Q1

Integration.

1a Let $u = \ln x$. Then $du = dx/x$ and so $\int \frac{\ln x}{x}\,dx = \int u du = u^2/2 + c$
$= (\ln x)^2/2 + c$

1b $I = xe^{2x}\,dx$. Use integration by parts;
$u = x, dv = e^{2x}\,dx$
$du = dx, v = e^{2x}/2$

So, $I = \int udv = uv - \int vdu$ (Formula)
$I = xe^{2x}/2 - \int e^{2x}/2\,dx = xe^{2x}/2 - e^{2x}/4 + c$

1c As there is an x^2 term on top and bottom we manipulate to form a '1' as shown,

$$\begin{aligned} I &= \int \frac{x^2}{1+4x^2}\,dx \\ &= \frac{1}{4}\int \frac{4x^2+1}{1+4x^2}\,dx - \frac{1}{16}\int \frac{1}{\frac{1}{4}+x^2}\,dx \\ &= \frac{1}{4}x - \frac{1}{16}.2.\tan^{-1}2x + c \\ &= \frac{x}{4} - \frac{1}{8}\tan^{-1}2x + c \end{aligned}$$

1d $I = \int_2^5 \frac{x-6}{x^2+3x-4}\,dx$
$= \int_2^5 \frac{a}{x+4} + \frac{b}{x-1}\,dx$
$= \int_2^5 \frac{a(x-1)+b(x+4)}{(x+4)(x-1)}\,dx$

So $a(x-1) + b(x+4) \equiv x - 6$ and hence $a+b = 1$ and $4b-a = -6$. Solving these two simultaneously we obtain, $a = 2, b = -1$ and hence,

$$\begin{aligned} I &= \int_2^5 \frac{2}{x+4} - \frac{1}{x-1}\,dx \\ &= 2\ln(x+4) - \ln(x-1)\,\Big|_2^5 \\ &= 2\ln 9 - \ln 4 - (2\ln 6 - \ln 1) \\ &= 4\ln 3 - 2\ln 2 - 2\ln 2 - 2\ln 3 \\ &= 2\ln 3 - 4\ln 2 \\ &\text{ or } \ln\left(\frac{9}{16}\right) \end{aligned}$$

1e
$$I = \int_1^{\sqrt{3}} \frac{1}{x^2\sqrt{1+x^2}}\,dx$$

Use the substitution $u = 1/x^2 = x^{-2}$. Then $du = -2x^{-3}dx$ and when $x = \sqrt{3} \implies u = 1/3$, when $x = 1 \implies u = 1/1^2 = 1$ so we have

$$I = -\frac{1}{2}\int \frac{x^3\,du}{x^2\sqrt{1+1/u}} = -\frac{1}{2}\int \frac{du}{\sqrt{\frac{1}{x^2}(1+1/u)}}$$
$$= -\frac{1}{2}\int_1^{1/3} \frac{du}{\sqrt{u+1}} = \frac{1}{2}\int_{1/3}^1 (u+1)^{-1/2}\,du$$
$$= (u+1)^{1/2}\Big|_{1/3}^1 = \sqrt{2} - 2/\sqrt{3}$$

Q2

Complex Numbers.

2a $i^9 = (i^2)^4 i = (-1)^4 i = i = 0 + 1i$ (in the form a+ib).

2b $\frac{-2+3i}{2+i} \times \frac{2-i}{2-i}$ (to 'realise' the denominator)
$= \frac{-4+6i+2i-3i^2}{4-i^2}$
$= \frac{8i-1}{5} = -\frac{1}{5} + \frac{8}{5}i$

2c

2.c.i 2.c.ii 2.c.iii

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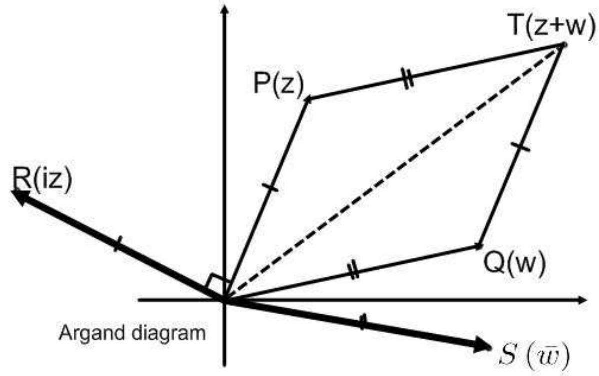
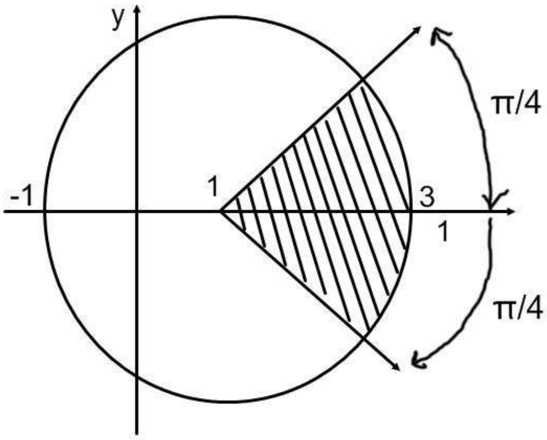
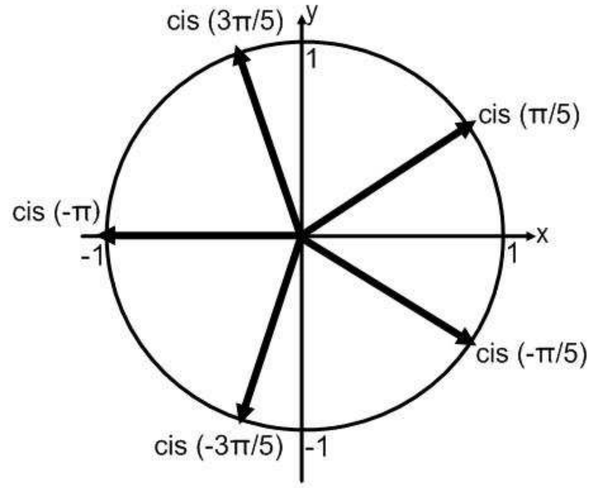


2d



2e 2.e.i We're solving $z^5 = -1$. Assume the form, $z = r\ \text{cis}\,\theta = r\cos\theta + i\sin\theta$.
$\therefore r^5(\ \text{cis}\,\theta)^5 = -1 \implies r^5\ \text{cis}\,5\theta = -1$ (By De Moivre's theorem)
$\therefore r = 1$ and $\ \text{cis}\,5\theta = -1 = \ \text{cis}\,(\pi + 2\pi n)$ where $n \in \mathbb{Z}$.
$5\theta = \pi + 2\pi n$
$\theta = \dfrac{(2n+1)\pi}{5}$ where $n \in \mathbb{Z}$. (this is all of them)
The five with principal arguments are for $n = 0, \pm 1, \pm 2$ we have the five roots,
$z = \ \text{cis}\,\frac{\pi}{5},\ \text{cis}\,\frac{3\pi}{5},\ \text{cis}\,\frac{-\pi}{5},\ \text{cis}\,\pi,\ \text{cis}\,\frac{-3\pi}{5}$
and these are equally spaced around the circle differing in arguments by $2\pi/5$,

2.e.ii



2f 2.f.i
Solve $(x+iy)^2 = 3+4i$
$x^2 - y^2 + 2xyi = 3 + 4i$
$\therefore x^2 - y^2 = 3$ and $2xy = 4$
Combining these by eliminating y we have,
$x^2 - 4/x^2 = 3$
$x^4 - 3x^2 - 4 = 0$ (quadratic in x^2
$x^2 = (3 \pm \sqrt{9+16})/2 = (3 \pm 5)/2 = 4, -1$
As x is real then $x = \pm 2$, so $y = \pm 2/2 = \pm 1$ and therefore the two square roots are
$z = \pm 2 \pm i.1 = \pm(2+i)$. (Check this by squaring it, you should get $3+4i$.)

2.f.ii We use the otherwise method; notice that $z = 1$ is a zero of $z^2 + iz - 1 - i$, dividing out we get,

$$\begin{array}{rl} & \underline{z+i+1} \\ z-1 & \big)z^2+iz-1-i \\ & \quad z^2 - z \\ \hline & (i+1)z-1-i \\ & (i+1)z-(1+i) \\ \hline & 0 \end{array}$$

Thus the equation becomes, $(z-1)(z+i+1) = 0$ and so the solutions are

$$z = 1, -i-1$$

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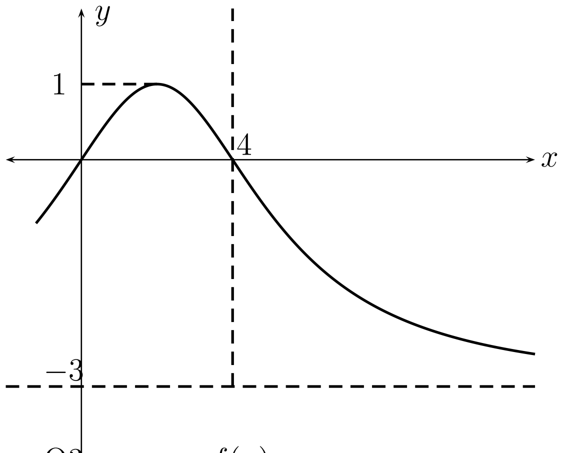
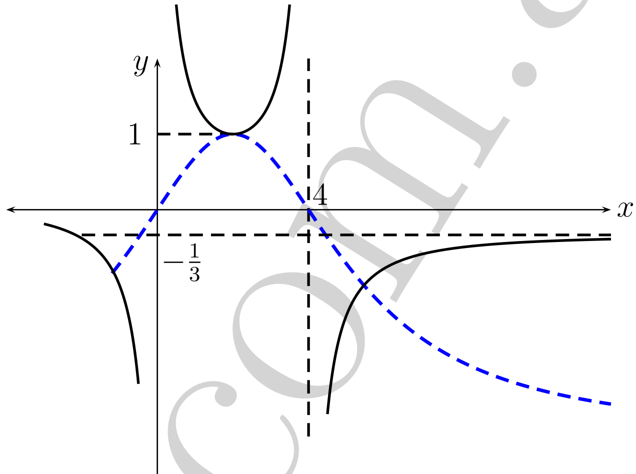
Q3

3a Graphs.

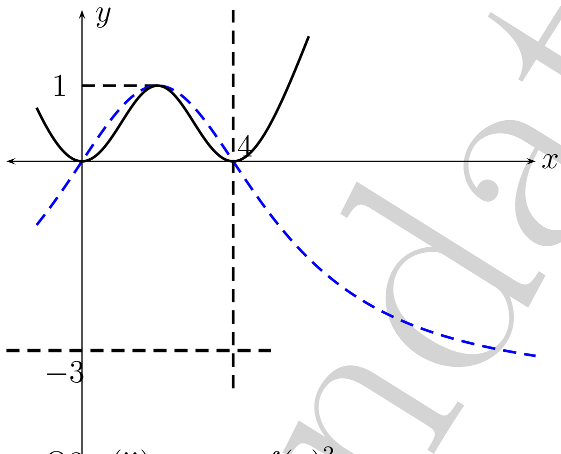
3.a.i 3.a.ii 3.a.iii



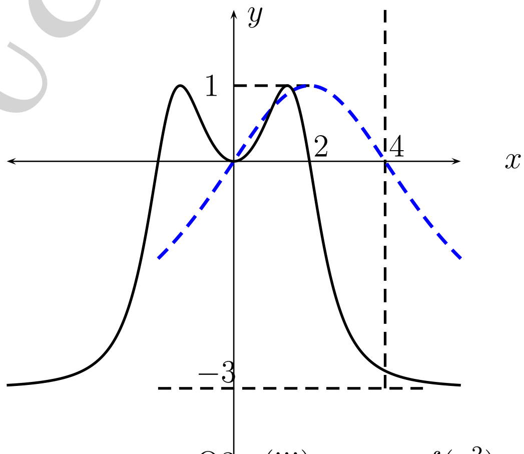
Q3a $y = f(x)$



Q3a (i) $y = \dfrac{1}{f(x)}$



Q3a (ii) $y = f(x)^2$



Q3a (iii) $y = f(x^2)$

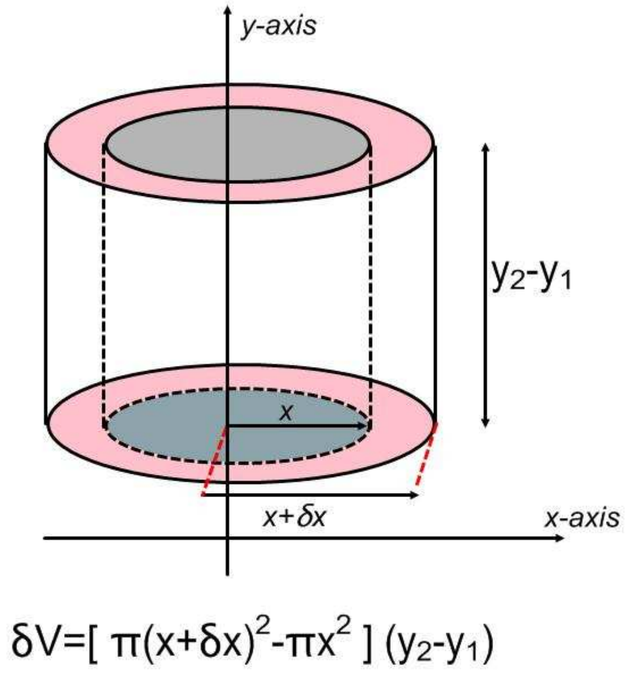
3b Graphs.

$x^2 + 2xy + 3y^2 = 18$. Differentiating implicitly regarding y as a function of x,
$2x + 2y + 2xy' + 6y.y' = 0$
$2y'(x + 3y) = -2(x + y)$
$y' = \dfrac{-x - y}{x + 3y}$
So $y' = 0$ (for stat pts)
$\implies -x - y = 0 \implies y = -x$, and substitution back into the equation gives,
$x^2 - 2x^2 + 3x^2 = 18$
$2x^2 = 18$
$x = \pm 3$
$y = \mp 3$ and hence the stat points are $(-3, 3), (3, -3)$ (and we have checked that at these points, $x + 3y \neq 0$]

3c Polynomials.

We have, $P(x) = x^3 + ax^2 + bx + 5$, $a, b \in \mathbb{R}$. If $(x - 1)^2$ is a factor then by theory, $P(1) = P'(1) = 0$ and $P'(x) = 3x^2 + 2ax + b$, so we have $1+a+b+5=0$ and $3+2a+b=0$. $\therefore b=-6-a \implies 3+2a+-6-a=0 \implies a=3 \implies b=-6-3=-9$

3d Volumes–cylindrical shells.



From theory we have the volume of the infinitesimal shell $\delta V = \pi(r_2^2 - r_1^2)h = (r_2+r_1)(r_2-r_1)(y_2-y_1)$ and $r_2 = x+\delta x$, $r_1 = x$

$\delta V = \pi(x+x+\delta x)(x+\delta x-x)(y_2-y_1)$
$\delta V = 2\pi x(x+1-(x-1)^2)\delta x$
(Ignoring δx^2 terms)
$V = 2\pi \int_0^3 x(x+1) - x(x-1)^2\, dx$ (Limits were obtained as; $x+1 = x^2+1-2x \implies x^2-3x=0 \implies x=0,3$)
$V = 2\pi \int_0^3 x^2+x-x^3+2x^2-x\, dx$
$V = 2\pi \int_0^3 3x^2 - x^3\, dx = 2\pi\left[x^3 - x^4/4\right]_0^3$
$= 2\pi(27-81/4) = \dfrac{27\pi}{2}$ units ^{3}

Q4

4a Conics.

4.a.i
Get y': $\dfrac{2x}{a^2} + \dfrac{2yy'}{b^2} = 0$
$\implies y' = -\left.\dfrac{b^2x}{a^2y}\right|_{(\text{ at } P)} = -\dfrac{b^2x_0}{a^2y_0}.$
So the normal gradient is the negative reciprocal of the tangent gradient, and we have
$y - y_0 = \dfrac{a^2y_0}{b^2x_0}(x-x_0)$ NORMAL

4.a.ii When $y=0$ (on x axis) then
$-y_0 = \dfrac{a^2y_0}{b^2x_0}(x-x_0)$
Solving this for x gives, $x = \dfrac{(a^2-b^2)x_0}{a^2}$
Recall that for an ellipse $b^2 = a^2(1-e^2) \implies b^2 = a^2 - a^2e^2 \implies a^2-b^2 = a^2e^2$ and therefore we have
$x = \dfrac{a^2e^2x_0}{a^2} = e^2x_0$ and so $N(e^2x_0, 0)$ as required.//

4.a.iii By the focus directrix definition we have $PS = ePM$ and $PS' = ePM'$. Therefore,

$$\frac{PS}{PS'} = \frac{ePM}{ePM'} = \frac{PM}{PM'}$$

and in terms of the coordinates we have
$\dfrac{PS}{PS'} = \dfrac{a/e - x_0}{x_0 + a/e} = \dfrac{a - ex_0}{ex_0 + a}$
$= \dfrac{e(a-ex_0)}{e(ex_0+a)} = \dfrac{NS}{NS'}$
(since $NS = ae - e^2x_0 = e(a-ex_0)$ and $NS' = e^2x_0 + ae = e(ex_0+a)$)

4.a.iv We have $\alpha = \angle S'PN$ and $\beta = \angle NPS$ and by the sine rule we have
$\dfrac{PS}{\sin\angle PNS} = \dfrac{NS}{\sin\beta}$ and $\dfrac{PS'}{\sin\angle PNS'} = \dfrac{NS'}{\sin\alpha}$
So by (iii) since $PS/NS = PS'/NS'$ then we have

$\dfrac{\sin\angle PNS}{\sin\beta} = \dfrac{\sin\angle PNS'}{\sin\alpha}$

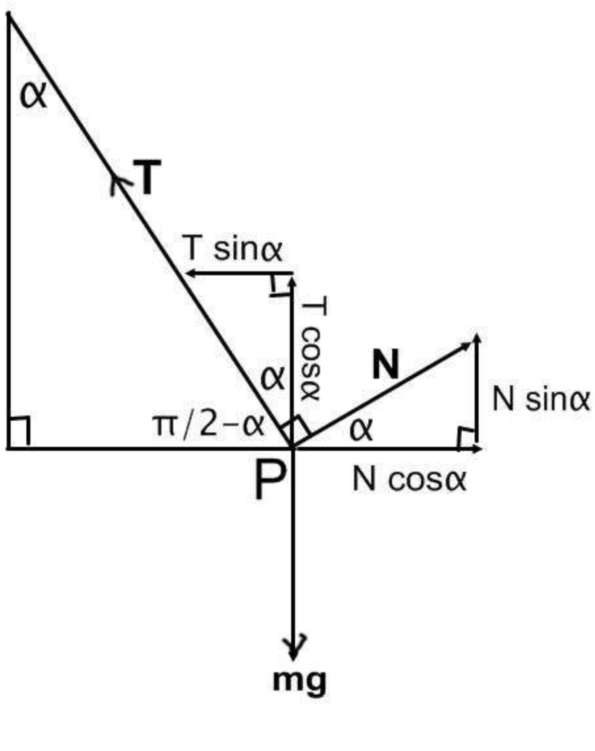
but $\angle PNS' = \pi - \angle PNS$ (since $\angle SNS'$ is a straight angle) and since $\sin(\pi - \angle PNS) = \sin\angle PNS$ we have

$\dfrac{\sin\angle PNS}{\sin\beta} = \dfrac{\sin(\pi - \angle PNS)}{\sin\alpha} = \dfrac{\sin\angle PNS}{\sin\alpha}$ and therefore

$\dfrac{1}{\sin\beta} = \dfrac{1}{\sin\alpha} \implies \sin\alpha = \sin\beta$ and as $\alpha, \beta < \pi/2$ (by construction) then we conclude that $\alpha = \beta$ as required.

4b Mechanics.

4.b.i



Resolve vertical forces:
$T\cos\alpha + N\sin\alpha = mg \quad \ldots(1)$
Resolve horizontal forces:
$T\sin\alpha - N\cos\alpha = mr\omega^2 \quad \ldots(2)$

4.b.ii (1) $\times \cos\alpha +$ (2) $\times \sin\alpha$ gives

$T = mg\cos\alpha + mr\omega^2\sin\alpha$
$= m(g\cos\alpha + r\omega^2\sin\alpha)$ as required.

(1) $\times \sin\alpha -$ (2) $\times \cos\alpha$ gives

$N = mg\sin\alpha - mr\omega^2\cos\alpha$
$= m(g\sin\alpha - r\omega^2\cos\alpha)$

(For both calculations we used $\cos^2\alpha + \sin^2\alpha = 1$)

4.b.iii

If $T = N$ then

$m(g\cos\alpha + r\omega^2\sin\alpha) = m(g\sin\alpha - r\omega^2\cos\alpha)$
$g\cos\alpha + r\omega^2\sin\alpha = g\sin\alpha - r\omega^2\cos\alpha$
$\omega^2 = \dfrac{g}{r} \times \dfrac{\sin\alpha - \cos\alpha}{\sin\alpha + \cos\alpha}$
$= \dfrac{g}{r}\left(\dfrac{\tan\alpha - 1}{\tan\alpha + 1}\right)$ as required.//

(here we dividing top/bottom by $\cos\alpha$ and using $\tan\alpha = \sin\alpha/\cos\alpha$)

4.b.iv

We know that $\omega^2 > 0$ and so the right hand side, in (iii), must be positive also.

Hence $\dfrac{\tan\alpha - 1}{\tan\alpha + 1} > 0$
$\therefore \tan\alpha > 1$ (since $\alpha > 0$ and so $\tan\alpha + 1 > 0$)

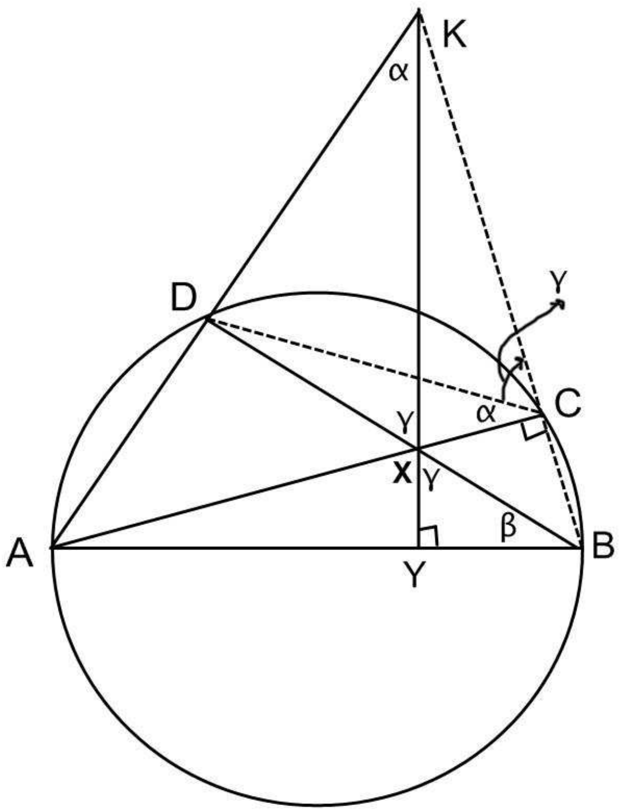
$\therefore \alpha > \tan^{-1}1$ and hence $\alpha > \pi/4$ and hence as obviously $\alpha < \pi/2$ we arrive at

$$\pi/4 < \alpha < \pi/2$$

Q5

5a Circle Geometry.

5.a.i



$\angle ADB = \pi/2$ (angle subtended by a diameter AB)
$\angle KXD = \angle YXB$ (vertically opposite angles are equal)
$\Delta KDX ||| \Delta BYX$ (AA - two corresponding angles are equal)
Hence the third corresponding angle is also equal and thus
$\alpha = \angle AKY = \angle XKD = \angle XBY = \angle ABD = \beta$, as required.

5.a.ii

We know $\alpha = \beta$ (corresponding angles in the similar triangles of part (i) are equal)

Also, $\angle DCA = \angle DBA = \alpha$ (angles on the circle subtended by same segment AD are equal)

$\therefore$ $DKCX$ is a cyclic quadrilateral as the interval DY subtends the same angle α at K and at B.

(Circle result: Angles on circumference of a circle subtended by same segment are equal)

5.a.iii

Firstly from the right-angled triangle ΔXYB we know that $\gamma + \beta = \pi/2$.
But $\alpha = \beta \implies \gamma + \alpha = \pi/2$.
Also, $\angle ACB = \pi/2$ (as AB is a diameter).
Hence, $\angle KCB = \angle KCA + \angle ACB = \alpha + \gamma + \pi/2 = \pi/2 + \pi/2 = \pi$.
Thus $\angle KCB = \pi$ is a straight angle and therefore K, C, B are on the same straight line and so are collinear, as required.

5b Integration.

5.b.i

$I_n = \int_0^1 x^{2n+1} e^{x^2}\, dx$. First we will use a simple substitution and then we use integration by parts to finish.
Let $w = x^2$, then $dw = 2x\, dx$ and the integral becomes
$I_n = \dfrac{1}{2} \displaystyle\int_0^1 w^n e^w\, dw$. Now for integration by parts using the formula $\int u\, dv = uv - \int v\, du$ we choose $u = w^n$ and $dv = e^w\, dw$.
Then $du = nw^{n-1}\, dw$ and $v = e^w$, so that

$$I_n = \frac{1}{2} w^n e^w \bigg|_0^1 - n \int_0^1 w^{n-1} e^w\, dw$$

$$= \frac{1}{2}(1^n.e^1 - 0) - nI_{n-1}$$

$= \dfrac{e}{2} - nI_{n-1}$ as required.

5.b.ii

$I_2 = e/2 - 2I_1$
$I_1 = e/2 - 1.I_0$

$$I_0 = 0.5 \int_0^1 w^0 e^w\, dw = 0.5\, e^w\big|_0^1 = \frac{e-1}{2}$$

Thus $I_1 = e/2 - (e-1)/2 = 1/2$ and $I_2 = e/2 - 2.1/2 = e/2 - 1$

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5c Graphs.

5.c.i

$f(x) = \dfrac{e^x - e^{-x}}{2} - x$

$f'(x) = \dfrac{e^x - (-1)e^{-x}}{2} - 1 = \dfrac{e^x + e^{-x}}{2} - 1$

$f''(x) = \dfrac{e^x - e^{-x}}{2}$

$f'''(x) = \dfrac{e^x + e^{-x}}{2}$

5.c.ii $f'''(x) > 0$ since $e^x > 0, e^{-x} > 0$ for all x

Also, $f''(0) = 0$ and so for $x > 0$ the function $f''(x)$ is increasing for all $x > 0$ (since the sign of f''' tells us how f'' is changing), whilst being equal to zero at $x = 0$, hence $f''(x) > 0$ for $x > 0$. [It is also obvious graphically if you graph and combine $y = e^x$ and $y = e^{-x}$.]

5.c.iii

By a similar argument as in (ii)

$f'(0) = 0$ and $f''(x) > 0$ for $x > 0$ (established in part (i))

Hence $f'(x)$ is an inceasing function so that $f'(x) > 0$ for $x > 0$ (as it started at zero when $x = 0$)

Hence $f(x)$ is an increasing function for $x > 0$, and since it starts at $x = 0$ (ie $f(0) = 0$), then we must have $f(x) > 0$ for $x > 0$ as required.

Q6

6a

The cross sectional area is

$A(x) = 2y(4 - x)$

$= 2(4 - x)\sqrt{4 - x} = (4 - x)^{3/2}$

$V = \int_0^4 A(x)\,dx$

$= 2\int_0^4 (4 - x)^{1.5}\,dx = 2\left[-\frac{2}{5}(4 - x)^{2.5}\right]_0^4$

$= (-4/5)[0 - 4^{5/2}] = \dfrac{4}{5} \times 32 = 128/5$ units^2

6b 6.b.i

Assume that α is a zero we have $P(\alpha) = \alpha^3 + q\alpha^2 + q\alpha + 1 = 0$.

Now, $P(\dfrac{1}{\alpha}) = \dfrac{1}{\alpha^3} + \dfrac{q}{\alpha^2} + \dfrac{q}{\alpha} + 1$

$= \dfrac{1}{\alpha^3}(1 + q\alpha + q\alpha^2 + \alpha^3)$

$= \dfrac{1}{\alpha^3}P(\alpha)$

$= \dfrac{0}{\alpha^3}$

$= 0$

and hence $\frac{1}{\alpha}$ is a zero.//

6.b.ii

Since α is not real and the coefficients of P are real then by theory the roots must occur in complex conjugate pairs, say, $\alpha, \bar{\alpha}$.

Q6.b.ii.1

Product of roots is $\alpha \times \bar{\alpha} \times (-1) = -d/a = -1/1 = -1$, and hence

$|\alpha|^2 = 1 \implies |\alpha| = 1$

(since for any complex number z, $|z| \geq 0$ and $z\bar{z} = |z|^2$).

Q6.b.ii.2

Sum roots is $\alpha + \bar{\alpha} + (-1) = -b/a = -q/1 = -q \implies \alpha + \bar{\alpha} = 1 - q$.

So $\text{Re}(\alpha) = \dfrac{1 - q}{2}$

(since for a complex number z, $z + \bar{z} = 2\,\text{Re}(z)$)

6c 6.c.i

$PQ^2 = OP^2 - r^2$ (Pythagoras)

$PQ^2 = x^2 + y^2 - r^2$ (using distance formula on OP)

Finaly, $PQ = \sqrt{x^2 + y^2 - r^2}$.

6.c.ii

$PQ = PR \implies \sqrt{x^2 + y^2 - r^2} = c - x$

(PR is just the difference in x coordinates)

$\therefore x^2 + y^2 - r^2 = c^2 + x^2 - 2xc$

$y^2 = r^2 + c^2 - 2xc$ as required.

6.c.iii

We put the parabola into standard form to read off the vertex and focal length;

$y^2 = -2cx + r^2 + c^2$

$= -2c\left(x - \frac{(r^2+c^2)}{2c}\right)$

This the focal length is $a = 2c/4 = c/2$ and the vertex is $\left(\frac{r^2+c^2}{2c}, 0\right)$.

Therefore, the focus is

$S\left(\frac{r^2+c^2}{2c} - \frac{c}{2}, 0\right) = S\left(\frac{r^2}{2c}, 0\right)$

6.c.iv The directrix of the locus is

$x = \frac{r^2+c^2}{2c} + c/2 = \frac{r^2+2c^2}{2c}$.

But we know that $PM = PS$ (definition of parabola, with M on directrix)

and $PQ = PR$ (from the locus definition)

Hence,

$|\,PS - PQ\,| = |\,PM - PR\,|$

(difference in the x-values of points M and R)

$= \left|\dfrac{r^2+2c^2}{2c} - c\right|$ which is independant of x as required.

Q7

7a 7.a.i Q7.a.i.1

$\ddot{x} = g - rv$

$\therefore v\dfrac{dv}{dx} = g - rv$

(since by theory $\ddot{x} = \dfrac{d}{dx}\left(v^2/2\right) = v\dfrac{dv}{dx}$)

$\therefore v\dfrac{dv}{dx} = \dfrac{g - rv}{v}$

$\dfrac{dx}{dv} = \dfrac{v}{g - rv}$

$x = \displaystyle\int \frac{v}{g - rv}\,dv$

$= -\dfrac{1}{r}\displaystyle\int \frac{-rv}{g - rv}\,dv$

$= -\dfrac{1}{r}\displaystyle\int \frac{g - rv}{g - rv} - \frac{g}{g - rv}\,dv$

$= -\dfrac{1}{r}[v + \dfrac{g}{r}\ln|g - rv|] + c$

Using intital conditions,

$0 = 0 - (g/v^2)\ln g + c \implies c = \dfrac{g}{r^2}\ln g$

Updating x,

$x = -\dfrac{1}{r}[v + \dfrac{g}{r}\ln|g - rv|] + \dfrac{g}{r^2}\ln g$

$= \dfrac{g}{r^2}\ln\left(\dfrac{g}{g - rv}\right) - \dfrac{v}{r}$

Q7.a.i.2

Hence.

$L = \dfrac{9.8}{0.2^2}\ln\left(\dfrac{9.8}{9.8 - 0.2\times 30}\right) - \dfrac{30}{0.2} = 82\text{m}$ (2 sf)

7.a.ii

$x = e^{-t/10}(29s_t - 10c_t) + 92$ (where $s_t = \sin t, c_t = \cos t$)

$\dot{x} = -e^{-t/10}(29s_t - 10c_t)/10 + e^{-t/10}(29c_t + 10s_t)$

$= e^{-t/10}(29c_t + 10s_t - 29s_t/10 + c_t)$

$= e^{-t/10}(30c_t + 71s_t/10)$

Solve for t where $\dot{x} = 0$

$30c_t + 71s_t/10 = 0$

$30c_t = -71s_t/10$

$\tan t = -300/71$

$\implies t = n\pi - \tan^{-1}(300/71)$ and we need $t > 0$ so require that

$n\pi > \tan^{-1}(300/71)$

$\implies n > \tan^{-1}(300/71)/\pi = 0.426...$

$\implies n \geq 1.$

Choosing $n = 1$ we have $t = \pi - \tan^{-1}(300/71)/ = 1.80$ (2dp)

Hence $x = e^{-1.8/10}(29\sin 1.8 - 10\cos 1.8) + 92 = 25.487 + 92 = 117.487$

and as he is 2m tall his head reaches down a distance of $117.487 + 2 = 119.487$m and so he stays dry. (i.e. out of the water).

7b

Let $z = \cos\theta + i\sin\theta$.

7.b.i

Using De Moivre's Theorem we have
$z^n = \cos n\theta + i\sin n\theta$ and
$z^{-n} = \cos(-n\theta) + i\sin(-n\theta)$
$= \cos n\theta - i\sin n\theta$ (since sine is an odd function and cosine is even function)

So, $z^n + z^{-n} = \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$
$= 2\cos n\theta$ as required.

7.b.ii

By part (i), with $n = 1$ we have, $z + z^{-1} = 2\cos\theta$, and hence
$(z+z^{-1})^{2m} = (2\cos\theta)^{2m}$ and expansion of the left side using the binomial theorem gives

$$(z+z^{-1})^{2m} = \sum_0^{2m} \binom{2m}{r} z^{2m-r}.z^{-r}$$

$$= \sum_0^{2m} \binom{2m}{r} z^{2m-2}$$

$$= \binom{2m}{0} z^{2m} + \binom{2m}{1} z^{2m-2} + \binom{2m}{2} z^{2m-4} + \binom{2m}{3} z^{2m-6} + \cdots + \binom{2m}{2m-1} z^{-2m+2} + \binom{2m}{2m} z^{-2m}$$

But $\binom{2m}{0} = \binom{2m}{2m}$, $\binom{2m}{1} = \binom{2m}{2m-1}$, $\binom{2m}{2} = \binom{2m}{2m-2}$, ...

and so

$$(z+z^{-1})^{2m} = \binom{2m}{0}(z^{2m} + z^{-2m}) + \binom{2m}{1}(z^{2m-2} + z^{-2m+2}) + \binom{2m}{2}(z^{2m-4} + z^{-2m+4}) + \cdots + \binom{2m}{m-1}(z^2 + z^{-2}) + + \binom{2m}{m} z^0$$

$$= 2\left[\binom{2m}{0}\cos 2m\theta + \binom{2m}{1}\cos 2(m-1)\theta + \binom{2m}{2}\cos 2(m-2)\theta + \cdots + \binom{2m}{m-1}\cos 2\theta\right] + \binom{2m}{m}$$ as required.

7.b.iii

Integrating the expression in (ii) and manipulating the contants we have
$\int_0^{\pi/2} \cos^{2m}\theta\,\theta$

$$= 2^{-2m+1}\left[\frac{\sin 2m\theta}{2m} + \binom{2m}{1}\frac{\sin(2m-2)\theta}{2m-2} + \binom{2m}{2}\frac{\sin(2m-4)\theta}{2m-4} + \cdots + \binom{2m}{m-1}\frac{\sin 2\theta}{2}\right]_0^{\pi/2} + 2^{-2m}\binom{2m}{m}\theta\Big|_0^{\pi/2}$$

$$= 2^{-2m+1}[0] + 2^{-2m}\binom{2m}{m}\frac{\pi}{2}$$

$$= \frac{\pi}{2^{2m+1}}\binom{2m}{m}$$

Q8

8a 8.a.i

Start
$\frac{1}{2}(\cot\theta/2 - \tan\theta/2)$
$= \frac{1}{2}\left(\frac{\cos\theta/2}{\sin\theta/2} - \frac{\sin\theta/2}{\cos\theta/2}\right)$
$= \frac{1}{2}\left(\frac{\cos^2\theta/2 - \sin^2\theta/2}{\sin\theta/2\cos\theta/2}\right)$
$= \frac{\cos\theta}{\sin\theta}$ (using double angle formulas)
$= \cot\theta$ and hence we are done as
$\frac{1}{2}(\cot\theta/2 - \tan\theta/2) = \cot\theta$ is equivalent to the identity we are asked to prove.

8.a.ii

We are required to prove:

$$\sum_{r=1}^{n} \frac{1}{2^{r-1}}\tan\frac{x}{2^r} = \frac{1}{2^{n-1}}\cot\frac{x}{2^n} - 2\cot x$$

Firstly, for $n = 1$,
$LHS = \frac{1}{2^{1-1}}.\tan\frac{x}{2} = \tan\frac{x}{2}$

$RHS = \frac{1}{2^0}\cot x/2 - 2\cot x$
$= \cot x/2 - 2\cot x$
$= \cot x/2 - (\cot x/2 - \tan x/2)$ by part (i).
$= \tan x/2 = LHS$
Hence the statement is true for $n = 1$.

Secondly, we prove the statement true for $n = k+1$ whenever the statement is true for $n = k$. So we assume that

$$\sum_{r=1}^{k} \frac{1}{2^{r-1}}\tan\frac{x}{2^r} = \frac{1}{2^{k-1}}\cot\frac{x}{2^k} - 2\cot x \ldots (*)$$

Now assuming (*) we prove that

$$\sum_{r=1}^{k+1} \frac{1}{2^{r-1}}\tan\frac{x}{2^r} = \frac{1}{2^k}\cot\frac{x}{2^{k+1}} - 2\cot x \ldots (**)$$

Start, $LHS = \sum_{r=1}^{k+1} \frac{1}{2^{r-1}}\tan\frac{x}{2^r}$

$= \dfrac{1}{2^{k-1}} \cot \dfrac{x}{2^k} - 2\cot x + \dfrac{1}{2^k} \tan \dfrac{x}{2^{k+1}}$ (by induction hypothesis)
$= \dfrac{1}{2^k}\left(\cot \dfrac{x}{2^k} + \tan \dfrac{x}{2^{k+1}}\right) - 2\cot x$
$= \dfrac{1}{2^k} \cot \dfrac{x}{2^{k+1}} - 2\cot x$
$= RHS$ and hence the statement is true for $n = k+1$ when it is true for $n = k$.
Finally, we conclude that by induction the statement is true for all positive integers.

> Remark: This is enough of a conclusion to a proof by induction—contrary to the advice of many textbooks. A reference is the 2007 examiners report, `http://www.boardofstudies.nsw.edu.au/hsc_exams/exam-papers-2007/pdf_doc/mathematics-ext2-notes-07.pdf`. The 2/3 unit syllabus confirms this view also - see page 47. JSH

8.a.iii

Firstly a preliminary, Recall that

$$\lim_{x\to 0} \frac{x}{\tan x} = 1$$

Hence

$$\lim_{x\to 0} \frac{\frac{x}{2^n}}{\tan \frac{x}{2^n}} = 1$$

We have, $\sum_{r=1}^{n} \frac{1}{2^{r-1}} \tan \frac{x}{2^r}$
$= \dfrac{1}{2^{n-1}} \cot \dfrac{x}{2^n} - 2\cot x$
$= \dfrac{2}{x} \dfrac{\frac{x}{2^n}}{\tan \frac{x}{2^n}} - 2\cot x$
$= \dfrac{2}{x} - 2\cot x$ by the above result.

8.a.iv

Substituting $x = \pi/2$ into the result of part (iii) we have

$\tan \pi/4 + \dfrac{1}{2}\tan \pi/8 + \dfrac{1}{4}\tan \pi/16 + \cdots$
$= \dfrac{2}{\frac{\pi}{2}} - 2\cot \dfrac{\pi}{2}$
$= \dfrac{4}{\pi} - 0$
$= \dfrac{4}{\pi}$

8b

Area under curve is
$\int_{n-1}^{n} \frac{1}{x} = \ln n - \ln n - 1 = \ln \frac{n}{n-1}$

Define A_L =lower rectangle area$= 1 \times 1/n = 1/n$
Define A_U =upper rectangle area$= 1 \times 1/(n-1) = 1/(n-1)$

Ths since by construction, $A_L < A < AU$ then we have $1/n < \ln \frac{n}{n-1} < 1/(n-1)$ and exponentiating (i.e. take the power of e) and use the fact that $e^{\ln x} = x$ to get
$e^{\frac{1}{n}} < \frac{n}{n-1} < e^{\frac{1}{(n-1)}}$
$e^1 < \left(\frac{n}{n-1}\right)^n < e^{\frac{n}{(n-1)}}$ (by taking power of n)
$e^{\frac{-n}{(n-1)}} < \left(\frac{n-1}{n}\right)^n < e^{-1}$
(We twice used the fact that for positive numbers a,b,c,d, that $a/b < c/d \implies d/c < b/a$)
Thus $e^{\frac{-n}{(n-1)}} < \left(1 - \frac{1}{n}\right)^n < e^{-1}$ as required.//

8c 8.c.i

The probability of A_1 winning first up equals p.
The probability of A_1 winning on the second round [after n people (including A_1's first attempt), failed to win] equals $q^n p$
The probability of A_1 winniung after two rounds is $q^n q^n p$

Thus, $W = p + pq^n + pq^{2n} + pq^{3n} + \cdots$
$W = p + pq^n(1 + q^n + q^{2n} + \cdots)$
$= p + pq^n \dfrac{1}{1-q^n}$ (using G.P. summation formula)
$= p + \dfrac{pq^n}{1-q^n}$
$= \dfrac{p - pq^n + pq^n}{1-q^n}$
$W = \dfrac{p}{1-q^n}$
Then manipulate this to get the requested format;
$W(1-q^n) = p$
$W - Wq^n = p$
$W = p + q^n W$ as required.

8.c.ii

$$\begin{aligned}
W_m &= p + pq^n + pq^{2n} + \cdots + pq^{(m-1)n} \\
&= \frac{p(q^{mn}-1)}{q^n - 1} \\
\frac{W_m}{W} &= \frac{p(q^{mn}-1)}{q^n-1} \times \frac{(1-q^n)}{p} \\
&= -q^{mn} + 1 \\
&= -\left(1 - \frac{1}{n}\right)^{nm} + 1
\end{aligned}$$

By part (b), taking the mth power, we have
$e^{\frac{-nm}{(n-1)}} < \left(1 - \frac{1}{n}\right)^{nm} < e^{-m}$
If n is large, that is, as $n \to \infty$, then $\frac{-nm}{(n-1)} \to -m$.
So, $\left(1 - \frac{1}{n}\right)^{nm} \to e^{-m}$ as $n \to \infty$.
Thus $-\left(1 - \frac{1}{n}\right)^{nm} + 1 \to -e^{-m} + 1$ as $n \to \infty$.
So $\frac{W_m}{W}$ is aproximately equal to $1 - e^{-m}$, as required.

Contact: Comments/corrections are appreciated,

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Regards,

Jan Hansen 3 November 2009.