomial expansion)

equating real parts: $\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

$\cos 5\theta = \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)(1-\cos^2\theta)$
$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta(1-2\cos^2\theta+\cos^4\theta)$
$=$ " " " $5\cos\theta - 10\cos^3\theta + 5\cos^5\theta$
$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$.

(e) unit circle: circle of radius 1 unit.

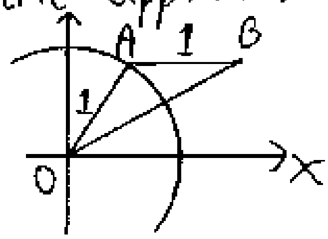
Algebraic approach:

let $z = \cos 2\theta + i\sin 2\theta$
$z+1 = \cos 2\theta + 1 + i\sin 2\theta$
$= 2\cos^2\theta - 1 + 1 + i2\sin\theta\cos\theta$
$= 2\cos^2\theta + i2\sin\theta\cos\theta$

$\arg(z+1) = \tan^{-1}\dfrac{2\sin\theta\cos\theta}{2\cos^2\theta}$
$= \tan^{-1}\tan\theta = \theta$.

$\therefore 2\arg(z+1) = \arg(z)$.

Geometric approach:



A $\Rightarrow z$
B $\Rightarrow z+1$

$\angle AOX = \theta = \arg(z)$.

ΔOAB is isosceles, $\angle AOB = \alpha$

also, $\angle OAB = 180^\circ - \theta$ since $AB \parallel OX$

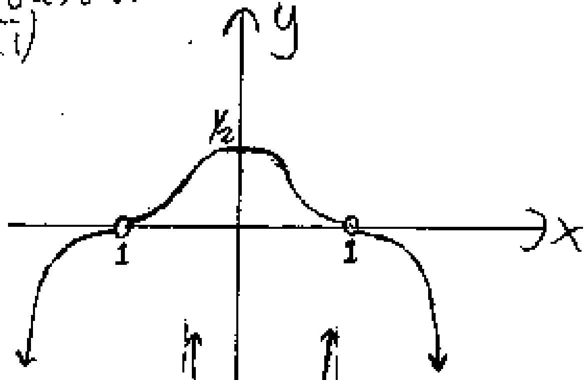
$\therefore \angle OAB + 2\alpha = 180^\circ$ ($\angle$ sum in ΔOAB)

$\alpha = \frac{1}{2}\theta$

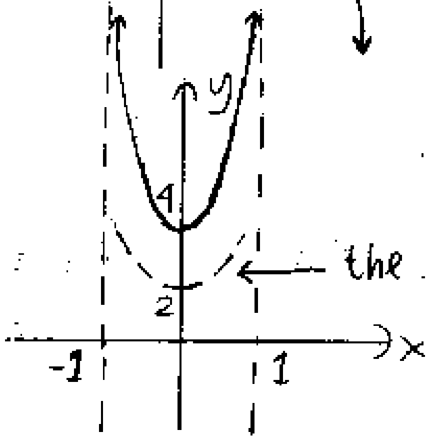
but $\arg(z+1) = \angle XOB = \theta - \alpha$
$= \frac{1}{2}\theta$.

Question 3.

(a) (i)



(ii)



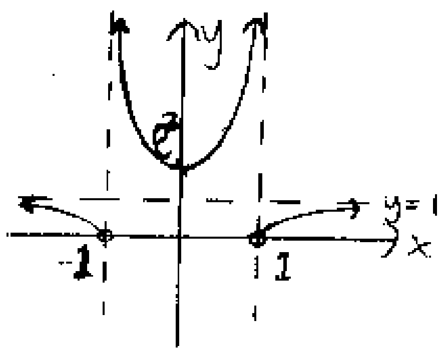
for $x > 1$ and $x < 1$,
$y = f(x) + |f(x)| = 0$

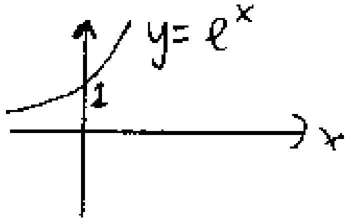
the $y = f(x)$ curve for comparison. It is less steep

(iii)

note: the middle part has asymptotes.

(iv)





note: the middle part of $y = e^{f(x)}$ is steeper than in part(iii)

(b) $e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{4}{9}} = \frac{\sqrt{5}}{3}$

foci: $(ae, 0)$ and $(-ae, 0)$
$(3\sqrt{5}, 0)$ and $(-3\sqrt{5}, 0)$

directrices: $y = a/e$ and $y = -a/e$
$y = 27/\sqrt{5}$ and $y = -27/\sqrt{5}$

(c)(i) from $y = (x-1)(3-x)$ we have $x^2 - 4x + (3+y) = 0$.

sum of roots: $(x_1 + x_2) = 4$ | $(x_2 - x_1)^2 = (x_1 + x_2)^2 - 4x_1x_2$

product of root $x_1 x_2 = (3+y)$ | $(x_2 - x_1) = 2\sqrt{1-y}$

note: x_1 and x_2 are the endpoints of line l.

$\delta A = \pi\left((3-x_1)^2 - (3-x_2)^2\right) = (x_2 - x_1)(6 - (x_1 + x_2))$ using $a^2 - b^2 = (a-b)(a+b)$

$= 4\pi\sqrt{1-y}$. Note: alternatively you can move y-axis to $x=3$.

Question 3

(c)(ii) the curve has a maximum at $x=\frac{3+1}{2}=2$

$y=(2-1)(3-2)=1$

Volume $=\int_0^1 4\pi\sqrt{1-y}\,dy$

let $u=1-y$

$du=-dy$

$=4\pi\int_1^0 u^{1/2}\cdot -du$

$=4\pi\left[\frac{2u^{3/2}}{3}\right]_0^1$

$=4\pi(2/3)=\frac{8\pi}{3}$ units^3

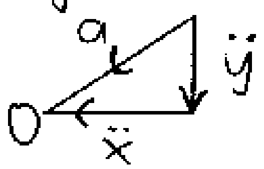
Question 4

(a) (i) $\dot{x}=-r\sin\theta\times w$, where $w=\frac{d\theta}{dt}$ and is a constant.

$\dot{y}=r\cos\theta\times w$

$\ddot{x}=-rw\cos\theta\times w$

$\ddot{y}=-rw\sin\theta\times w$



$a=\sqrt{(-rw^2\cos\theta)^2+(-rw^2\sin\theta)^2}$

$=rw^2$, since $\sin^2\theta+\cos^2\theta=1$

$\therefore F=ma$

$=mrw^2$, directed towards O.

(ii) $\frac{Am}{r^2}=mrw^2$

$\frac{A}{r^3}=w^2 \Rightarrow r=\sqrt[3]{\frac{A}{w^2}}$

(b) (i) $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

differentiating: $\frac{2x}{a^2}-\frac{2y}{b^2}\frac{dy}{dx}=0$

$\frac{dy}{dx}=\frac{b^2}{a^2}\frac{x}{y}$

$=\frac{b}{a\sin\theta}$ at P.

tangent: $y-y_1=m(x-x_1)$

$y-b\tan\theta=\frac{b}{a\sin\theta}(x-a\sec\theta)$

$a\sin\theta\, y-ab\sin\theta\tan\theta=bx-ab\sec\theta$

or $bx\sec\theta-ay\tan\theta=ab$

(which will be easier for later parts)

(b)(ii) asymptotes are $y = \pm \frac{b}{a} x$

put $y = \frac{b}{a} x$ into tangent equation:

$$a\sin\theta\left(\frac{b}{a}x\right) - ab\sin\theta\tan\theta = bx - ab\sec\theta$$

$$\sin\theta\, x - a\sin\theta\tan\theta - x + a\sec\theta = 0$$

$$(\sin\theta - 1)x = a(\sin\theta\tan\theta - \sec\theta)$$

$$= a\left(\frac{\sin^2\theta}{\cos\theta} - \frac{1}{\cos\theta}\right)$$

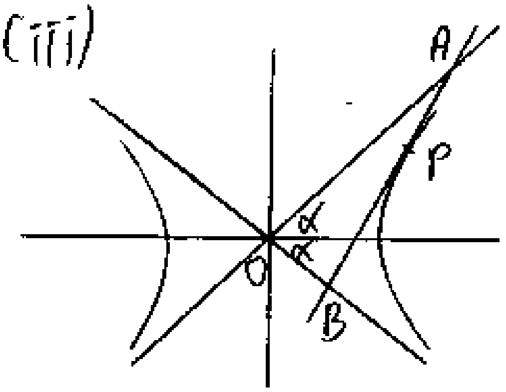
$$= a(-\cos\theta)$$

$$x = \frac{a\cos\theta}{1-\sin\theta}, \quad y = \frac{b}{a}x = \frac{b\cos\theta}{1-\sin\theta}$$

putting $y = -\frac{b}{a}x$, we can see $x = \frac{a\cos\theta}{1+\sin\theta}$, $y = \frac{-b\cos\theta}{1+\sin\theta}$

Alternatively, we can substitute A and B into the equations of asymptotes and tangents.

(iii)



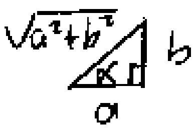
$\alpha = \tan^{-1}\frac{b}{a}$

using sine rule for area,

area $\Delta OAB = \frac{1}{2}\, OA \times OB \times \sin 2\alpha$

$$OA = \sqrt{\frac{a^2\cos^2\theta + b^2\cos^2\theta}{(1-\sin\theta)^2}} = \frac{\sqrt{a^2+b^2}\cos\theta}{(1-\sin\theta)}$$

$$OB = \frac{\sqrt{a^2+b^2}\cos\theta}{(1+\sin\theta)}$$



$$\sin 2\alpha = 2\sin\alpha\cos\alpha = 2\frac{ab}{a^2+b^2}$$

$$\text{area } \Delta OAB = \frac{1}{2} \times \frac{\sqrt{a^2+b^2}\cos\theta}{1-\sin\theta} \times \frac{\sqrt{a^2+b^2}\cos\theta}{1+\sin\theta} \times \frac{2ab}{a^2+b^2}$$

$$= \frac{ab\cos^2\theta}{1-\sin^2\theta} = ab$$

Alternative approach to finding area ΔOAB:

area = $\frac{1}{2}$ base × perpendicular height.

(use the perpendicular distance formula)

Question 4

(c)(i) Each person can choose from n doors.
Repetition is allowed.
Number of ways $= n^n$

(ii) This event is the complementary event of each door being chosen (by exactly one person).

$$P = 1 - \frac{n!}{n^n} \quad \text{OR} \quad \frac{n^n - n!}{n^n}$$

Question 5

(a)(i) $S_1 = \alpha + \beta + \gamma = 0$ because the coefficient of x^2 is 0.

$S_2 = \alpha^2 + \beta^2 + \gamma^2$
$= (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$
$= (0)^2 - 2(p)$
$= -2p.$

$S_3 = \alpha^3 + \beta^3 + \gamma^3$
$= -p(0) - 3q$
$= -3q$

putting $x = \alpha, \beta, \gamma$ in turn:

$$\begin{aligned} \alpha^3 + p\alpha + q &= 0 \\ \beta^3 + p\beta + q &= 0 \\ \gamma^3 + p\gamma + q &= 0 \quad + \end{aligned}$$

($x = \alpha$ is a root of $x^3 + px + q = 0$)

$$\alpha^3 + \beta^3 + \gamma^3 = -p(\alpha + \beta + \gamma) - 3q$$

(ii) $\alpha^3 + p\alpha + q = 0 \Rightarrow \alpha^3 = -p\alpha - q$

$\times \alpha^{n-3}$: $\alpha^n = -p\alpha^{n-2} - q\alpha^{n-3}$

similarly: $\beta^n = -p\beta^{n-2} - q\beta^{n-3}$

$\gamma^n = -p\gamma^{n-2} - q\gamma^{n-3} \quad +$

$$\alpha^n + \beta^n + \gamma^n = -p(\ldots) - q(\ldots)$$

$$S_n = -pS_{n-2} - qS_{n-3}$$

(iii) put $n = 5$: $S_5 = -pS_3 - qS_2$
but from (i), $S_2 = -2p$ and $S_3 = -3q$
hence $S_5 = \left(\frac{1}{2}S_2\right)S_3 + \left(\frac{1}{3}S_3\right)S_2 = \frac{5}{6}S_2S_3$

$$\therefore \frac{S_5}{5} = \left(\frac{S_2}{2}\right)\left(\frac{S_3}{3}\right)$$

Question 5

(b) (i) $\ddot{x} = -k\dot{x}$

$\frac{dv}{dt} = -kv$ (we want v in terms of t so we let $\ddot{x} = dv/dt$)

$\frac{dt}{dv} = -\frac{1}{k}\frac{1}{v} \Rightarrow t = -\frac{1}{k}\ln v + C$

initially: $0 = -\frac{1}{k}\ln(u\cos\alpha) + C \Rightarrow C = \frac{1}{k}\ln(u\cos\alpha)$

$\therefore t = -\frac{1}{k}\ln\left(\frac{v}{u\cos\alpha}\right)$

$e^{-kt} = \frac{v}{u\cos\alpha} \Rightarrow v = u e^{-kt}\cos\alpha$.

(ii) differentiating: $\dot{y} = \frac{1}{k}\left(-k(ku\sin\alpha + g)e^{-kt}\right)$

$= -(ku\sin\alpha + g)e^{-kt}$

$= -k\dot{y} - g$, by inspection.

It also satisfies initial condition:

at $t=0$: $\dot{y} = \frac{1}{k}\left((ku\sin\alpha + g)e^{0} - g\right)$

$= u\sin\alpha$ as expected.

(iii) max height: $\dot{y} = 0$

$(ku\sin\alpha + g)e^{-kt} = g$

$e^{-kt} = \frac{g}{ku\sin\alpha + g}$

$-kt = \ln(\dots)$

$t = -\frac{1}{k}\ln\left(\frac{g}{ku\sin\alpha + g}\right)$

(iv) integrating $\dot{x} = u e^{-kt}\cos\alpha$

$x = -\frac{u}{k}e^{-kt}\cos\alpha + C$.

initially $x=0 \Rightarrow C = \frac{u}{k}\cos\alpha$.

as $t \to \infty$, $x = -\frac{u}{k}\cos\alpha\, e^{-\infty} + \frac{u}{k}\cos\alpha$

$= \frac{u}{k}\cos\alpha$, which is the limiting value.

Question 6

(a) (i) LHS $= \cos(ax+bx) + \cos(ax-bx)$

$= \cos ax\cos bx - \sin ax\sin bx + \cos ax\cos bx + \sin ax\sin bx$

$= 2\cos ax\cos bx = $ RHS.

(ii) $= \frac{1}{2}\int 2\cos 3x\cos 2x\,dx = \frac{1}{2}\int(\cos 5x + \cos x)\,dx$

$= \frac{1}{2}\left[\frac{\sin 5x}{5} + \sin x\right] + C$

$= \frac{1}{10}\sin 5x + \frac{1}{5}\sin x + C$.

(b) S_n is a sequence defined by:

$S_1 = 1$
$S_2 = 2$
$S_n = S_{n-1} + (n-1)S_{n-2}$ for $n > 2$.

(i) $S_3 = S_2 + (3-1)S_1$
$= 2 + 1 = 3.$

$S_4 = S_3 + (4-1)S_2$
$= 3 + 3(2) = 9$

(ii) $LHS^2 = (\sqrt{x})^2 + 2x\sqrt{x} + x^2$
$= x + x^2 + 2x\sqrt{x}$
$= x(x+1) + 2x\sqrt{x}$
$= RHS^2 + 2x\sqrt{x}, x \geq 0$
$\therefore LHS \geq RHS.$

Alternatively, this is triangle inequality in a right-angled triangle. (also prove for when $x=0$)

(iii) for $n=1$, $S_1 \geq \sqrt{1!}$ by inspection.
for $n=2$, $S_2 \geq \sqrt{2!}$ since $2 > \sqrt{2}$ by definition
for $n > 2$, we need to show that
$S_n = S_{n-1} + (n-1)S_{n-2} \geq \sqrt{n!}$

Let k be a positive integer, $k \geq 2$. If $S_n \geq \sqrt{n!}$ for all integers $n \leq k$ then $S_n \geq \sqrt{n!}$ for $n = 1, 2, 3, \ldots, k$.
Consider S_{k+1}.

$S_{k+1} = S_k + (k+1-1)S_{k-1}$
$= S_k + kS_{k-1}$
$\geq \sqrt{k!} + k\sqrt{(k-1)!}$
$= \sqrt{k}\sqrt{(k-1)!} + k\sqrt{(k-1)!}$
$= (\sqrt{k} + k)\sqrt{(k-1)!}$
$\geq \sqrt{k(k+1)}\sqrt{(k-1)!}$ from part (ii)
$= \sqrt{k(k+1)(k-1)!} = \sqrt{(k+1)!}$

$\therefore S_n \geq \sqrt{n!}$ for $n \leq k$ implies $S_n \geq \sqrt{n!}$ for $n = k+1$, where $k \geq 2$. But $S_n \geq \sqrt{n!}$ for $n=1$ and $n=2$.
Hence $S_n \geq \sqrt{n!}$ for all positive integers n.

(c)(i) LHS − RHS $= \frac{x+y}{2} - \sqrt{xy}$

$= \frac{1}{2}(x+y-2\sqrt{x}\sqrt{y})$

$= \frac{1}{2}(\sqrt{x}-\sqrt{y})^2 \geq 0$ as required.

(ii) from (i): $x+y \geq 2\sqrt{xy}$, $x \geq 0$ and $y \geq 0$, also $\sqrt{xy} \geq 0$.

so
$$\left.\begin{aligned} a^4+b^4 &\geq 2a^2b^2 \\ a^4+c^4 &\geq 2a^2c^2 \\ b^4+c^4 &\geq 2b^2c^2 \end{aligned}\right\} \begin{aligned} &2(a^4+b^4+c^4) \geq \\ &\quad 2(a^2b^2+a^2c^2+b^2c^2) \\ &a^4+b^4+c^4 \geq a^2b^2+a^2c^2+b^2c^2 \end{aligned}$$

(iii) similarly, using $x+y \geq 2\sqrt{xy}$ or otherwise using (ii),

$(ab)^2 + (ac)^2 \geq 2(ab)(ac)$

$(ab)^2+(bc)^2 \geq 2(ab)(bc)$

$(ac)^2+(bc)^2 \geq 2(ac)(bc)$ +

$\div 2$: $(ab)^2+(ac)^2+(bc)^2 \geq a^2bc+b^2ac+c^2ab$

[If one of the 3 terms on RHS is −ve then LHS > RHS anyway]

(iv) $abcd$

$= abc(a+b+c)$

$= a^2bc + b^2ac + c^2ab$.

$\therefore a^2b^2 + a^2c^2 + b^2c^2 \geq abcd$.

But $a^4+b^4+c^4 \geq a^2b^2+a^2c^2+b^2c^2$

so $a^4+b^4+c^4 \geq abcd$.

Question 7.

(a)(i) curve is $y = x(3-x^2) = x(\sqrt{3}+x)(\sqrt{3}-x)$

$\delta V = 2\pi r h \,\delta x = 2\pi x|y|\,\delta x$

$= 2\pi x \cdot x(3-x^2)\,\delta x$, for $0 \leq x \leq \sqrt{3}$ (by inspection)

Volume $= \lim_{\delta x \to 0} \sum_{x=0}^{\sqrt{3}} \delta V = \int_0^{\sqrt{3}} 2\pi x^2(3-x^2)\,dx$

$= 2\pi \left[\frac{3x^3}{3} - \frac{x^5}{5}\right]_0^{\sqrt{3}} = 2\pi \times \frac{6}{5}\sqrt{3}$

$= \frac{12}{5}\sqrt{3}\pi$ units^3

(b)(i) $\angle P$ is common

$\angle SAP = \angle BSP$ (alternate segment theorem)

Hence, $\Delta ASP \parallel\!\mid \Delta SBP$ (2 angles are equal)

Question 7

(b)(ii) ratios of sides:

$$\frac{SP}{BP} = \frac{AP}{SP} \Rightarrow SP^2 = AP \times BP \quad -(1)$$

Looking at the derivation, result (1) is not unique to a circle with a particular radius, and also holds for the bigger circle.

$$TP^2 = AB \times BP \quad -(2)$$

From (1) and (2), $PT = PS$

(iii) DT will pass through the centre of C_2 if $\angle PTD = 90°$. (i.e. DT is on top of the radius from T).
Let's prove that $\Delta SPD \equiv \Delta TPD$.

$\angle P$ is common
$SP = TP$ (from (ii))
DP is common
} equal sides and equal included angles $\Rightarrow$ proven.

$\therefore \angle PTD = \angle PSD$ (corresponding angles)
$= 90°$ ($SD \perp SP$)

(c)(i) By inspection, this means we need to show:

$$\cos\left(\frac{\alpha}{2^n}\right)\sin\left(\frac{\alpha}{2^n}\right) = \frac{1}{2}\sin\left(\frac{\alpha}{2^{n-1}}\right)$$

$LHS = \frac{1}{2}\sin\left(\frac{2\alpha}{2^n}\right)$ using double angle result.
$= \frac{1}{2}\sin\left(\frac{\alpha}{2^{n-1}}\right) = RHS.$

note: P_{n-1} is $\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{4}\right)\ldots\cos\left(\frac{\alpha}{2^{n-1}}\right)$

(ii) from (i): $P_n = \dfrac{P_{n-1}\sin\left(\frac{\alpha}{2^{n-1}}\right)}{2\sin\left(\frac{\alpha}{2^n}\right)}$

where $P_{n-1} = \dfrac{P_{n-2}\sin\left(\frac{\alpha}{2^{n-2}}\right)}{2\sin\left(\frac{\alpha}{2^{n-1}}\right)}$ etc. and finally

$$P_{n-(n-2)} = \frac{P_1 \sin\left(\frac{\alpha}{2}\right)}{2\sin\left(\frac{\alpha}{2^2}\right)} = \frac{\cos\left(\frac{\alpha}{2}\right)\sin\left(\frac{\alpha}{2}\right)}{2\sin\left(\frac{\alpha}{2^2}\right)}$$
$$= \frac{\sin\alpha}{2.2\sin\left(\frac{\alpha}{2^2}\right)}$$

Alternatively, expand P_n up to P_0 and find P_0 ($\neq \cos\frac{\alpha}{2^0}$)

In the process, $\sin\left(\frac{\alpha}{2^{n-1}}\right), \sin\left(\frac{\alpha}{2^{n-2}}\right), \ldots, \sin\left(\frac{\alpha}{2^{n-(n-2)}}\right)$ will cancel out. The 2 in the denominator will be $2 \cdot 2^{n-2} \cdot 2 = 2^n$. Hence the given result is true.

Question 7

(c) (iii) LHS $= \dfrac{\sin\alpha}{P_n} = \sin\alpha \times \dfrac{1}{P_n}$

$= \sin\alpha \times \dfrac{2^n \sin\left(\frac{\alpha}{2^n}\right)}{\sin\alpha}$ from (ii)

$= 2^n \sin\left(\frac{\alpha}{2^n}\right)$

$< 2^n \dfrac{\alpha}{2^n}$ since $\sin x < x$ for $x > 0$

$= \alpha$, as required

Question 8

(a) (i) let a and b be positive integers.

Any $(k, 2k)$ pair can be written as

$(3a+1, 3b+2)$ or $(3a+2, 3b+1)$ or $(3a, 3b)$

Hence $1 + w^k + w^{2k}$ is either

$1 + w^{3a} + w^{3b}$ or $1 + w^{3a+1} + w^{3b+2}$ or $1 + w^{3a+2} + w^{3b+1}$

$1+1+1$ or $1 + w + w^2$ since $w^3 = 1$, $w^{3a} = 1$ and $w^{3b} = 1$

let $w = \text{cis}\,\theta$

$\theta = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$ since $w \neq 1$

when $w = \text{cis}\frac{2\pi}{3}$,

$1 + w + w^2 = 1 + \text{cis}\frac{2\pi}{3} + \text{cis}\frac{4\pi}{3}$

$= 1 - \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} - \cos\frac{\pi}{3} - i\sin\frac{\pi}{3}$

$= 0$

when $w = \text{cis}\frac{4\pi}{3}$,

$1 + w + w^2 = 1 + \text{cis}\frac{4\pi}{3} + \text{cis}\frac{8\pi}{3} = 0$

$\therefore$ Possible values are 3 or 0.

(ii) $(1+w)^n = {}^nC_0 w^0 + {}^nC_1 w^1 + {}^nC_2 w^2 + \ldots + {}^nC_n w^n$

$(1+w^2)^n = {}^nC_0 (w^2)^0 + {}^nC_1 (w^2)^1 + \ldots + {}^nC_n (w^2)^n$

(i) Alternative approach :

$(w^k - 1)(1 + w^k + w^{2k}) = w^k + w^{2k} + w^{3k} - (1 + w^k + w^{2k})$

$= (w^3)^k - 1$

$= 0$ since $w^3 = 1$

$\therefore$ either $(w^k - 1) = 0$ or $(1 + w^k + w^{2k}) = 0$

$w^k = 1$

$1 + w^k + w^{2k} = 3$ or $1 + w^k + w^{2k} = 0$.

(a) (iii) We need to show: ${}^nC_0 + {}^nC_3 + {}^nC_6 + \ldots + {}^nC_{3l} = \frac{1}{3}\left(2^n + (1+w)^n + (1+w^2)^n\right)$

$(1+w)^n + (1+w^2)^n$
$= {}^nC_0(w^0+w^0) + {}^nC_1(w^1+w^2) + {}^nC_2(w^2+w^4) + \ldots +$
${}^nC_{3l}(w^{3l}+w^{6l}) + \ldots + {}^nC_n(w^n+w^{2n})$

[but from (i), $w^k + w^{2k}$ is either:
$= 2$ if k is a multiple of 3, inc. $k=0$
or $= -1$ if it's not]

$= 2({}^nC_0 + {}^nC_3 + {}^nC_6 + \ldots + {}^nC_{3l}) - ({}^nC_1 + {}^nC_2 + {}^nC_4 + {}^nC_5 + \ldots + {}^nC_n)$
$= 2({}^nC_0 + {}^nC_3 + \ldots + {}^nC_{3l}) - (2^n - ({}^nC_0 + {}^nC_3 + \ldots + {}^nC_{3l}))$

since $2^n = \sum_{k=0}^{n} {}^nC_k$

$= 3({}^nC_0 + {}^nC_3 + \ldots + {}^nC_{3l}) - 2^n$

$\therefore$ RHS in the statement to be proven
$= \frac{1}{3}\left(2^n + [3({}^nC_0 + {}^nC_3 + \ldots + {}^nC_{3l}) - 2^n]\right)$
$=$ LHS

(iv) By inspection, we need to prove that
$2 = (1+w)^n + (1+w^2)^n$ — ①

More careful inspection suggests that we can't use binomial theorem like in (iii).

from (i) we have $1 + w + w^2 = 0$.
$1 + w = -w^2$
$1 + w^2 = -w$.

RHS in ① $= (-w^2)^n + (-w)^n$, but $n = 6m$, $m = 0, 1, 2, \ldots$
$= (-w^2)^{6m} + (-w)^{6m}$
$= w^{12m} + w^{6m}$
$= 2$ using $w^k + w^{2k} = 2$ result.
$=$ LHS.

Question 8

(b)(i) By inspection of the given result, and by trial and error principle,

$$I_n = \frac{q^{2n}}{n!}\left[\sin x\left(\frac{\pi^2}{4}-x^2\right)^n\right]_{-\pi/2}^{\pi/2} - \frac{q^{2n}}{n!}\int_{-\pi/2}^{\pi/2} n\left(\frac{\pi^2}{4}-x^2\right)^{n-1}\sin x.-2x.dx$$

by letting $u = \left(\frac{\pi^2}{4}-x^2\right)^n$ and $\frac{dv}{dx} = \cos x$

$$\frac{du}{dx} = n.-2x.\left(\frac{\pi^2}{4}-x^2\right)^{n-1} \text{ and } v = \sin x$$

(note: if we let $u = \left(\frac{\pi^2}{4}-x^2\right)^n\cos x$, we will need to use the product rule to differentrate and we will still have $\left(\frac{\pi^2}{4}-x^2\right)^n$ in $\frac{du}{dx}$)

$$I_n = \frac{2q^{2n}}{(n-1)!}\int_{-\pi/2}^{\pi/2}\left(\frac{\pi^2}{4}-x^2\right)^{n-1}x.\sin x.dx$$

Now we let $u_1 = \left(\frac{\pi^2}{4}-x^2\right)^{n-1}x$ and $\frac{dv_1}{dx} = \sin x$

$$\frac{du_1}{dx} = (-2x)(n-1)\left(\frac{\pi^2}{4}-x^2\right)^{n-2}(x) + \left(\frac{\pi^2}{4}-x^2\right)^{n-1}$$

$$v_1 = -\cos x$$

$$I_n = \frac{2q^{2n}}{(n-1)!}\left[-x\cos x\left(\frac{\pi^2}{4}-x^2\right)^{n-1}\right]_{-\pi/2}^{\pi/2} - \frac{2q^{2n}}{(n-1)!}\int_{-\pi/2}^{\pi/2}-\cos x\left(\frac{\pi^2}{4}-x^2\right)^{n-1}dx$$

$$- \frac{2q^{2n}}{(n-1)!}\int_{-\pi/2}^{\pi/2}(n-1)(2x)(\cos x)(x)\left(\frac{\pi^2}{4}-x^2\right)^{n-2}dx$$

① — $I_n = \frac{2q^{2n}}{(n-1)!}\int_{-\pi/2}^{\pi/2}\left(\frac{\pi^2}{4}-x^2\right)^{n-1}\cos x\,dx - \frac{4q^{2n}}{(n-2)!}\int_{\pi/2}^{\pi/2}x^2\left(\frac{\pi^2}{4}-x^2\right)^{n-2}\cos x\,dx$

(note: if we let $u_1 = \left(\frac{\pi^2}{4}-x^2\right)^{n-1}\cos x$, $\frac{du_1}{dx}$ will contain $\cos x$ and $\sin x$, so will $\int v_1\frac{du_1}{dx}dx$)

(ii) We replace the x^2 in the second integrand.

$$I_n = \frac{2q^{2n}}{(n-1)!}\int_{-\pi/2}^{\pi/2}(\ldots)^{n-1}\cos x\,dx - \frac{4q^{2n}}{(n-2)!}\int_{\pi/2}^{\pi/2}\frac{\pi^2}{4}(\ldots)^{n-2} + \frac{4q^2}{(n-2)!}\int_{-\pi/2}^{\pi/2}(\ldots)^{n-1}dx$$

$$= \frac{2q^{2n}}{(n-1)!}\times\frac{(n-1)!}{q^{2n-2}}I_{n-1} - \frac{4q^{2n}}{(n-2)!}\times\frac{p^2}{4q^2}\times\frac{(n-2)!}{q^{2n-4}}I_{n-2} + \frac{4q^{2n}}{(n-2)!}\times\frac{(n-1)!}{q^{2n-2}}I_{n-1}$$

(remembering $\pi = p/q$)

$$= 2q^2I_{n-1} - p^2q^2I_{n-2} + 4(n-1)q^2I_{n-1}$$

$$= (4n-2)q^2I_{n-1} - p^2q^2I_{n-2}$$

(iii) $I_0 = 2$ and $I_1 = 4q^2$ (given in question)

$I_2 = (4n-2)q^2I_1 - p^2q^2I_0$, all terms are integers so I_3 is an integer.

∴ All terms in the recurrence relation $I_n = (4n-2)q^2I_{n-1} - p^2q^2I_{n-2}$ will be integers, hence I_n is an integer for $n = 0, 1, 2, 3, \ldots$

Question 8

(b) (iv) The integral I_n is from $x = -\pi/2$ to $x = \pi/2$

For $-\pi/2 \leqslant x \leqslant \pi/2$, $x^2 \leqslant \frac{\pi^2}{4}$ and $0 \leqslant \cos x \leqslant 1$.

$$0 \leqslant \left(\frac{\pi^2}{4} - x^2\right)^n \cos x \leqslant \left(\frac{\pi^2}{4}\right)^n$$

Lower bound for I_n:

$$\frac{q^{2n}}{n!}\int_{-\pi/2}^{\pi/2} 0 \, dx = 0$$

Upper bound for I_n:

$$\frac{q^{2n}}{n!}\int_{-\pi/2}^{\pi/2}\left(\frac{\pi^2}{4}\right)^n dx$$

$$= \frac{q^{2n}}{n!}\left[\left(\frac{\pi^2}{4}\right)^n x\right]_{-\pi/2}^{\pi/2}$$

$$= \frac{q^{2n}}{n!}\cdot\pi\left(\frac{\pi^2}{4}\right)^n, \text{ where } \pi = \frac{p}{q}$$

$$= \frac{q^{2n}}{n!}\cdot\frac{p}{q}\cdot\frac{p^{2n}}{4^n q^{2n}}$$

$$= \frac{p}{q}\left(\frac{p}{2}\right)^{2n}\frac{1}{n!}$$

$$\therefore \quad 0 < I_n < \frac{p}{q}\left(\frac{p}{2}\right)^{2n}\frac{1}{n!}$$

(the integrand curves reach equality only at certain points so the areas under them can't be equal)

(v) Putting this result to the result in (iv),

$$0 < I_n < 1$$

Hence I_n can't be an integer.

But, from (iii) it is an integer.

$\therefore$ we have a contradiction

$\therefore$ π can't be rational (can't be written as $\frac{p}{q}$)