HSC 2003 - 4 Unit Solutions

Question 1

(a) $\left[-(1+e^{x})^{-1}\right]_0^1 = \frac{-1}{1+e} + \frac{1}{2} = \frac{e-1}{2(1+e)}$

(b) $\int x^3 \log_e x \, dx = \frac{x^4}{4}.\log_e x - \int \frac{x^4}{4}.\frac{1}{x}\, dx$

$= \frac{x^4}{4}\log_e x - \frac{1}{16}x^4 + C$

(c) $\int \frac{dx}{\sqrt{(x-1)^2+4}} = \ln\left((x-1) + \sqrt{x^2-2x+5}\right) + C$

(d) (i) $ax^2 + 4a + bx^2 - x - bx + 1 = 5x^2 - 3x + 13$

$a + b = 5 \quad , \quad b + 1 = 3$

$b = 2, \ a = 3$

(ii) $\int \frac{5x^2-3x+13}{(x-1)(x^2+4)}\,dx = \int \frac{3}{x-1} + \frac{2x-1}{x^2+4}\,dx$

$= 3\ln(x-1) + \ln(x^2+4)$
$- \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$

(e) $\int_0^{\frac{3}{\sqrt{2}}} \frac{dx}{(4-x^2)^{\frac{3}{2}}}$

$x = 3\sin\theta \rightarrow dx = 3\cos\theta \, d\theta$

At $x = 0, \ \theta = 0$

At $x = \frac{3}{\sqrt{2}} \rightarrow \theta = \pi/4$

$= \int_0^{\frac{\pi}{4}} \frac{3\cos\theta}{(9 - 9\sin^2\theta)^{\frac{3}{2}}}\,d\theta$

$= \int_0^{\frac{\pi}{4}} \frac{3\cos\theta}{(3\cos\theta)^3}\,d\theta$

$= \frac{1}{9}\int_0^{\frac{\pi}{4}} \sec^2\theta \, d\theta = \frac{1}{9}\left[\tan\theta\right]_0^{\pi/4} = \frac{1}{9}$

Question 2

(a) (i) $z\overline{w} = (2+i)(1+i) = 1+3i$

(ii) $\frac{4}{z} \times \frac{\bar{z}}{\bar{z}} = \frac{8-4i}{5} = \frac{8}{5} - \frac{4}{5}i$

(b) (i) $\sqrt{2}\left[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right]$

(ii) $z^4 + 4 = 4\left[\cos 3\pi + i\sin 3\pi\right] + 4 = 4\times -1 + 4 = 0$

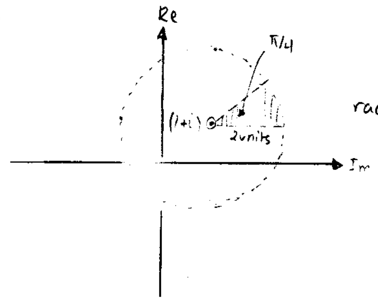
(iii) Coefficients of z^4+4 are real $\therefore \bar{\alpha} = -1-i$ is a root of $z^4+4=0$.

$\therefore (z-\alpha)(z-\bar{\alpha})$ is a real quadratic factor of z^4+4.

i.e. $(z-\alpha)(z-\bar{\alpha}) = z^2 - 2\mathrm{Re}(\alpha)z + \alpha\bar{\alpha}$

$= z^2 + 2z + 2$

(c)



radius = 2 units

(d) $(\cos 5\theta + i\sin 5\theta)$

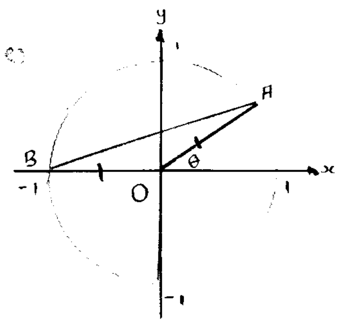
$= \cos^5\theta + 5\cos^4\theta\, i\sin\theta - 10\cos^3\theta\sin^2\theta - 10\cos^2\theta\, i\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta$

i.e. $\cos 5\theta = \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2$

$= 11\cos^5\theta - 10\cos^3\theta + 5\cos\theta(1 - 2\cos^2\theta + \cos^4\theta)$

$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

(e)



$\angle BOA = 180° - \theta$

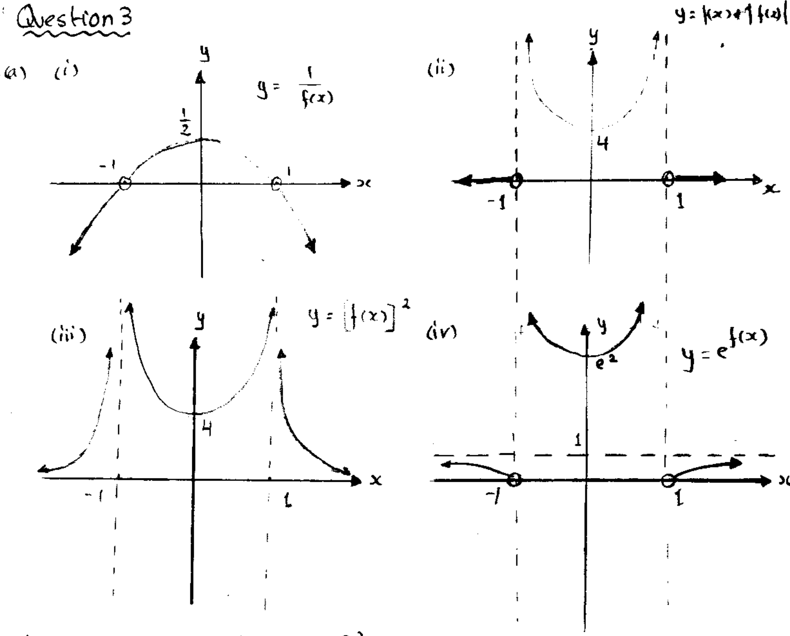
$\angle OAB = \angle OBA \quad \left\{\begin{array}{l}\text{base } \angle\text{'s of} \\ \text{isos. } \Delta\end{array}\right\}$

$\therefore \angle OAB = \angle OBA = \frac{\theta}{2} \quad (\angle \text{ sum of } \Delta)$

$\therefore 2\arg(z+1) = \arg z$

Question 3

(a)



(b) $a = 3, \ b = 2, \quad b^2 = a^2(1-e^2)$

$4 = 9(1-e^2)$

$\therefore e = \frac{\sqrt{5}}{3}$

Foci are $(\pm ae, 0) = (\pm\sqrt{5}, 0)$

Directrices are $x = \pm\frac{a}{e} = \pm\frac{9}{\sqrt{5}}$

(c)(i) $y = (x-1)(3-x) = -x^2 + 4x - 3$

i.e. $x^2 - 4x + 3 + y = 0$

$x = \frac{4 \pm \sqrt{16 - 4.1.(3+y)}}{2} = 2 \pm \sqrt{1-y}$

$\therefore r_1 = 3 - (2 + \sqrt{1-y}) = 1 - \sqrt{1-y}$

$r_2 = 3 - (2 - \sqrt{1-y}) = 1 + \sqrt{1-y}$

$A = \pi r_2^2 - \pi r_1^2 = \pi\left[2 - y + 2\sqrt{1-y} - (2 - y - 2\sqrt{1-y})\right] = 4\pi\sqrt{1-y}$

(ii) $V = 4\pi\int_0^1 \sqrt{1-y}\,dy$

$= 4\pi\left[-\frac{2}{3}(1-y)^{\frac{3}{2}}\right]_0^1 = \frac{8\pi}{3} \text{ units}^3$

Question 4

(a)(i) $x = r\cos(\omega t)$, $\dot{x} = -r\omega\sin(\omega t)$, $\ddot{x} = -r\omega^2\cos(\omega t)$

$y = r\sin(\omega t)$, $\dot{y} = r\omega\cos(\omega t)$, $\ddot{y} = -r\omega^2\sin(\omega t)$

$F = ma = m\sqrt{(\ddot{x})^2 + (\ddot{y})^2} = m\sqrt{r^2\omega^4(\cos^2\omega t + \sin^2\omega t)} = mr\omega^2$

(ii) $mr\omega^2 = \frac{Am}{r^2} \rightarrow r^3 = \frac{A}{\omega^2} \rightarrow r = \sqrt[3]{\frac{A}{\omega^2}}$

(b)(i) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \rightarrow \frac{2x}{a^2} - \frac{2y}{b^2}\cdot y' = 0 \rightarrow y' = \frac{b^2x}{a^2y}$

At P gradient of tangent $= \frac{b\sec\theta}{a\tan\theta}$

Equation of tangent: $(y - b\tan\theta) = \frac{b\sec\theta}{a\tan\theta}(x - a\sec\theta)$

i.e. $bx\sec\theta - ay\tan\theta = ab(\sec^2\theta - \tan^2\theta) = ab$

(ii) Asymptotes are $y = \pm\frac{b}{a}x$, substituting $y = \frac{b}{a}x$ gives

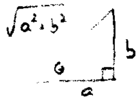
$bx\sec\theta - bx\tan\theta = ab$

$x = \frac{a}{\sec\theta - \tan\theta} = \frac{a}{\frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta}} = \frac{a\cos\theta}{1-\sin\theta}$, $y = \frac{b\cos\theta}{1-\sin\theta}$

when $y = -\frac{b}{a}x$ gives: $x = \frac{a}{\frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}} = \frac{a\cos\theta}{1+\sin\theta}$, $y = \frac{-b\cos\theta}{1+\sin\theta}$

i.e. $A\left(\frac{a\cos\theta}{1-\sin\theta}, \frac{b\cos\theta}{1-\sin\theta}\right)$ and $B\left(\frac{a\cos\theta}{1+\sin\theta}, \frac{-b\cos\theta}{1+\sin\theta}\right)$

(iii) Let $\angle BOA = 2\theta$, where $\theta = \tan^{-1}\frac{b}{a}$



Area of $\Delta OAB = \frac{1}{2}(OA)(OB)\sin 2\theta$

$= \frac{1}{2}\sqrt{\frac{b^2\cos^2\theta}{(1-\sin\theta)^2} + \frac{a^2\cos^2\theta}{(1-\sin\theta)^2}}\sqrt{\frac{b^2\cos^2\theta}{(1+\sin\theta)^2} + \frac{a^2\cos^2\theta}{(1+\sin\theta)^2}}\cdot 2\sin\theta\cos\theta$

$= \frac{\sqrt{(a^2+b^2)\cos^2\theta}}{1-\sin\theta}\cdot\frac{\sqrt{(a^2+b^2)\cos^2\theta}}{1+\sin\theta}\cdot\frac{b}{\sqrt{a^2+b^2}}\cdot\frac{a}{\sqrt{a^2+b^2}}$

$= \frac{(a^2+b^2)\cos^2\theta\cdot ab}{(1-\sin^2\theta)(a^2+b^2)} = ab$ units^2

(c) (i) n^n

(ii) P(at least one door will not be chosen)

$= 1 - $ P(every door is chosen)

$= 1 - \frac{n!}{n^n}$

Question 5

(a) (i) $S_1 = \alpha + \beta + \gamma = 0$

$S_2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma) = -2p$

$S_3 = \alpha^3 + \beta^3 + \gamma^3$

Noting $x^3 = -px - q$ $\therefore$ $\alpha^3 = -p\alpha - q$

$\beta^3 = -p\beta - q$

$\gamma^3 = -p\gamma - q$

$\therefore S_3 = \alpha^3 + \beta^3 + \gamma^3 = -p(\alpha+\beta+\gamma) - 3q = -3q$

(ii) $-pS_{n-2} - qS_{n-3}$

$= -p(\alpha^{n-2} + \beta^{n-2} + \gamma^{n-2}) - q(\alpha^{n-3} + \beta^{n-3} + \gamma^{n-3})$

$= -p\alpha^{n-2} - q\alpha^{n-3} - p\beta^{n-2} - q\beta^{n-3} - p\gamma^{n-2} - q\gamma^{n-3}$

$= -\alpha^{n-3}(p\alpha + q) - \beta^{n-3}(p\beta + q) - \gamma^{n-3}(p\gamma + q)$

$= -\alpha^{n-3}(-\alpha^3) - \beta^{n-3}(-\beta^3) - \gamma^{n-3}(-\gamma^3)$

$= \alpha^n + \beta^n + \gamma^n$

$= S_n$ for $n \geq 3$

(iii) $S_5 = -pS_3 - qS_2$

$\alpha^5 + \beta^5 + \gamma^5 = \frac{S_2}{2}\cdot S_3 + \frac{S_3}{2}\cdot S_2 = \frac{5S_2S_3}{6}$

i.e. $\frac{\alpha^5 + \beta^5 + \gamma^5}{5} = \left(\frac{\alpha^2+\beta^2+\gamma^2}{2}\right)\left(\frac{\alpha^3+\beta^3+\gamma^3}{3}\right)$

(b) (i) $\frac{dv}{dt} = -kv$

$\frac{dt}{dv} = -\frac{1}{kv}$

Integrating both sides w.r.t. v, gives:

$t = -\frac{1}{k}\ln(v) + C$, at $t=0$, $v = u\cos\alpha$

$\therefore C = \frac{1}{k}\ln(u\cos\alpha)$

i.e. $t = \frac{1}{k}\ln(u\cos\alpha) - \frac{1}{k}\ln(v) = \frac{1}{k}\ln\left(\frac{u\cos\alpha}{v}\right)$

$kt = \ln\left(\frac{u\cos\alpha}{v}\right)$

$e^{kt} = \frac{u\cos\alpha}{v}$ i.e. $v = e^{-kt}u\cos\alpha$ or $\dot{x} = e^{-kt}u\cos\alpha$

(ii) $\dot{y} = \frac{1}{k}\left((ku\sin\alpha + g)e^{-kt} - g\right)$ ......①

$\ddot{y} = -(ku\sin\alpha + g)e^{-kt}$

$= -(k\dot{y} + g)$ From ① $k\dot{y} + g = (ku\sin\alpha + g)e^{-kt}$

$= -k\dot{y} - g$

(iii) Maximum height reached when $\dot{y} = 0$

i.e. $g = (ku\sin\alpha + g)e^{-kt}$

$e^{kt} = \frac{ku\sin\alpha + g}{g}$

$\therefore t = \frac{1}{k}\ln\left(\frac{ku\sin\alpha}{g} + 1\right)$

(iv) $x = \int u\cos\alpha e^{-kt}dt = -\frac{1}{k}u\cos\alpha e^{-kt} + C$

At $t=0$, $x=0$ $\therefore$ $C = \frac{1}{k}u\cos\alpha$

i.e. $x = \frac{1}{k}u\cos\alpha(1 - e^{-kt})$

As $t \to \infty$, $x \to \frac{1}{k}u\cos\alpha$

Question 6

(a) (i) $\cos(a+b)x + \cos(a-b)x$

$= \cos ax\cos bx - \sin ax\sin bx$

$+ \cos ax\cos bx + \sin ax\sin bx$

$= 2\cos ax\cos bx$

(ii) $\int\cos 3x\cos 2x\,dx = \frac{1}{2}\int\cos 5x + \cos x\,dx$

$= \frac{1}{2}\left[\frac{1}{5}\sin 5x + \sin x\right] + C$

$= \frac{1}{10}\sin 5x + \frac{1}{2}\sin x + C$

(b) (i) $S_3 = S_2 + 2S_1 = 2 + 2\times 1 = 4$

$S_4 = S_3 + 3S_2 = 4 + 3\times 2 = 10$

(ii) $(x+\sqrt{x})^2 = x^2 + x + 2x\sqrt{x}$

since $x \geq 0$ $\therefore$ $2x\sqrt{x} \geq 0$

i.e. $(x+\sqrt{x})^2 \geq x^2 + x$

$\therefore x + \sqrt{x} \geq \sqrt{x(x+1)}$

(iii) Step 1: Prove true for $n=1, n=2$

$S_1 = 1 \geq \sqrt{1!} = 1$

$S_2 = 2 \geq \sqrt{2!} = \sqrt{2}$

$\therefore$ Result is true for $n=1, 2$

Step 2: Assume result true for $n=k-1, n=k$

i.e. $S_{k-1} \geq \sqrt{(k-1)!}$ and $S_k \geq \sqrt{k!}$

Step 3: Prove true for $n=k+1$

$S_{k+1} = S_k + kS_{k-1}$

$\geq \sqrt{k!} + k\sqrt{(k-1)!}$ (from step 2)

$= \sqrt{(k-1)!}\,[\sqrt{k}+k]$

$\geq \sqrt{(k-1)!}\,\sqrt{k(k+1)}$ from (ii)

$= \sqrt{(k+1)!}$

Step 4: If result is true for $n=k-1, n=k$ it is also true for $n=k+1$. Since result is true for $n=1$ and $n=2$ it is also true for $n=3, n=4$ and so on for all positive integers of n.

(c) (i) $(\sqrt{x}-\sqrt{y})^2 \geq 0$ i.e. $x+y \geq 2\sqrt{xy} \rightarrow \frac{x+y}{2} \geq \sqrt{xy}$

(ii) Using (i), let $x=a^4$, $y=b^4$

i.e. $\frac{a^4+b^4}{2} \geq a^2b^2$, similarly

$\frac{a^4+c^4}{2} \geq a^2c^2$ and $\frac{b^4+c^4}{2} \geq b^2c^2$

Adding gives: $a^4+b^4+c^4 \geq a^2b^2+a^2c^2+b^2c^2$

(iii) Using (ii) let $a=\sqrt{ab}$, $b=\sqrt{ac}$, $c=\sqrt{bc}$

i.e. $a^2b^2+a^2c^2+b^2c^2 \geq (ab)(ac)+(ab)(bc)+(ac)(bc)$

$= a^2bc + b^2ac + c^2ab$

(iv) $a^2b^2+a^2c^2+b^2c^2 \geq a^2bc+b^2ac+c^2ab$

$= abc(a+b+c) = abcd$

Question 7

(a) $V = \int_0^b 2\pi\,(\text{shell radius})(\text{shell height})\,dx$

$= 2\pi \int_0^{\sqrt{3}} xy\,dx$

$= 2\pi \int_0^{\sqrt{3}} 3x^2 - x^4\,dx = 2\pi\left[x^3 - \frac{x^5}{5}\right]_0^{\sqrt{3}} = \frac{12\pi\sqrt{3}}{5}$ units^3

(b) (i) $\angle BSP = \angle SAB$ ($\angle$ b/w tang. & chord = $\angle$ in alt. seg.)

$\angle SPA = \angle SPB$ (common $\angle$)

$\therefore \Delta ASP \,|||\, \Delta SBP$ (AAA)

(ii) $\frac{SP}{AP} = \frac{BP}{SP}$ (corresp. sides of ||| Δ's in = proportion)

i.e. $SP^2 = AP \times BP$

Looking at C_2:

$PT^2 = AP \times BP$ {Square of tang. from external pt = product of the intercept of the secant passing through that pt}

Hence $SP^2 = PT^2$ i.e. $SP = PT$ as both are positive

(iii) $\angle DSP = 90°$ (given)

$\angle DPS = \angle DPT$ (given PD bisects $\angle SPT$)

$PS = PT$ (from (ii))

Side PD common

$\therefore \Delta SPD \equiv \Delta DPT$ (SAS)

$\therefore \angle DTP = 90°$ {equivalent $\angle$'s in congruent Δ's}

Since DT meets the tangent at $90°$

$\therefore$ DT is diameter of C_2 and hence it passes through the centre of C_2.

(c) (i) $P_n \sin\left(\frac{\alpha}{2^n}\right) = P_{n-1}\cdot\cos\left(\frac{\alpha}{2^n}\right)\sin\left(\frac{\alpha}{2^n}\right)$

$= P_{n-1}\cdot\frac{1}{2}\sin\left(\frac{2\alpha}{2^n}\right)$

$= \frac{1}{2}P_{n-1}\sin\left(\frac{\alpha}{2^{n-1}}\right)$

(ii) $P_n \sin\left(\frac{\alpha}{2^n}\right) = \frac{1}{2}P_{n-1}\sin\left(\frac{\alpha}{2^{n-1}}\right)$, similarly

$P_{n-1}\sin\left(\frac{\alpha}{2^{n-1}}\right) = \frac{1}{2}P_{n-2}\sin\left(\frac{\alpha}{2^{n-2}}\right)$

Extending this result gives:

$P_n \sin\left(\frac{\alpha}{2^{n-1}}\right) = \frac{1}{2^n}P_0 \sin\alpha$

Noting $P_1 \sin\left(\frac{\alpha}{2}\right) = \frac{1}{2}P_0\sin\alpha$

$2\cos\frac{\alpha}{2}\sin\left(\frac{\alpha}{2}\right) = P_0\sin\alpha$

$\therefore P_0 = 1$

i.e. $P_n = \frac{\sin\alpha}{2^n \sin\left(\frac{\alpha}{2^{n-1}}\right)}$

(iii) $\sin\alpha < \alpha$ for $\alpha > 0$, given

i.e. $\sin\left(\frac{\alpha}{2^n}\right) < \frac{\alpha}{2^n}$

$2^n \sin\left(\frac{\alpha}{2^n}\right) < \left(\frac{\alpha}{2^n}\right)2^n = \alpha$

From (ii):

$\frac{\sin\alpha}{P_n} = 2^n \sin\left(\frac{\alpha}{2^{n-1}}\right) < \alpha$

$\therefore \frac{\sin\alpha}{\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{4}\right)\cdots\cos\left(\frac{\alpha}{2^n}\right)} < \alpha$

Question 8

(b) (i) $$I_n = \frac{q^{2n}}{n!}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\pi^2}{4}-x^2\right)^n \cos x \, dx$$

$$= \frac{q^{2n}}{n!}\left[\sin x \left(\frac{\pi^2}{4}-x^2\right)^n\right]_{-\pi/2}^{\pi/2} - \frac{q^{2n}}{n!}\int_{-\frac{\pi}{2}}^{\pi/2} n\left(\frac{\pi^2}{4}-x^2\right)^{n-1} \cdot \sin x \cdot -2x$$

$$= \frac{2q^{2n}}{(n-1)!}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\pi^2}{4}-x^2\right)^{n-1} x\sin x \, dx$$

Integrating by parts again gives:

$$I_n = \frac{2q^{2n}}{(n-1)!}\left[0 - \int -\cos x\left[x\cdot(n-1)\left(\frac{\pi^2}{4}-x^2\right)^{n-2} \cdot -2x + \left(\frac{\pi^2}{4}-x^2\right)^{n-1}\cdot 1\right] dx\right.$$

$$= \frac{2q^{2n}}{(n-1)!}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\pi^2}{4}-x^2\right)^{n-1} \cos x \, dx - \frac{4q^{2n}}{(n-2)!}\int_{-\frac{\pi}{2}}^{\pi/2} x^2\left(\frac{\pi^2}{4}-x^2\right)^{n-2} dx$$

(ii) $$I_n = \frac{2q^{2n}}{(n-1)!}\int\left(\frac{\pi^2}{4}-x^2\right)^{n-1}\cos x \, dx - \frac{4q^{2n}}{(n-2)!}\int\left[\frac{\pi^2}{4} - \left(\frac{\pi^2}{4}-x^2\right)\right]\left(\frac{\pi^2}{4}-x^2\right)^{n-2}\cos x \, dx$$

$$= \frac{2q^{2n}}{(n-1)!} \times \frac{(n-1)!}{q^{2n-2}} \times I_{n-1} - \frac{4q^{2n}}{(n-2)!}\left[\frac{\pi^2}{4}\cdot\frac{(n-2)!}{q^{2n-4}}\cdot I_{n-2} - \frac{(n-1)!}{q^{2n-2}}\cdot I_{n-1}\right]$$

$$= 2q^2 I_{n-1} - q^4\left(\frac{p}{q}\right)^2 I_{n-2} + 4q^2(n-1) I_{n-1} = q^2(4n-2) I_{n-1} - p^2q^2 I_{n-2}$$

(iii) $I_2 = (4n-2)q^2 I_1 - p^2q^2 I_0 = (4n-2)q^2 \cdot 4q - 2p^2q^2$

since p, q and n are integers $\therefore$ I_2 is also an integer

likewise $I_3 = (4n-2)q^2 I_2 - p^2q^2 I_1$ is an integer as both I_2 and I_1 are integers

Hence it follows from above that I_n is an integer for $n = 0, 1, 2, \ldots$

(iv) For $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, $0 \leq \left(\frac{\pi^2}{4}-x^2\right)^n \cos x \leq \left(\frac{\pi^2}{4}\right)^n$

so an upper bound on the value of I_n is given by:

$$\frac{q^{2n}}{n!}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{\pi^2}{4}\right)^n dx = \left(\frac{\pi^2}{4}\right)^n \cdot \frac{q^{2n}}{n!}\cdot\pi = \left(\frac{p^2}{4q^2}\right)^n \cdot \frac{q^{2n}}{n!}\cdot\frac{p}{q} = \left(\frac{p}{2q}\right)^n \cdot \frac{q^{2n}}{n!}\cdot\frac{p}{q} = \left(\frac{p}{2}\right)^{2n}\cdot\frac{1}{n!}\cdot\frac{p}{q}$$

$f(x) = \left(\frac{\pi^2}{4}-x^2\right)^n \cos x$ is an even function and $f(x) \geq 0$ for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ $\therefore 0 < I_n$

Hence $0 < I_n < \left(\frac{p}{2}\right)^{2n}\cdot\frac{1}{n!}\cdot\frac{p}{q}$

(v) From (iv) $0 < I_n < 1$ which proves than I_n can't be an integer, so we have a contradiction. We are thus forced to conclude that π is irrational.

Question 8

(a)

(i) $(\omega^k - 1)(1 + \omega^k + \omega^{2k}) = (\omega^k)^3 - 1 = (\omega^3)^k - 1 = 0$

$\therefore \quad \omega^k = 1 \quad \text{or} \quad 1 + \omega^k + \omega^{2k} = 0$

$\therefore$ 2 possible values of $1 + \omega^k + \omega^{2k}$ are 0 and 3.

(ii) $(1+w)^n = {}^nC_0 + {}^nC_1 w + {}^nC_2 w^2 + \cdots + {}^nC_n w^n$

$(1+w^2)^n = {}^nC_0 + {}^nC_1 w^2 + {}^nC_2 w^4 + \cdots + {}^nC_n w^{2n}$

(iii) $(1+w)^n + (1+w^2)^n$

$= 2\,{}^nC_0 + {}^nC_1(w + w^2) + {}^nC_2(w^2 + w^4) + {}^nC_3(w^3 + w^6) + \cdots + {}^nC_{3\ell}(w^{3\ell} + w^{6\ell}) + \cdots + {}^nC_n(w^n + w^{2n})$

Noting $w^3 = 1$ and $w^k + w^{2k} = -1$ (from (i)) gives:

$(1+w)^n + (1+w^2)^n = 2({}^nC_0 + {}^nC_3 + {}^nC_6 + \cdots + {}^nC_{3\ell}) - ({}^nC_1 + {}^nC_2 + {}^nC_4 + {}^nC_5 + \cdots + {}^nC_n)$

$= 2({}^nC_0 + {}^nC_3 + {}^nC_6 + \cdots + {}^nC_{3\ell}) - [2^n - ({}^nC_0 + {}^nC_3 + {}^nC_6 + \cdots + {}^nC_{3\ell})]$

$= 3({}^nC_0 + {}^nC_3 + {}^nC_6 + \cdots + {}^nC_{3\ell}) - 2^n$

Note: ${}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n = 2^n$

$\therefore \quad {}^nC_0 + {}^nC_3 + {}^nC_6 + \cdots + {}^nC_{3\ell} = \frac{1}{3}\left(2^n + (1+w)^n + (1+w^2)^n\right)$

(iv) Let $n = 6M$ where M is an integer. Thus:

$(1+w)^{6M} = (-w^2)^{6M} = (w^{12})^M = (w^3)^{4M} = 1$ i.e. $(1+w)^n = 1$, also

$(1+w^2)^{6M} = (-w)^{6M} = (w^6)^M = (w^3)^{2M} = 1$ i.e. $(1+w^2)^n = 1$

i.e. ${}^nC_0 + {}^nC_3 + {}^nC_6 + \cdots + {}^nC_n = \frac{1}{3}(2^n + 2)$ [substituting results into (iii)]