HSC 2005 MATHEMATICS EXTENSION 2 (4 unit) EXAM : ANSWERS/SOLUTIONS.

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Question 1

Q1.a Let $u = \sin\theta$. Then $du = \cos\theta\, d\theta$ and so $I = \int \frac{1}{u^5}\, du = \frac{u^{-4}}{-4} + c = \mathbf{\frac{-1}{4\sin^4\theta} + c}$

Q1.b

Q1.b.i $\frac{5x}{x^2 - x - 6} \equiv \frac{a}{x-3} + \frac{b}{x+2} \equiv \frac{a(x+2) + b(x-3)}{(x-3)(x+2)}$. So $5x \equiv a(x+2) + b(x-3)$. Putting $x = 3$ gives $5.3 = a.5 + 0$ and so $a = 3$. Putting $x = -2$ gives $-10 = 0 + b(-2-3)$ so $b = 2$. So $\mathbf{a = 3, b = 2}$.

Q1.b.ii $I = \int \frac{5x}{x^2 - x - 6}\, dx = \int \frac{3}{x-3} + \frac{2}{x+2}\, dx = 3\ln|x-3| + 2\ln|x+2| + c.$

Q1.c $I = \int_1^e x^7 \ln x\, dx$. Put $u' = x^7$, $v = \ln x$. Then $u = x^8/8$, $v' = 1/x$. So, $I = uv - \int uv' = \frac{x^8}{8}\ln x - \int \frac{x^7}{8}\, dx = \left[\frac{x^8}{8}\ln x - \frac{x^8}{64}\right]_{x=1}^{x=e} = \mathbf{\frac{7e^8 + 1}{64}}$.

Q1.d $I = \int \frac{dx}{\sqrt{4x^2 - 1}} = \frac{1}{2}\int \frac{dx}{\sqrt{x^2 - \left(\frac{1}{2}\right)^2}} = \frac{1}{2}\ln\left(x + \sqrt{x^2 - \frac{1}{4}}\right) + c$

$= \frac{1}{2}\ln\left(2x + \sqrt{4x^2 - 1}\right) + c'$

Q1.e

Q1.e.i

$t = \tan\theta/2$.

$\frac{dt}{d\theta} = \frac{1}{2}\sec^2\ \theta/2 = \frac{1}{2}(\tan^2\theta/2 + 1) = \frac{1}{2}(t^2 + 1)$. We can also write $2\, dt = (t^2 + 1)\, d\theta$

Q1.e.ii $\frac{2t}{1+t^2} = \frac{2\tan\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}} = \frac{2\sin\frac{\theta}{2} \big/ \cos\frac{\theta}{2}}{\sec^2\frac{\theta}{2}} = \frac{2\sin\frac{\theta}{2}\ \cos^2\frac{\theta}{2}}{\cos\frac{\theta}{2}} = 2\sin\frac{\theta}{2}\ \cos\frac{\theta}{2} = \sin\theta$

Q1.e.iii $\int \mathrm{cosec}\ \theta\, d\theta = \int \frac{d\theta}{\sin\theta} = \int \frac{d\theta(1+t^2)}{2t} = \int \frac{dt}{t} = \ln t + c = \ln(\tan\theta/2) + c$

Question 2

Q2.a $z = 3 + i$, $w = 1 - i$

Q2.a.i $2z + iw = 2(3+i) + i(1-i) = 6 + 2i + i - i^2 = 6 + 3i + 1 = 7 + 3i$

Q2.a.ii $\bar{z}w = (\overline{3+i})(1-i) = (3-i)(1-i) = 2 - 4i$

Q2.a.iii $\frac{6}{w} = \frac{6}{(1-i)} \cdot \frac{(1+i)}{(1+i)} = \frac{6+6i}{(1-i)(1+i)} = \frac{6+6i}{2} = 3 + 3i$

Q2.b $\beta = 1 - i\sqrt{3}$

Q2.b.i $|\beta| = \sqrt{1+3} = 2$, and since β lies in quadrant 4 we have, $\arg\beta = -\tan^{-1}\sqrt{3} = -\dfrac{\pi}{3}$. So $\beta = 2 \text{ cis } \left(-\dfrac{\pi}{3}\right)$

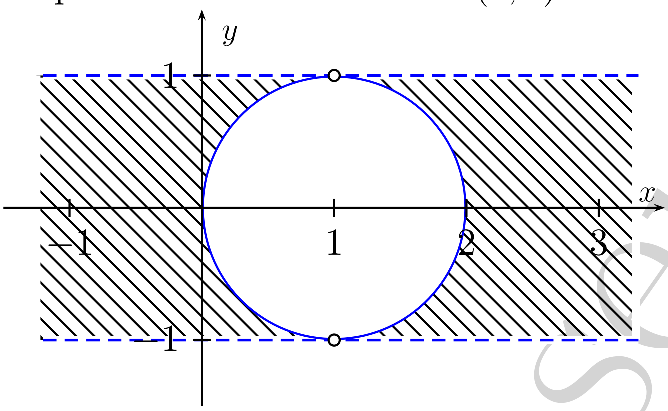
Q2.b.ii $\beta^5 = 2^5 \left(\text{ cis } \dfrac{-\pi}{3} \right)^5 = 32 \text{ cis } \dfrac{-5\pi}{3} = 32 \text{ cis } \dfrac{\pi}{3}$

Q2.b.iii $\beta^5 = 32(\cos\pi/3 + i\sin\pi/3) = 32\left(1/2 + i\sqrt{3}/2\right) = 16 + 16\sqrt{3}i$

Q2.c Firstly, $|z - 1 \geq 1$ are all the points which lie on or outside the circle of radius 1, centee $(1, 0)$. Let $z = x + iy$, then

$$\begin{aligned} |z - \bar{z}| &< 2 \\ |x + iy - (x - iy)| &< 2 \\ |2iy| &< 2 \\ |y| &< 1 \\ -1 < y &< 1 \end{aligned}$$

So the region is the points above the line $y = -1$ and below the line $y = 1$ and excluding all points inside the circle at $(1, 0)$ with radius 1, and excluding the points $(1, 1), (1, -1)$.



Q2.d

Q2.d.i The angle between OQ and line l is $\arg z_2 - \alpha$, and the angle between line l and OP is $\alpha - \arg z_1$. Putting these equal, $\arg z_2 - \alpha = \alpha - \arg z_1$ which gives $\arg z_1 + \arg z_2 = 2\alpha$ as required.

Q2.d.ii We have $z_1 = |z_1| \text{ cis}(\arg z_1)$, $z_2 = |z_2| \text{ cis}(\arg z_2)$. Multiplying and using $|z_1| = |z_2|$, we get $z_1 z_2 = |z_1||z_2| \text{ cis}(\arg z_1) \text{ cis}(\arg z_2) = |z_1|^2 \text{ cis}(\arg z_1 + \arg z_2) = |z_1|^2 \text{ cis } 2\alpha$ or in long format $z_1 z_2 = |z_1|^2(\cos 2\alpha + i \sin 2\alpha)$ as required.

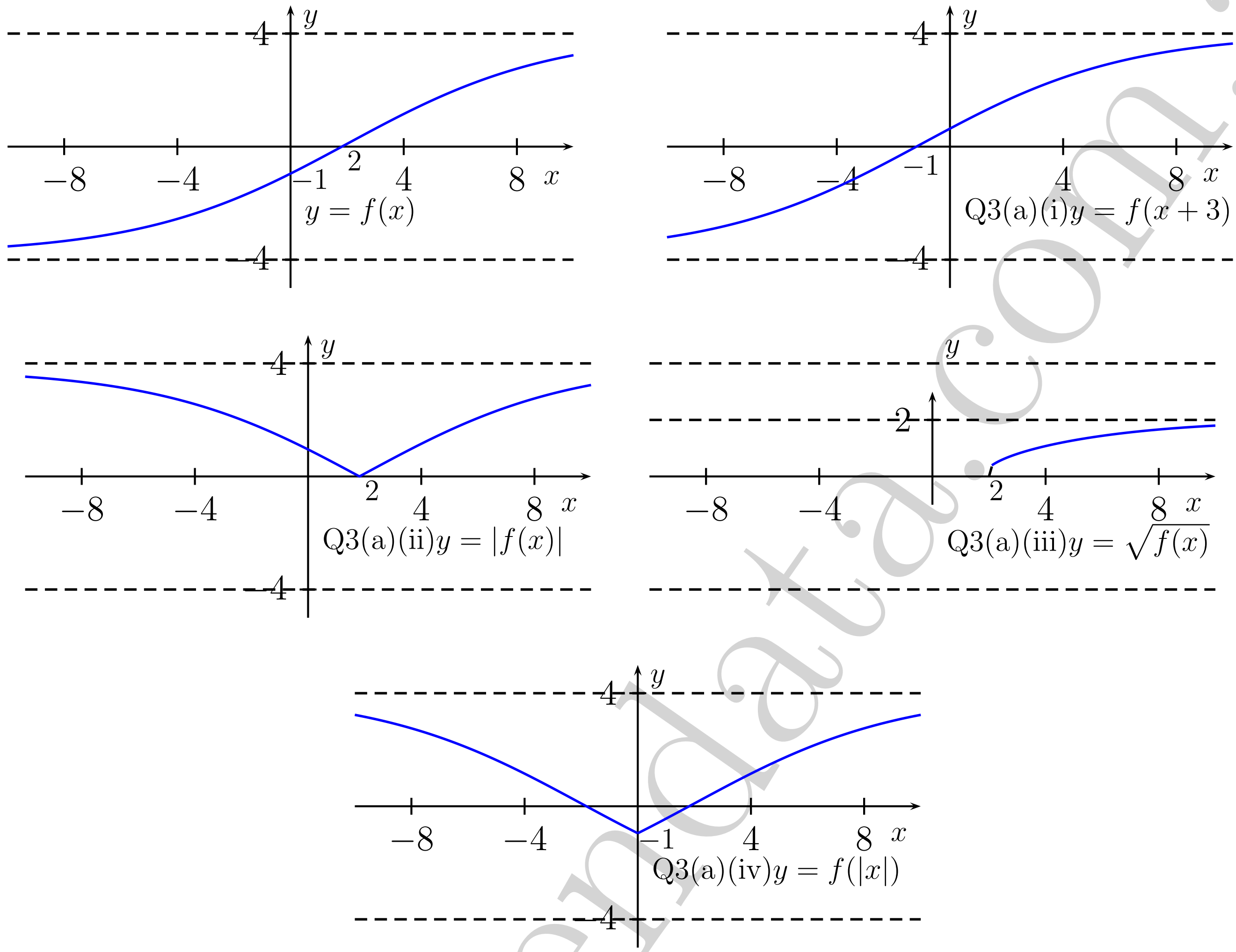
Q2.d.iii

$z_1 z_2 = |z_1|^2 \text{ cis } 2\pi/4 = |z_1|^2 i$

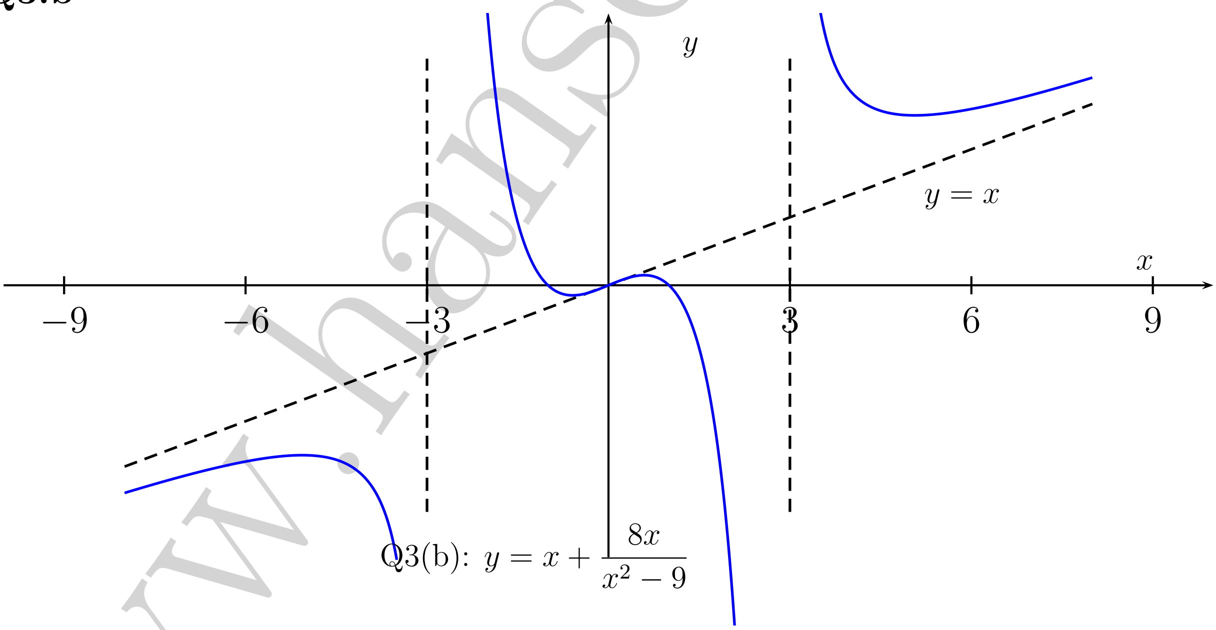
As z_1 varies. $|z_1|$ takes all values $|z_1| > 0$ ($|z_1| \neq 0$ since the argument of z_1 is not undefined.). Therefore the locus of R is $y > 0$ (the positive y-axis).

Question 3

Q3.a



Q3.b



Q3.c We have, $x^3 - 4xy + y^3 = 1$. Differentiate, regarding y as a function of x to obtain, $3x^2 - 4y - 4xy' + 3y^2y' = 0$ and solving for y', $y' = \dfrac{4y - 3x^2}{3y^2 - 4x}$, so at $(2, 1)$ the gradient of

the normal is $m = \left.\dfrac{4x-3y^2}{4y-3x^2}\right|_{(2,1)} = \dfrac{8-3}{4-12} = -\dfrac{5}{8}$.
The equation of the normal is $y-1 = (-5/8)(x-2)$ so finally $\mathbf{5x+8y-18=0}$

Q3.d Resolve Horizontal: $\dfrac{mv^2}{r} = N\sin\theta$
Resolve Vertical: $N\cos\theta = mg$
So, $\sin\theta = \dfrac{mv^2}{rN}$, $\quad\cos\theta = \dfrac{mg}{N}$, upon squaring,

$$\begin{aligned} 1 &= \left(\frac{mv^2}{rN}\right)^2 + \frac{m^2g^2}{N^2} \\ N^2 &= \frac{m^2v^4}{r^2} + m^2g^2 \\ &= m^2\left(g^2+\frac{v^4}{r^2}\right) \\ N &= m\sqrt{g^2+\frac{v^4}{r^2}} \end{aligned}$$

Question 4

Q4.a

Q4.a.i

$$\begin{aligned} \delta V &= \pi(x+\delta x+x)(x+\delta x-x)y = \pi e^{-x^2}(2x\delta x+\delta x^2) \\ \delta V &= 2\pi x e^{-x^2}\,\delta x \\ V &= 2\pi\int_0^N xe^{-x^2}\,dx \\ V &= \pi\int_0^{N^2} e^{-u}\,du = \pi\left[-e^{-u}\right]_0^{N^2} = \pi\left[-e^{-N^2} - -e^0\right] = \pi\left(\mathbf{1-\frac{1}{e^{N^2}}}\right) \end{aligned}$$

(where we used the subst. $u = x^2, dx = 2x\,dx$ and ignoring $(\delta x)^2$ terms as per theory)

Q4.a.ii As $N\to\infty$, then $V\to\pi$.

Q4.b $x^4+px^3+qx^2+rx+s=0$

Q4.b.i sum and product of root equations are, $\Sigma\alpha = -p, \Sigma\alpha\beta = q, \Sigma\alpha\beta\gamma = -r, \alpha\beta\gamma\delta = s$

Q4.b.ii

$$\begin{aligned} (\Sigma\alpha)^2 &= \Sigma\alpha^2 + 2\Sigma\alpha\beta \\ \Sigma\alpha^2 &= (\Sigma\alpha)^2 - 2\Sigma\alpha\beta \\ &= (-p)^2 - 2.q \\ &= p^2-2q \end{aligned}$$

Q4.b.iii

$\Sigma\alpha^2 = 9-2.5 = -1$. The sum of 4 real numers squared must be ≥ 0 so there must exist non real roots.

Q4.b.iv

$P(0) = -8$, $P(1) = 2$. This shows there is at least one real root as the polynomial has to cross the x-axis somewhere between0 and 1. But $P(x)$ has real coefficients so any non real roots occur in complex conjugate pairs. Hence from (iii) we deduce that either there are two non real roots or four non real roots. Hence since there is at least one real root, we must have precisely two real roots.

Q4.c

Q4.c.i Substitute the point so, $-b^2x_1(-b) = (a^2-b^2)x_1y_1$. Factorise, $x_1(b^3-(a^2-b^2)y_1) = 0$. If $x_1 \neq 0$, then $y_1 = \dfrac{b^3}{a^2-b^2}$. If $x_1 = 0$ then $0/a^2 + y_1^2/b^2 = 1$, so $y_1 = \pm b$.

Q4.c.ii

We have

$$\begin{aligned} y_1 = \tfrac{b^3}{a^2-b^2} &< b \\ \therefore b^2 &< a^2 - b^2 \\ 2b^2 &< a^2 \\ 2a^2(1-e^2) &< a^2 \\ 1-e^2 &< 1/2 \\ e^2 &> 1 - 1/2 \\ e^2 &> \tfrac{1}{2} \\ e &> \tfrac{1}{\sqrt{2}} \end{aligned}$$

Question 5

Q5.a

Q5.a.i Area $= 0.5ad = 0.5bc$, and Pythagoras says $a^2 = b^2 + c^2$ so, $a^2d^2 = b^2c^2$ becomes $b^2c^2 = d^2(b^2 + c^2)$

Q5.a.ii

By (i) we know

$AC^2.AB^2 = AP^2(AC^2 + AB^2)$. Dividing throughout by $AP^2AC^2AB^2$ we have $\dfrac{1}{AP^2} = \dfrac{1}{AB^2} + \dfrac{1}{AC^2}$. Multiply by h^2 to get $\dfrac{h^2}{AP^2} = \dfrac{h^2}{AB^2} + \dfrac{h^2}{AC^2}$.

But $\tan^2\alpha = \dfrac{h^2}{AB^2}$, $\tan^2\beta = \dfrac{h^2}{AC^2}$, $\tan^2\gamma = \dfrac{h^2}{AP^2}$.

Hence, $\tan^2\gamma = \tan^2\alpha + \tan^2\beta$ as required.

Q5.b

Q5.b.i Ferdinand wouldn't score the goal last as he would already have lost so the only possibilitites are these: FMMMMM, MFMMMM, MMFMMM, MMMFMM, MMMMFM. Using the formula will be better in general though; Fix an M at the end. In how many ways can you arrange 4M's and one F? In $\frac{5!}{1!4!} = 5$ ways.

Q5.b.ii Using the method in one applied to the different possible numbers of F's and M's we have

$2\left(1 + \dfrac{5!}{4!} + \dfrac{6!}{4!2!} + \dfrac{7!}{3!4!} + \dfrac{8!}{4!4!}\right)$ which equals 252

Q5.c

Q5.c.i Since $f' > 0$ the inverse exists so we may write $x = f^{-1}(y)$ Consider the rectangle with corners $(0, 0)$ (a, b). The area of this rectangle is ab and is made up of the area under the curve $y = f(x)$ between $x = 0$ and $x = a$ plus the area 'under' the curve $x = f^{-1}(y)$ between $y = 0$ and $y = b$. Hence $ab = \int_0^a f(x)\,dx + \int_0^b f^{-1}(y)\,dy$ but y is a dummy variable so $\int_0^b f^{-1}(y)\,dy = \int_0^b f^{-1}(x)\,dx$. Hence $ab = \int_0^a f(x)\,dx + \int_0^b f^{-1}(x)\,dx$ which is the required result.

Q5.c.ii

$I = \int_0^2 \sin^{-1}\frac{x}{4}\,dx$. With $x = \sin^{-1}\frac{y}{4}$ we have $y = 4\sin x$ so $f(x) = 4\sin x$

$$\begin{aligned} I &= ab - \int_0^a f(x)\,dx \\ &= 2.\pi/6 - \int_0^{\pi/6} 4\sin x\,dx \\ &= \pi/3 + 4\cos x|_0^{\pi/6} \end{aligned}$$

$I = \frac{\pi}{3} + 2\sqrt{3} - 4$

Q5.d

Q5.d.i

$AD = 2y$ and denoting theheight of the rectangle ABCD by h_x

$$\begin{aligned} \text{Area ABCD} &= \text{base } \times \text{height} \\ &= 2\sqrt{9 - x^2}h_x \\ &= 2\sqrt{9 - x^2}x\tan 60 \\ &= 2x\sqrt{27 - 3x^2} \end{aligned} \quad \text{as required.}$$

Q5.d.ii

$V = 2\sqrt{3}\displaystyle\int_0^3 x\sqrt{9 - x^2}\,dx$. Put $u = 9 - x^2, du = -2x\,dx$

Then, $V = -\sqrt{3}\displaystyle\int_9^0 u^{1/2}\,du = \sqrt{3}\left[\frac{2}{3}u^{3/2}\right]_0^9 = 18\sqrt{3}$

Question 6

Q6.a

Q6.a.i Step 1: $I_0(x) = \int_0^x t^0 e^{-t}\,dt = [-e^{-t}]_0^x = 1 - e^{-x}$ and $0![1 - e^{-x}(1)] = 1 - e^{-x}$ so LHS = RHS and so its true for $n = 0$.

Step 2: Assume the result is true for $n = k$. so assume that

$I_k(x) = k!\left[1 - e^{-x}\left(1 + x + \cdots + \dfrac{x^k}{k!}\right)\right]$

Step 3: Prove the result true for $n = k + 1$, so prove that

$I_{k+1}(x) = (k + 1)!\left[1 - e^{-x}\left(1 + x + \cdots + \dfrac{x^{k+1}}{(k + 1)!}\right)\right]$

Start,

$I_{k+1}(x) = \int_0^x t^{k+1}e^{-t}\,dt$. For integrtaion by parts, let $u = t^{k+1}$, and $v' = e^{-t}$, so, $u' = (k + 1)t^k$, $v = -e^{-t}$. Then,

$I_{k+1}(x) = \left[-t^{k+1}.e^{-t}\right]_0^x + \displaystyle\int (k + 1)t^k e^{-t}\,dt$

$= -x^{k+1}e^{-x} + (k+1)I_k(x)$
$= -x^{k+1}e^{-x} + (k+1)k!\left[1 - e^{-x}\left(1 + x + \cdots + \frac{x^k}{k!}\right)\right]$
$= -x^{k+1}e^{-x}\frac{(k+1)!}{(k+1)!} + (k+1)!\left[1 - e^{-x}\left(1 + x + \cdots + \frac{x^k}{k!}\right)\right]$
$= (k+1)!\left[1 - e^{-x}\left(1 + x + \cdots + \frac{x^{k+1}}{(k+1)!}\right)\right]$
Hence it's true for $n = k+1$.
Step 4: Since its true for $n = 0$ and it is true for $n = k+1$ ewhenevr it is true for $n = k$ then it is true for all integers $n \geq 0$.

Q6.a.ii $\int_0^1 t^n e^{-t}\,dt = I_n(1)$ The integrand is non negative so the value of the integral must also be non negative.

Also, $e^t \geq 1$ for all $t \geq 0$, so $\frac{1}{e^t} \leq 1$, and hence $\dfrac{t^n}{e^t} \leq t^n$ Therefore,
$I_n(1) = \int_0^1 t^n e^{-t}\,dt \leq \int_0^1 t^n\,dt = \frac{1}{n+1}$
so we have $0 \leq I_n(1) \leq \dfrac{1}{n+1}$ as required.

Q6.a.iii By (ii) we have
$0 \leq n!\left[1 - e^{-1}\left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}\right)\right] \leq \frac{1}{n+1}$
Therefore dividing throughout by $n!$ we have
$0 \leq \left[1 - e^{-1}\left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}\right)\right] \leq \frac{1}{(n+1)n!} = \frac{1}{(n+1)!}$ as required.

Q6.a.iv As $n \to \infty$ the right hand side of the result in (iii) approaches zero, and hence
$\lim_{n\to\infty}\left[1 - e^{-1}\left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}\right)\right] = 0$
Hence $\lim_{n\to\infty}\left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}\right) = e$

Q6.b $n > 2$, $\omega^n = 1$. So $0 = \omega^n - 1 = (\omega - 1)(1 + \omega + \omega^2 + \cdots + \omega^{n-1})$ and as $\omega \neq 1$ we must have $1 + \omega + \omega^2 + \cdots + \omega^{n-1} = 0$.

Q6.b.i Expanding $(1 + 2\omega + 3\omega^2 + 4\omega^3 + \cdots + n\omega^{n-1})(\omega - 1)$ we have
$\omega + 2\omega^2 + 3\omega^3 + 4\omega^4 + \cdots + n\omega^n - 1 - 2\omega - 3\omega^2 - 4\omega^3 - \cdots - n\omega^{n-1}$
$= -\omega - \omega^2 - \omega^3 - \cdots - \omega^{n-1} + n\omega^n - 1$
$= 0 + n\omega^n = n$
since $\omega^n = 1$ and $-\omega - 1 - \omega^2 - \omega^3 - \cdots - \omega^{n-1} = -(1 + \omega + \omega^2 + \cdots + \omega^{n-1}) = -0 = 0.$

Q6.b.ii With $z = \operatorname{cis}\theta$, $z^2 = \operatorname{cis}2\theta = \cos 2\theta + i\sin 2\theta$. Using the given identity
$\frac{1}{z^2 - 1} = \frac{1}{\cos 2\theta + i\sin 2\theta - 1} = \frac{(\operatorname{cis}\theta)^{-1}}{\operatorname{cis}\theta - (\operatorname{cis}\theta)^{-1}} = \frac{\operatorname{cis}(-\theta)}{\operatorname{cis}\theta - \operatorname{cis}(-\theta)}$
$= \frac{\cos\theta - i\sin\theta}{\cos\theta + i\sin\theta - \cos\theta + i\sin\theta} = \frac{\cos\theta - i\sin\theta}{2i\sin\theta}$

Q6.b.iii Using part (ii),

$$\frac{1}{\omega-1}=\frac{1}{\cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n}-1}=\frac{\cos\frac{\pi}{n}-i\sin\frac{\pi}{n}}{2i\sin\frac{\pi}{n}}\times\frac{i}{i}=\frac{i\cos\frac{\pi}{n}+\sin\frac{\pi}{n}}{-2\sin\frac{\pi}{n}}=-\frac{1}{2}-\frac{i}{2}\cot\frac{\pi}{n}$$

Hence $\text{Re}\left(\frac{1}{\omega-1}\right)=-\frac{1}{2}$

Q6.b.iv With $\omega=\ \text{cis}\,\frac{2\pi}{5}$ and by part (i),

$1+2\omega+3\omega^2+4\omega^3+5\omega^4=\dfrac{5}{\omega-1}$

Expanding,

$\dfrac{5}{\omega-1}=1+2\ \text{cis}\,\dfrac{2\pi}{5}+3\ \text{cis}\,\dfrac{4\pi}{5}+4\ \text{cis}\,\dfrac{6\pi}{5}+\ \text{cis}\,\dfrac{8\pi}{5}$

$=\left(1+2\cos\dfrac{2\pi}{5}+3\cos\dfrac{4\pi}{5}+4\cos\dfrac{6\pi}{5}+\cos\dfrac{8\pi}{5}\right)+i(...)$

$\therefore\quad 1+2\cos\frac{2\pi}{5}+3\cos\frac{4\pi}{5}+4\cos\frac{6\pi}{5}+\cos\frac{8\pi}{5}=5\,\text{Re}\left(\frac{1}{\omega-1}\right)=5.-1/2=-5/2$ as required.

Q6.b.v Using ASTC the expression in part (iv) becomes

$1+2\cos\frac{2\pi}{5}-3\cos\frac{\pi}{5}-4\cos\frac{\pi}{5}+5\cos\frac{2\pi}{5}=-5/2.$

$7\cos\frac{2\pi}{5}-7\cos\frac{\pi}{5}=-7/2$ which simplifies to $2\cos^2\frac{\pi}{5}-\cos\frac{\pi}{5}-1/2=0$. Solving,

$\cos\dfrac{\pi}{5}=\dfrac{1\pm\sqrt{1+4.1/2.2}}{4}=\dfrac{1\pm\sqrt{5}}{4}.$

But $\cos\frac{\pi}{5}>0$ hence $\cos\dfrac{\pi}{5}=\dfrac{1+\sqrt{5}}{4}$

Question 7

Q7.a

Q7.a.i $\angle BNP=\angle BMP=90^o$, so $BNPM$ is cyclic quadrilateral since opposite angles add to 180^o

Q7.a.ii $\angle BNM=\angle BPM$ (Angles on same chord)

Also, $\angle BAP=\angle BPM$ (Angle between tangent and chord is equal to angle on chord)

$\therefore\quad \angle BNM=\angle BAP$ which shows that $MN\parallel PA$ (corresponding angles are equal) as required.

Q7.a.iii $\dfrac{r}{s}=\dfrac{p+q}{u}$ (since $\Delta TMN\ |||\ \Delta TPA$)

$\therefore\quad \dfrac{s}{u}=\dfrac{r}{p+q}<\dfrac{r}{p}$ (since $q>0$)

Q7.a.iv

Noting that ΔTBM is a right angled triangle with hypotenuse p, the longest side, we see that $p>r$ so $\frac{r}{p}<1$. So $\quad\dfrac{s}{u}<\dfrac{r}{p}<1$. So $\frac{s}{u}<1$ and $s<u$ as required.

Q7.b Firstly since $\ddot{x}=-\dfrac{k}{x^2}$, $k>0$ then this tells us that $mR\omega^2=mk/R^2$ (with $x=R$).

Q7.b.i Now, $\omega=2\pi f=2\pi/T$ so $mR\omega^2=mR.4\pi^2/T^2=mk/R^2$ and solving we have $k=\dfrac{4\pi^2R^3}{T^2}$

Q7.b.ii Using $\ddot{x} = \frac{d}{dx}\left(v^2/2\right)$ we have $\frac{d}{dx}\left(v^2/2\right) = -\frac{k}{x^2} = -kx^{-2}$ and integrating with respect to x we have $v^2/2 = kx^{-1} + c$. We are given that when $x = R, v = 0$ so $0 = k/R + c$ and so $v^2/2 = \frac{k}{x} - \frac{k}{R} = k\left(\frac{1}{x} - \frac{1}{R}\right)$

Inserting the value for k, $v^2/2 = \frac{4\pi^2 R^3}{T^2}\left(\frac{R-x}{Rx}\right)$ and finally simplifying to get

$v^2 = \frac{8\pi^2 R^2}{T^2}\left(\frac{R-x}{x}\right)$.

Q7.b.iii $v = \sqrt{\frac{8\pi^2 R^2}{T^2}}\sqrt{\left(\frac{R-x}{x}\right)}$

Writing $v = \frac{dx}{dt}$ and inverting we have,

$\frac{dt}{dx} = \frac{T}{2\sqrt{2}\pi R}\sqrt{\frac{x}{R-x}}$

$t = \frac{T}{2\sqrt{2}\pi R}\int_0^R \sqrt{\frac{x}{R-x}}\,dx$

$= \frac{T}{2\sqrt{2}\pi R}\left[R\sin^{-1}\left(\sqrt{\frac{x}{R}}\right) - \sqrt{x(R-x)}\right]_0^R$

$= \frac{T}{2\sqrt{2}\pi R}\left[(R\sin^{-1}(1) - \sqrt{0}) - (R\sin^{-1}0 - \sqrt{0})\right] = \frac{T}{2\sqrt{2}\pi R}\frac{R\pi}{2} = \frac{T}{4\sqrt{2}}$ as required.

Question 8

Q8.a

Q8.a.i $f(x) = \frac{a+b}{3(ab)^{1/3}}x^{-1/3} + \frac{x^{2/3}}{3(ab)^{1/3}}$

Differentiate and simplify to get

$3(ab)^{1/3}f'(x) = \frac{2x-(a+b)}{3x^{4/3}}$ For stationary points $f'(x) = 0$ and this oocurs when $x = \frac{a+b}{2}$. and using the first or second derivative test is easily seen to give a minimum as required.

Q8.a.ii

When $x = \frac{a+b}{2}$, $f(x) = \frac{(a+b)^{2/3}}{2^{2/3}(ab)^{1/3}}$ and since this value was shown to be the minimum of the function we know that for any c we have

$\frac{a+b+c}{3(abc)^{1/3}} \geq \frac{(a+b)^{2/3}}{2^{2/3}(ab)^{1/3}}$. Cubing we get, $\left(\frac{a+b+c}{3(abc)^{1/3}}\right)^3 \geq \frac{(a+b)^2}{2^2(ab)} = \left(\frac{a+b}{2\sqrt{ab}}\right)^2$ as required.

Now since $\frac{a+b}{2} \geq \sqrt{ab}$ (assumed) then $\frac{a+b}{2\sqrt{ab}} \geq 1$ and so also $\left(\frac{a+b}{2\sqrt{ab}}\right)^2 \geq 1$.

Hence, $\left(\dfrac{a+b+c}{3(abc)^{1/3}}\right)^3 \geq \left(\dfrac{a+b}{2\sqrt{ab}}\right)^2 \geq 1$ and so $\dfrac{a+b+c}{3(abc)^{1/3}} \geq 1$

or equivalently $\dfrac{a+b+c}{3} \geq \sqrt[3]{abc}$

Q8.a.iii

Let the roots be $\alpha, \beta, \gamma > 0$. Then using sum/product of roots formulae we have $\alpha+\beta+\gamma = p$, $\alpha\beta\gamma = r$ and by part (ii) we have $\dfrac{p}{3} \geq \sqrt[3]{r}$ and so $p^3 \geq 27r$ as required.

Q8.a.iv If this polynomial had three positive real roots then $2^3 \geq 27$ which is a contradiction so we cannot have three positive roots. But since $p(0) = -1$ and $p(2) > 0$ there must be at least one positive real root. Noting the coefficients of the polynomial are real and since the product of the roots is positive then either the remaining roots are both negative or they are non real complex conjugate roots. But $p'(x) > 0$ for $x \leq 0$ we can see there are no negative roots (since $p(0) = -1$). So there is precisely one positive real root.

Q8.b

Given $A(a\sec\theta, b\sec\theta)$, $B(a\sec\theta, -b\sec\theta)$, $P(a\sec\theta, b\tan\theta)$

Q8.b.i $AP^2 \times PB^2 = (0^2+b^2(\tan\theta-\sec\theta)^2)(0^2+b^2(\tan\theta+\sec\theta)^2) = b^4(\tan^2\theta-\sec^2\theta)^2 = b^4.1 = b^4$ and so $AP \times PB = b^2$ as required.

Q8.b.ii Using the sine rule.

$\dfrac{CP}{\sin(\frac{\pi}{2}+\beta)} = \dfrac{AP}{\sin(\alpha-\beta)}$,

so $CP = \dfrac{AP\ \cos\beta}{\sin(\alpha-\beta)}$

Similarly, $\dfrac{PB}{\sin(\alpha+\beta)} = \dfrac{PD}{\sin(\frac{\pi}{2}-\beta)}$,

so $PD = \dfrac{PB\ \sin(\frac{\pi}{2}-\beta)}{\sin(\alpha+\beta)} = \dfrac{PB\ \cos\beta}{\sin(\alpha+\beta)}$ as required.

Q8.b.iii

$CP \times PD = \dfrac{AP.PB.\cos^2\beta}{\sin(\alpha-\beta)\ \sin(\alpha+\beta)} = \dfrac{b^2.\cos^2\beta}{\sin(\alpha-\beta)\ \sin(\alpha+\beta)}$

and since $\tan\beta = b/a$ is constant then this only depends upon the angle α and not on θ and therefore not on the position of the point P.

Q8.b.iv We have, $CP \times PD = p(q+r)$ and $DQ \times QC = q(r+p)$

But by part (iii) these are equal to each other so we have

$p(q+r) = q(r+p)$

$pq + pr = qr + qp$

$p = q$ as required.

Q8.b.v For some value of θ it is evident that $P = T = Q$ so that in the limit since $p = q$ we have $VT = TU$.

The End