HSC - Mathematics Extension 2 Exam Solutions 2004

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THE MERIDIAN INTERNATIONAL SCHOOL

① (a) $I = \int x e^{3x}\,dx \qquad u' = x,\ v' = e^{3x}$

$= \frac{1}{3} x e^{3x} - \int \frac{e^{3x}}{3}\,dx$

$= \frac{1}{3} x e^{3x} - \frac{e^{3x}}{9} + C$

(b) $u = -\cos x \quad du = \sin x\,dx$

$I = \int_0^{\pi/4} \frac{\sin x}{\cos^3 x}\,dx = -\int \frac{du}{u^3}$

$= \frac{1}{2}\left[\frac{1}{u^2}\right] = \frac{1}{2}\left[\frac{1}{\cos^2 x}\right]_{x=0}^{x=\pi/4}$

$= \frac{1}{2}(2-1) = \frac{1}{2}.$

(c) $I = \int \frac{dx}{\sqrt{5+4x-x^2}}$

$= \int \frac{dx}{\sqrt{9-(x-2)^2}}$

$= \sin^{-1}\left(\frac{x-2}{3}\right) + C$

(d) i) Combine denominators to get,

$x^2 - 7x + 4 = a(x-1)^2 + b(x^2-1) - (x+1)$

Use $x = 0, -1$ to obtain

$a = 3 \ , \ b = -2$

ii) $I = \int \frac{3}{(x+1)} - \frac{2}{(x-1)} - \frac{1}{(x-1)^2}\,dx$

$= 3\ln(x+1) - 2\ln(x-1) + \frac{1}{(x-1)} + C$

(e) $dx = 2\cos\theta\,d\theta$

$I = \int_0^{\pi/2} \frac{4\sin^2\theta\, 2\cos\theta\,d\theta}{2\cos\theta}$

$= 2\int_0^{\pi/2} \sin^2\theta\,d\theta$

$I = 2\int_0^{\pi/2} 1 - \cos 2\theta\,d\theta$

$= 2\left(\theta - \frac{\sin 2\theta}{2}\right)_0^{\pi/2}$

$= \pi.$

② (a) i) $zw = 5 + 5i$

ii) $\overline{\left(\frac{10}{z}\right)} = \overline{\left(\frac{10}{1+2i} \times \frac{1-2i}{1-2i}\right)}$

$= \frac{10+20i}{1+4} = 2 + 4i$

(b) i) $\frac{\alpha}{\beta} = \frac{1+\sqrt{3}i}{1+i} \times \frac{1-i}{1-i}$

$= \frac{1+\sqrt{3}}{2} + i\frac{(\sqrt{3}-1)}{2}$

(ii) $|\alpha| = \sqrt{3+1} = 2$

$\arg\alpha = \tan^{-1}\sqrt{3} = \pi/3$

$\therefore \alpha = 2 \operatorname{cis}\frac{\pi}{3}$

(iii) $\left|\alpha/\beta\right| = |\alpha|/|\beta| = \frac{2}{\sqrt{2}} = \sqrt{2}$

$\operatorname{Arg}(\alpha/\beta) = \arg\alpha - \arg\beta$

$= \pi/3 - \pi/4$

$= \pi/12$

$\therefore \alpha/\beta = \sqrt{2} \operatorname{cis}\pi/12.$

(iv) Equating imaginary parts from i; iii we have $\frac{\sqrt{3}-1}{2} = \sqrt{2}\sin\pi/12$

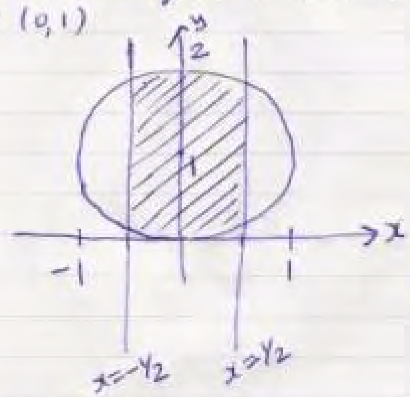
$\therefore \sin\pi/12 = \frac{\sqrt{3}-1}{2\sqrt{2}}$ or $\frac{\sqrt{6}-\sqrt{2}}{4}.$

2/10

② (c) Put $z = x+iy$

$\therefore |z+\bar{z}| \le 1 \Rightarrow |x| \le \frac{1}{2}$

and $|z-i| \le 1$ is the inside and on the circle of radius 1, centred at (0,1)



(d) Since A, B, C are on the circle we have

$r = |z| = |w| = |z+w|$

(i) ACBO is a rhombus and OC = OA as A, C are on a circle.

$\therefore$ ΔOBC, ΔOCA are both equilateral.

$\therefore \angle COA = \angle BOC = \pi/3$

So $\angle AOB = \pi/3 + \pi/3 = \frac{2\pi}{3}$ as required

(ii) $|z^3| = |z|^3 = |w|^3 = |w^3|$ by assumption

$\arg(w^3) - \arg(z^3)$

$= 3(\arg w - \arg z)$

$= 2\pi$ using i)

hence $\operatorname{Arg}(w^3) = \operatorname{Arg}(z^3)$

(principal arguments are equal)

Thus since their moduli and arguments are equal then $z^3 = w^3$ as required.

(iii) by (ii) $z^3 - w^3 = 0$

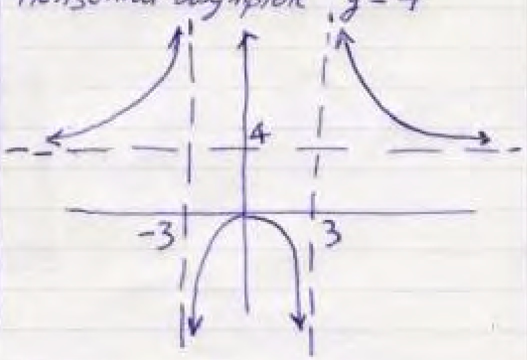
so $(z-w)(z^2+w^2+zw) = 0$

but $z \ne w$ implying $z^2 + w^2 + zw = 0.$

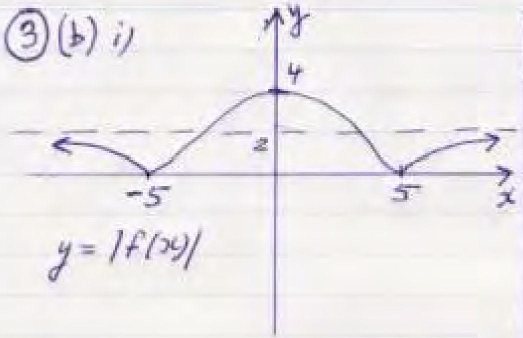
③ (a) $y = 4x^2/(x^2-9)$

Vertical asymptotes $x = 3, x = -3$

Horizontal asymptote $y = 4$

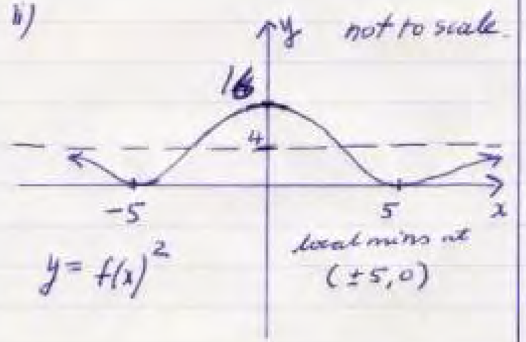


(3) (b) i)



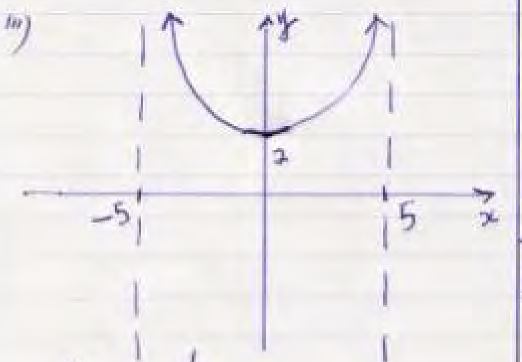
$y = |f(x)|$

ii)



$y = f(x)^2$

iii)



$y = \dfrac{1}{\sqrt{f(x)}}$

(c) $x^2 - xy + y^3 = 5$

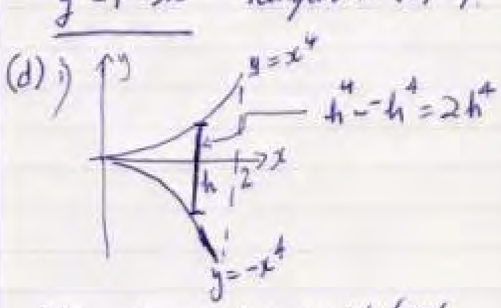
$2x - y - xy' + 3y^2 y' = 0$

$y' = \dfrac{y-2x}{3y^2 - x}$

$\therefore y + 1 = \left(\dfrac{-1-4}{3-2}\right)(x-2)$

$\underline{y = 9 - 5x}$ tangent at $(2,-1)$.

(d) i)



The base of the equilateral triangle has length $2h^4$

$\text{Area} = \frac{1}{2} \cdot 2h^4 \cdot 2h^4 \cdot \sin 60^\circ$

$= 2h^8 \cdot \sqrt{3}/2$

$= h^8 \sqrt{3}$ as required.

(ii)

typical slice with volume

$\delta V = \sqrt{3}\, x^8 \,\delta x$

Volume is $V = \lim\limits_{\delta x \to 0} \sum\limits_{x=0}^{2} \sqrt{3}\, x^8 \,\delta x$

$V = \sqrt{3} \displaystyle\int_0^2 x^8 \, dx = \underline{\dfrac{2^9 \sqrt{3}}{9}}$ units^3

(4) (a) i) $\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2$

$= \alpha\beta\gamma(\alpha+\beta+\gamma)$

and $\alpha+\beta+\gamma = -b/a = -7/3$

$\alpha\beta + \alpha\gamma + \beta\gamma = c/a = 11/3$

$\alpha\beta\gamma = -d/a = -51/3 = -17$

So $\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2 = -17 \cdot -7/3 = \underline{\dfrac{119}{3}}$

(ii) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \beta\gamma + \alpha\gamma)$

$= 49/9 - 2 \cdot \frac{11}{3} = \underline{-17/9}$

(4) a) iii) From part ii) we have

$\alpha^2 + \beta^2 + \gamma^2 < 0$

So α, β, γ cannot all be real.

Since $p(x)$ has real coefficients we know that non-real roots occur in complex conjugate pairs.

Hence there is precisely one real root.

[Alternatively, one may notice that $p(-3) = 0$ and upon division obtain

$p(x) = (x+3)(3x^2 - 2x + 17)$

where the quadratic factor has $\Delta < 0$

Hence the only real root is $x = -3$.]

(b) Let $\alpha = \angle AHE$, $\beta = \angle DCE$

(i) $\gamma = \angle HBD$.

Then, $\angle BHD = \alpha$ (vertically opposite angles)

$\angle CAL = \gamma$
$\angle ALB = \beta$ } angles on same chord

$\gamma + \alpha = \pi/2$ angles sum to π^c in ΔBHD

$\gamma + \beta = \pi/2$ angles sum to π^c in ΔAEL.

$\therefore \alpha = \beta$

That is $\angle AHE = \angle DCE$.

(ii) We know that

$\angle DCE = \angle BLA$ (angles on same chord)

Hence ΔAHL is isosceles so that

$AH = AL$.

(iii) Similarly $AH = AM$. $\therefore AM = AL$

(iv) Apply the result of ii), iii) to the vertices B and C to get

$BM = BK$ and $CK = CL$

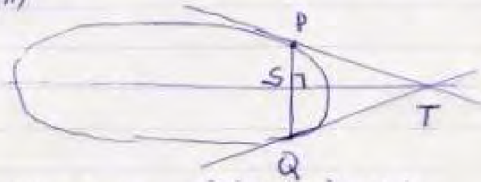
Hence the arc BC subtended by chords BK, KC is half the arc MKL (subtended by chords MB, BK, KC, CL).

(c) (i) Since $S(ae, 0)$ lies on PQ

then $\frac{x_0}{a^2} \cdot ae + 0 = 1$ $\therefore x_0 = a/e$

so that T lies on the directrix $x = a/e$

(ii)



When $x = ae$, $a^2e^2/a^2 + y^2/b^2 = 1$

$\therefore P(ae, b\sqrt{1-e^2})$

So, $\dfrac{PS}{ST} = \dfrac{b\sqrt{1-e^2}}{a(\frac{1}{e} - e)} = \dfrac{be}{a} \dfrac{1}{\sqrt{1-e^2}}$

$= \dfrac{b}{a} \cdot e \cdot \dfrac{a}{b}$

$\underline{\dfrac{PS}{ST} = e}$

(iii) $\tan \angle STP = PS/ST = e < 1$

($0 < e < 1$ for ellipse)

and since $y = \tan x$ is an increasing function this implies that

$\angle STP < \tan^{-1} 1 = \pi/4$

Therefore $\angle PTQ = 2\angle STP < 2 \cdot \frac{\pi}{4}$

$\therefore \angle PTQ < \pi/2$ as required.

(iv) $\text{Area} = ST \times PS = e \cdot ST^2$

$= e \cdot a^2 (\frac{1}{e} - e)(1/e - e)$

$= a^2(1-e^2)(\frac{1}{e} - e)$

$= b^2 (\frac{1}{e} - e)$

(since for an ellipse have $b^2 = a^2(1-e^2)$) //

HSC - 2004 - Ext2 - solutions

5/10

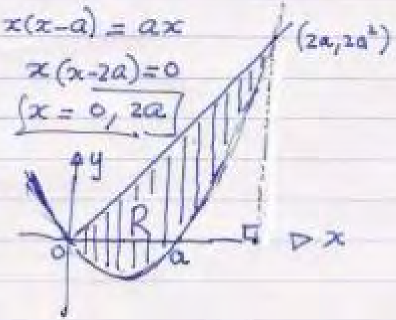
Q5

(a)(i) $x(x-a) = ax$

$x(x-2a)=0$

$\boxed{x = 0,\ 2a}$

(ii)



(iii) $V = \int_0^{2a} 2\pi(x+2a)(2ax-x^2)\,dx$

$= 2\pi \int_0^{2a} (-x^3+4a^2x)\,dx$

$= 2\pi \left[-\frac{1}{4}x^4 + 2a^2x^2 \right]_0^{2a}$

$= 2\pi\ (4a^4)$

$= \boxed{8\pi a^4}$

(b) (i) r students can occupy the 1st room in $\binom{n}{r}$ ways

, the remaining $n-r$ students occupy the 2nd room in one way.

$\therefore$ the number of ways both rooms can be occupied

$\sum_{r=1}^{n-1} \binom{n}{r} = \sum_{r=0}^{n} \binom{n}{r} - 2$

$= \boxed{2^n - 2}$

(ii) The # of ways 2 students can occupy the 1st two rooms is $\binom{5}{1}\binom{4}{1} = 20$

The # of ways 3 students can occupy the 1st two rooms is

$\binom{5}{1}\binom{4}{2} + \binom{5}{2}\binom{3}{1} = 30+30 = 60$

The # of ways 4 students can occupy the 1st two rooms is

$\binom{5}{1}\binom{4}{3} + \binom{5}{2}\binom{3}{2} + \binom{5}{3}\binom{2}{1}$

$= 20+30+20 = 70$

$\therefore$ The # of ways 5 students can be placed in three rooms so that no room is empty is

$20+60+70 = \boxed{150}$.

6/10

5 (c) (i)

The resultant force acting on the marble is null since $\omega=0$, so are its horizontal and vertical components $N\sin\theta - mr\omega^2$ and $N\cos\theta - mg$ respectively.

(ii) Either the marble is stationary at the bottom of the sphere ($\omega=0$, $\theta=0$) or

$\cos\theta = \frac{mg}{N}$

$= \frac{mg}{m\frac{r}{\sin\theta}\omega^2}$

$= \frac{gR}{\omega^2}$

(iii) from (ii),

$\omega = \sqrt{\frac{g}{R}\sec\theta} > \sqrt{\frac{g}{R}}$

since $\sec\theta > 1$.

Q6 a) (i)

$\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx = \int_{\pi}^{0} \frac{d(\cos x)}{1+\cos^2 x}$

$= \tan^{-1}(\cos x)\Big|_{\pi}^{0}$

$= \tan^{-1}(1) - \tan^{-1}(-1)$

$= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right)$

$= \boxed{\frac{\pi}{2}}$.

(ii) $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = \int_0^{\pi} \frac{(\pi-u)\sin u}{1+\cos^2 u}\,du$

$= \pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx - \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx$

$\therefore \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = \left(\frac{\pi}{2}\right)\left(\frac{\pi}{2}\right)$

$= \boxed{\frac{\pi^2}{4}}$.

b) (i) $\ddot{x} = g - kB = 0$

$\therefore k = \frac{g}{B}$

(ii) $\frac{d}{dt}(v e^{kt}) = \left(kv + \frac{dv}{dt}\right)e^{kt}$

$= (kv + \ddot{x})e^{kt}$

$= g e^{kt}$.

(iii) from (ii),

$v e^{kt} = \int g e^{kt}\,dt$

$= B e^{kt} + C$

and the point $(0, A)$ fits if

$C = A - B$, hence

$v = B - (B-A)e^{-\frac{gt}{B}}$.

(iv) $x = \int \left(B - (B-A)e^{-\frac{gt}{B}}\right) dt$

$= Bt + \frac{B}{g}(B-A) + C_1$

and the point $(0,0)$ fits

if $C_1 = -\frac{B}{g}(B-A)$, hence

$x = Bt - \frac{B}{g}(B-A)\left(1 - e^{-\frac{gt}{B}}\right)$.

(v)

$h + BT - \frac{B}{g}(B+A)\left(1 - e^{-\frac{gT}{B}}\right)$

$= BT - \frac{B}{g}(B-A)\left(1 - e^{-\frac{gT}{B}}\right)$

$h = \frac{2AB}{g}\left(1 - e^{-\frac{gT}{B}}\right)$

$T = -\frac{B}{g}\ln\left(1 - \frac{gh}{2AB}\right)$

$= \frac{B}{g}\ln\left(\frac{2AB}{2AB - gh}\right)$.

(vi) from (v),

$2AB - gh > 0$ i.e. $\boxed{A > \frac{gh}{2B}}$

Q7 (a) (i) $(a-1)^2 \geq 0$

$a^2 + 1 \geq 2a$

$a + \frac{1}{a} \geq 2$.

(ii) for $n = 1$, $a_1\left(\frac{1}{a_1}\right) = 1$.

Suppose $(a_1 + \cdots + a_k)\left(\frac{1}{a_1} + \cdots \frac{1}{a_k}\right) \geq k^2$

then $(a_1 + \cdots + a_k + a_{k+1})\left(\frac{1}{a_1} + \cdots + \frac{1}{a_k} + \frac{1}{a_{k+1}}\right)$

$= (a_1 + \cdots + a_k)\left(\frac{1}{a_1} + \cdots + \frac{1}{a_k}\right) + a_{k+1}\left(\frac{1}{a_1} + \cdots + \frac{1}{a_k}\right)$

$+ \frac{1}{a_{k+1}}(a_1 + \cdots + a_k) + 1$

$\geq k^2 + \left(\frac{a_1}{a_{k+1}} + \frac{a_{k+1}}{a_1}\right) + \cdots + \left(\frac{a_k}{a_{k+1}} + \frac{a_{k+1}}{a_k}\right) + 1$

$\geq k^2 + 2k + 1$

$= (k+1)^2$

with the use of part (i).

(iii) $\frac{1}{\operatorname{cosec}^2\theta} + \frac{1}{\sec^2\theta} + \frac{1}{\cot^2\theta} = \sin^2\theta + \cos^2\theta + \tan^2\theta$

$= 1 + \tan^2\theta$

$= \sec^2\theta$

$\therefore (\operatorname{cosec}^2\theta + \sec^2\theta + \cot^2\theta)\left(\frac{1}{\operatorname{cosec}^2\theta} + \frac{1}{\sec^2\theta} + \frac{1}{\cot^2\theta}\right)$

$= \sec^2\theta\,(\operatorname{cosec}^2\theta + \sec^2\theta + \cot^2\theta) \geq 9$

by (ii), i.e.

$\operatorname{cosec}^2\theta + \sec^2\theta + \cot^2\theta \geq 9\cos^2\theta$.

b) (i) $z^2 - 2\cos\alpha\, z + 1 = 0$

$z = \cos\alpha \pm \sqrt{\cos^2\alpha - 1}$

$= \cos\alpha \pm i\sin\alpha$

$z^n = \cos(n\alpha) \pm i\sin(n\alpha)$

by de Moivre's,

$\therefore z^n + \frac{1}{z^n} = 2\cos(n\alpha)$.

More directly, $z + \frac{1}{z} = 2\cos\alpha$

$\Rightarrow \arg(z) = \pm\alpha$ (by above)

$\Rightarrow \arg(z^n) = \pm n\alpha$

$\Rightarrow z^n + \frac{1}{z^n} = 2\cos(n\alpha)$

(ii) $w^3 + w^2 - 2w - 2 = (w^2 - 2)(w + 1)$

$= \left(z^2 + \frac{1}{z^2}\right)\left(z + \frac{1}{z} + 1\right)$

$= z + \frac{1}{z} + z^2 + \frac{1}{z^2} + z^3 + \frac{1}{z^3}$.

(iii) $\cos\alpha + \cos(2\alpha) + \cos(3\alpha)$

$= \frac{1}{2}\left(z + \frac{1}{z} + z^2 + \frac{1}{z^2} + z^3 + \frac{1}{z^3}\right)$

$= \frac{1}{2}(w^2 - 2)(w + 1) = 0$

$w = -1, \pm\sqrt{2}$

$\cos\alpha = \frac{1}{2}w = -\frac{1}{2}, \pm\frac{1}{\sqrt{2}}$

i.e. $\alpha = \frac{\pi}{4}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{4\pi}{3}, \frac{7\pi}{4}$.

Otherwise $\cos\alpha + \cos(2\alpha) + \cos(3\alpha)$

$= \cos(2\alpha) + 2\cos\alpha\cos(2\alpha)$

$= \cos(2\alpha)(1 + 2\cos\alpha)$

$\therefore \cos(2\alpha) = 0$ or $\cos\alpha = -\frac{1}{2}$.

$\frac{9}{10}$

<u>Q8</u> a) (i) Area $\Delta OPP' = \frac{1}{2} \times p \times \frac{1}{p} = \frac{1}{2}$

(ii) Area region OPQ = Area region $OPQ - \frac{1}{2} + \frac{1}{2}$

= Area region OPQ − area $\Delta OQQ'$ + area $\Delta OPP'$

= Area region $Q'QPP'$.

(iii) Let M' be the foot of the perpendicular from M to the x-axis, then $\Delta ORR' \sim \Delta OMM'$

$$\therefore \frac{r}{\frac{1}{r}} = \frac{\frac{p+q}{2}}{\frac{1}{2}\left(\frac{1}{p}+\frac{1}{q}\right)}$$

$$= pq$$

or $r^2 = pq$.

(iv) Area region $Q'QRR'$ − Area region $R'RPP'$

$$= \int_q^r \frac{dx}{x} - \int_r^p \frac{dx}{x}$$

$$= \ln\left(\frac{r}{q}\right) - \ln\left(\frac{p}{r}\right)$$

$$= \ln\left(\frac{r^2}{pq}\right)$$

$$= \ln 1$$

$$= 0.$$

(v) Area region ORQ = Area region SRQ + Area ΔOSQ

= Area region $SRQ + \frac{1}{2}$ − Area $\Delta OSQ'$

= Area region SRQ + Area region $QSRR'$

= Area region $Q'QRR'$.

Likewise, Area region ORP = Area region $R'RPP'$.

Hence, by (iv), Area region ORQ = Area region ORP.

$\frac{10}{10}$

b) (i) $$I_n + I_{n+2} = \int_0^{\frac{\pi}{4}} \tan^n x\,(1+\tan^2 x)\,dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^n x\; d(\tan x)$$

$$= \frac{1}{n+1} \tan^{n+1} x \Big|_0^{\frac{\pi}{4}}$$

$$= \frac{1}{n+1}.$$

$I_n \leq I_{n+1}$ follows from (i) with $I_n \geq 0$.

(ii) $$J_n - J_{n-1} = (-1)^n I_{2n} - (-1)^{n-1} I_{2n-2}$$

$$= (-1)^n (I_{2n} + I_{2n-2})$$

$$= (-1)^n \frac{1}{(2n-2)+1}$$

$$= \frac{(-1)^n}{2n-1}$$

(iii) $$J_n = J_{n-1} + \frac{(-1)^n}{2n-1}$$

$\therefore$ $J_1 = J_0 - 1$

$J_2 = J_0 - 1 + \frac{1}{3}$

$\vdots$

$$J_m = J_0 + \sum_{n=1}^{m} \frac{(-1)^n}{2n-1}$$

$$= I_0 + \sum_{n=1}^{m} \frac{(-1)^n}{2n-1}$$

$$= \int_0^{\frac{\pi}{4}} dx + \sum_{n=1}^{m} \frac{(-1)^n}{2n-1}$$

$$= \frac{\pi}{4} + \sum_{n=1}^{m} \frac{(-1)^n}{2n-1}.$$

(iv) $u = \tan x$

$dx = \frac{du}{1+u^2}$

$$\therefore I_n = \int_0^1 \frac{u^n\,du}{1+u^2}$$

(v) $I_n \geq 0$ since $\frac{u^n}{1+u^2} \geq 0$, $\forall n$

and $$I_n \leq \int_0^1 u^n\,du = \frac{1}{n+1}$$

Since $1+u^2 \geq 1$.

Hence

$$\lim_{n\to\infty} |J_n| = \lim_{n\to\infty} I_{2n}$$

$$\leq \lim_{n\to\infty} \frac{1}{2n+1}$$

$$= 0$$

implying

$$\lim_{n\to\infty} J_n = 0$$