2001 HSC, ME 2.

Question 1

(a) $\int_0^{\pi/4} \tan^3 x \sec^2 x \, dx$ let $v = \tan x$.

$= \left[\frac{\tan^4 x}{4}\right]_0^{\pi/4}$

$= \frac{1}{4}$

(b) $\int \frac{dx}{\sqrt{x^2-4x+1}}$

x^2-4x+1
$= x^2-4x+2^2-3$
$= (x-2)^2-3$

$= \int \frac{dx}{\sqrt{(x-2)^2-3}}$

$= \ln\left((x-2)+\sqrt{(x-2)^2-3}\right)+c$, using $\int \frac{1}{\sqrt{x^2-a^2}} = \ldots$ result.

(c) $\int_e^4 \frac{\ln x}{x^2} \, dx$

$\ln x$ can't be integrated, so let $u = \ln x$ and $\frac{dv}{dx} = x^{-2}$.
$\frac{du}{dx} = \frac{1}{x}$ $\quad v = -x^{-1}$

$= \left[-\frac{\ln x}{x}\right]_e^4 - \int_e^4 -1 \, dx$

$= \frac{-\ln 4}{4} - \frac{1}{e} + (4-e)$

(d) $\int_2^3 \frac{1+x}{\sqrt{x-1}} \, dx$

let $u = \sqrt{x-1}$
$u^2 = x-1 \Rightarrow 1+x = u^2+2$
$\frac{du}{dx} = \frac{1}{2}(x-1)^{-1/2}$
$x=3,\ u=\sqrt{2}$
$x=2,\ u=1$

$= \int_1^{\sqrt{2}} (u^2+2)\, 2\,du$

$= \left[\frac{u^3}{3}+2u\right]_1^{\sqrt{2}}$

$= \frac{2\sqrt{2}}{3}+2\sqrt{2} - \left(\frac{1}{3}+2\right)$

$= \frac{8\sqrt{2}-7}{3}$

(e)(i) $5x^2-3x+1 \equiv (ax+1)(x-2)+b(x^2+1)$

put $x=i$: $5i^2-3i+1 = (ai+1)(i-2)$, where $i=\sqrt{-1}$
$= ai^2-2ai+i-2$

equating coefficients of i terms, $a=2$.

$\therefore 5x^2-3x+1 \equiv (2x+1)(x-2)+b(x^2+1)$.

put $x=-\frac{1}{2}$: $1.25+3(0.5)+1 = b(0.25+1)$
$b=3$.

Question 1

(e) (ii) $\int \frac{5x^2-3x+1}{(x^2+1)(x-2)}\,dx$

$= \int \left(\frac{2x+1}{x^2+1} + \frac{3}{x-2} \right) dx$ from (i)

$= \int \left(\frac{2x}{x^2+1} + \frac{1}{x^2+1} + \frac{3}{x-2} \right) dx$

$= \ln(x^2+1) + \tan^{-1} x + 3\ln(x-2) + C$

Question 2

(a) $zw = (2+3i)(1+i)$

$= 2+2i+3i-3 = -1+5i$.

$\frac{1}{w} = \frac{1}{(1+i)} \times \frac{(1-i)}{(1-i)} = \frac{1-i}{1^2+1^2} = \frac{1}{2} - \frac{1}{2}i$

(b)(i) $1+\sqrt{3}i$. modulus $= \sqrt{1^2+(\sqrt{3})^2} = 2$

argument $= \tan^{-1}\frac{\sqrt{3}}{1} = \pi/3$ (since it's in first quadrant)

$\therefore 1+\sqrt{3}i = 2 \operatorname{cis} \pi/3$.

(ii) using De Moivre's theorem:

$(2 \operatorname{cis} \pi/3)^{10} = 2^{10} \operatorname{cis} \frac{10\pi}{3} = 2^{10} \operatorname{cis} \frac{4\pi}{3}$

$= 1024 \left(\cos\frac{4\pi}{3} + i \sin\frac{4\pi}{3} \right)$

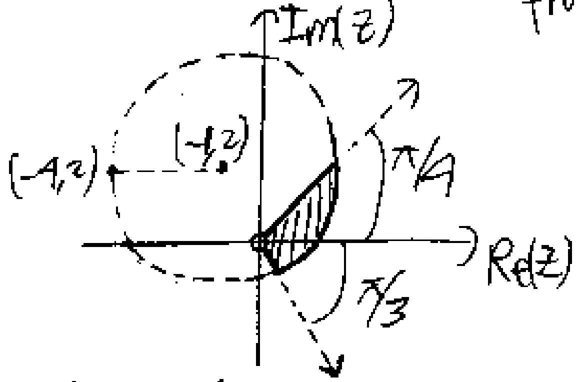
$= 1024 \left(-0.5 + i \times -\frac{\sqrt{3}}{2} \right)$

$= -512 - 512\sqrt{3}i$.

(c) $|z-(-1+2i)| \leq 3$ and $\arg z \geq -\frac{\pi}{3}$ and $\arg z \leq \frac{\pi}{4}$.

circle centred at $(-1,2)$ with radius 3.

ray starting from origin, angle $-\frac{\pi}{3}$



notice $(0,0)$ is not part of the region.

(d) $z^4 = -1 \Rightarrow r^4 \operatorname{cis} 4\theta = -1$, which is $1 \operatorname{cis} \pi$

$r^4 = 1$, $r = 1$ since $r > 0$

$4\theta = \pi + 2n\pi$, $n \in Z^+$

put $n=0$, $\theta = \pi/4$.

put $n=1$, $\theta = 3\pi/4$.

put $n=2$, $\theta = 5\pi/4 = -\pi/4$.

put $n=3$, $\theta = 7\pi/4 = -3\pi/4$.

we know there are 4 roots, so $z = \operatorname{cis} \pm \pi/4$, $\operatorname{cis} \pm 3\pi/4$.

Question 2

(e) (i) $(z_1 - z_2)^2 = -(z_3 - z_2)^2$ because $z_1 - z_2$ represents the vector $\vec{BA}$, and $z_3 - z_2$ represents the vector $\vec{CB}$, and since $\vec{BA} = i\,\vec{CB}$ [i is a rotation of $\pi/2$ anticlockwise]

$(z_1 - z_2) = i\,(z_3 - z_2)$

squaring, $(z_1 - z_2)^2 = -1\ (z_3 - z_2)^2$.

(ii) D can be produced by $\vec{OB} + \vec{BA} + \vec{BC}$, $\vec{BC} = -\vec{CB}$.

$z_2 + (z_1 - z_2) - (z_3 - z_2)$

$= z_1 + z_2 - z_3$.

Question 3

(a) $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$

(i) put $y=0$, $x^2 = 9 \Rightarrow x = \pm 3$ are the x-intercepts.

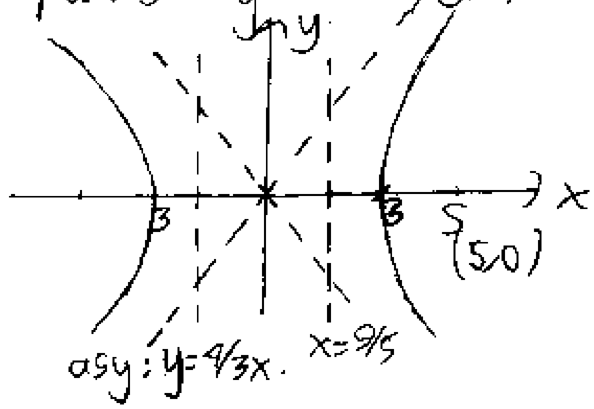
$e = \sqrt{1 + \dfrac{16}{9}} = 5/3$

$S = (\pm ae, 0) = (\pm 5, 0)$

(ii) directrices: $x = \pm a/e \Rightarrow x = \pm 9/5$

asymptotes: $y = \pm b/a\ x = \pm 4/3\ x$.

(iii)



(b)(i) we have $(\alpha+\beta+\gamma)^2 = (\alpha+\beta)^2 + 2(\alpha+\beta)\gamma + \gamma^2$

$= \alpha^2+\beta^2+\gamma^2 + 2(\alpha\beta+\beta\gamma+\gamma\alpha)$

so $\alpha\beta+\beta\gamma+\gamma\alpha = \frac{1}{2}(9-1) = 4$

we also have $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma} = \dfrac{\alpha\beta+\beta\gamma+\gamma\alpha}{\alpha\beta\gamma}$

so $\alpha\beta\gamma = 4/2 = 2$

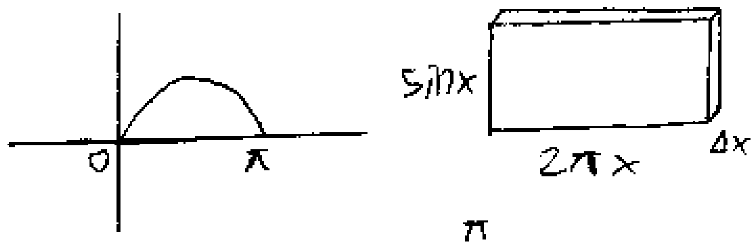
α, β, γ are the roots of $x^3 - 3x^2 + 4x - 2 = 0$ because the sums of the products of roots taken 1, 2, 3 at a time follow the pattern of $-b/a$, c/a, $-d/a$.

(ii) try $x=1$; $P(x)=0$. let $\alpha = 1$.

$\beta + \gamma = 2$ and $\beta\gamma = 2 \Rightarrow \beta(2-\beta) = 2 \Rightarrow \beta = 1+i$, $\gamma = 1-i$.

Question 3

(c)



$\Delta V = 2\pi x \sin x \, \Delta x$

$$\text{Volume} = \lim_{\Delta x \to 0} \sum_{x=0}^{\pi} 2\pi x \sin x \, \Delta x$$ ⇐ no of marks of 4 indicates you need this line.

$$= \int_0^{\pi} 2\pi x \sin x \, dx$$

let $u = x$ and $\frac{dv}{dx} = \sin x$

$\frac{du}{dx} = 1$ $v = -\cos x$

$$= 2\pi \times \left(\left[-x\cos x \right]_0^{\pi} - \int_0^{\pi} -\cos x \, dx \right)$$

$$= 2\pi \left(\pi + \left[\sin x \right]_0^{\pi} \right)$$

$$= 2\pi(\pi + 0) = 2\pi^2 \text{ units}^3$$

Question 4

(a) (i) $f(x)$ has turning points when $f'(x)$ crosses the x-axis.

at $x = -1$: maximum

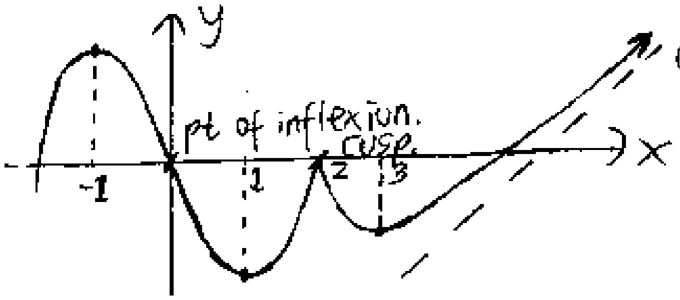
at $x = 1$: minimum

at $x = 3$: minimum.

(ii) → there's a TP at $x=0$ for $f'(x)$, so $f(x)$ will have a point of inflexion there, since $f''(x)$ changes sign around it.

→ $f(x)$ is given as continuous but $f'(x)$ isn't, so at $x=2$ there is a sharp edge in $f(x)$, otherwise $f'(x)$ would be defined at $x=2$.

→ the asymptote of $f(x)$ has a gradient of 1 (imagine integrating line $y=1$).

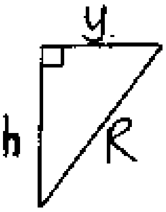


Question 4.

(b)(i) area of outer circle:

πy^2

$= \pi(R^2 - h^2)$.



using Pythagoras' theorem,

$y^2 = R^2 - h^2$.

area of inner circle:

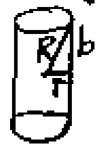
πr^2.

$\therefore$ area of cross-section $= \pi(R^2 - h^2 - r^2)$

(ii) $\text{Volume} = 2 \lim_{\Delta h \to 0} \sum_{h=0}^{b} \pi(R^2 - h^2 - r^2)\Delta h$, 2b is the length of the cylindrical hole,

(notice that the upper limit is not R, because some of the top and bottom ends of the sphere will be removed completely, without holes.)

$\text{Volume} = 2\pi \int_0^b \left((R^2 - r^2) - h^2\right) dh$, but $R^2 - r^2 = b^2$; looking at the cylinder,



$= 2\pi \int_0^b (b^2 - h^2)\,dh$

$= 2\pi \left[b^2 h - \frac{h^3}{3} \right]_0^b$

$= 2\pi \left[b^3 - \tfrac{1}{3} b^3 \right] = \frac{4\pi}{3} b^3$ units^3.

(c) let $f(x) = \tan^{-1} \frac{x}{x+1} + \tan^{-1} \frac{1}{2x+1}$ — ①

$f'(x) = \frac{1}{1 + \left(\frac{x}{x+1}\right)^2} \cdot \frac{d}{dx}\left(\frac{x}{x+1}\right) + \frac{1}{1 + \left(\frac{1}{2x+1}\right)^2} \cdot \frac{d}{dx}\left(\frac{1}{2x+1}\right)$

$= \frac{1}{(x+1)^2 + \frac{x^2 (x+1)^2}{(x+1)^2}} + \frac{-2}{(2x+1)^2 + \frac{(2x+1)^2}{(2x+1)^2}}$

$\frac{d}{dx}\left(\frac{x}{x+1}\right) = \frac{(x+1)(1) - (x)(1)}{(x+1)^2} = \frac{1}{(x+1)^2}$

$= \frac{1}{(x+1)^2 + x^2} - \frac{2}{(2x+1)^2 + 1}$

$= 0$, by inspection of the denominators in expanded forms.

Since $f'(x) = 0$, $f(x)$ never increases nor decreases, and consists of only horizontal line(s).

Examination of ① shows that $f(x)$ is undefined at $x+1=0$ and $2x+1=0$. for $2x+1 > 0$, there is only one horizontal line, hence $f(x)$ is a constant.

Putting random x in this domain, $f(0) = \pi/4 \Rightarrow$ constant $= \pi/4$.

Question 5

(a) (i) differentiating $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$,

$$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{b^2}{a^2}\frac{x}{y}$$

$$\text{at } P,\ \frac{dy}{dx}=-\frac{b^2}{a^2}\frac{a\cos\theta}{b\sin\theta}=-\frac{b}{a}\cot\theta$$

tangent at P:

$$y-b\sin\theta=-b/a\cot\theta\,(x-a\cos\theta)$$

$$\div b:\quad \frac{y}{b}-\sin\theta=-\frac{x\cot\theta}{a}+\cos\theta\cot\theta$$

$$\times\sin\theta:\quad \frac{y\sin\theta}{b}+\frac{x\cos\theta}{a}=\cos^2\theta+\sin^2\theta$$

$$=1$$

(ii) show $m_{QR}=-b/a\cot\theta$ (m of tangent at P)

$$m_{QR}=\frac{y_R-y_Q}{x_R-x_Q}=\frac{b\sin(\theta-\psi)-b\sin(\theta+\psi)}{a\cos(\theta-\psi)-a\cos(\theta+\psi)}$$

$$=\frac{b}{a}\left(\frac{\sin(\theta-\psi)-\sin(\theta+\psi)}{\cos(\theta-\psi)-\cos(\theta+\psi)}\right)$$

$$=\frac{b}{a}\,\frac{-\cos\theta}{\sin\theta}$$ by the sums-to-products formulae, or by the compound angle formulae

(iii) equation of line OP:

$$y-y_1=m(x-x_1)$$

$$y-b\sin\theta=\frac{b}{a}\tan\theta\,(x-a\cos\theta)\quad \text{—} \ (1)$$

midpoint M of QR: $x_m=\dfrac{a\cos(\theta+\psi)+a\cos(\theta-\psi)}{2}$

$$=\frac{a(2\cos\theta\cos\psi)}{2}=a\cos\theta\cos\psi$$

$$y_m=\frac{b\sin(\theta+\psi)+b\sin(\theta-\psi)}{2}$$

$$=\frac{b(2\sin\theta\cos\psi)}{2}=b\sin\theta\cos\psi$$

put x_m, y_m into (1): $b\sin\theta\cos\psi-b\sin\theta=\frac{b}{a}\tan\theta(a\cos\theta\cos\psi-a\cos\theta)$

$$b\sin\theta(\cos\psi-1)=b\sin\theta(\cos\psi-1)\ ✓$$

Question 5.

(b)(i) resistive force $= -kv^2$ (given in question).

$$F_{net} = F - kv^2, \quad F_{net} = m\ddot{x}$$

$$\frac{dv}{dt} = \frac{1}{m}(F - kv^2) \quad — (1)$$

(ii) Do we want v as a function of t? Probably not because we know no information about t. Maybe we want x as a function of **v**...

$$v\frac{dv}{dx} = \frac{1}{m}(F - kv^2)$$

$$\frac{dv}{dx} = \frac{1}{vm}(F - kv^2)$$

$$\frac{dx}{dv} = \frac{vm}{F - kv^2}$$

$$x = -\frac{m}{2k}\int \frac{-2kv}{F - kv^2}\,dv$$

$$= -\frac{m}{2k}\ln(F - kv^2) + c$$

$$\text{distance} = \left[-\frac{m}{2k}\ln(F - kv^2)\right]_{v_1}^{v_2}$$

$$= -\frac{m}{2k}\ln\left(\frac{F - kv_2^2}{F - kv_1^2}\right)$$

$$= \frac{m}{2k}\ln\left(\frac{F - kv_1^2}{F - kv_2^2}\right), \text{ using } -\ln x = \ln x^{-1}$$

(c)(i) $\dfrac{{}^{22}C_4 \times {}^{18}C_5 \times {}^{13}C_6 \times {}^{7}C_7}{4!}$ (in the numerator we assume the order of the groups matters.)

(ii) placing groups: $1 \times 3 \times 2 \times 1$

placing people within the groups: $4! \times 5! \times 6! \times 7!$

$\therefore$ no of ways $= 3! \times 4! \times 5! \times 6! \times 7! = {}^7P_5$

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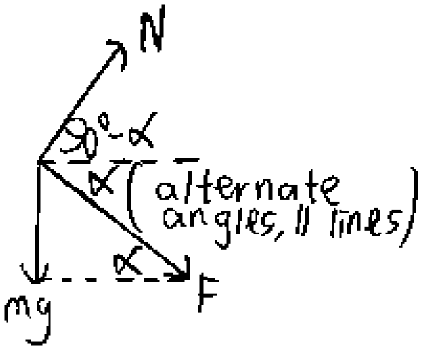
(a) (i)



horizontally:

$N\cos(90°-\alpha) + F\cos\alpha = \frac{mv^2}{r}$

$F\cos\alpha = \frac{mv^2}{r} - N\sin\alpha$ — ①

vertically:

$N\cos\alpha = mg + F\sin\alpha$

$F\sin\alpha = N\cos\alpha - mg$ — ②

(ii) ① × $\cos\alpha$ + ② × $\sin\alpha$: (we want to eliminate N)

$F(\cos^2\alpha \quad \frac{mv^2}{r}\cos\alpha - N\sin\alpha\cos\alpha + N\cos\alpha\sin\alpha - mg\sin\alpha$

$F = \frac{mv^2}{r}\cos\alpha - mg\sin\alpha$

$= \frac{m}{r}(v^2 - gr\tan\alpha)\cos\alpha$

(iii) $r = 200$, $V = 100$ km/h $= 250/9$, $F = 0$, $g = 9.8$

$0 = \frac{m}{200}\left(\frac{250^2}{81} - 1960\tan\alpha\right)\cos\alpha$

RHS will be zero when either $\frac{m}{200}$, (...) OR $\cos\alpha$ is zero, but only the (...) can be zero.

$1960\tan\alpha = \frac{250^2}{81}$

$\tan\alpha = 3125/7938$

$\alpha = 21°$

(b)(i) From the invisible diagram,

$\angle TAF = \angle ADB$ (alternate segment theorem).

$= \angle DEC$ (alternate angles on parallel lines)

$= \angle AET$ (vertically opposite angles).

$\therefore \angle TAF = \angle AET$ } $\Delta TFA ||| \Delta TAE$
$\angle T$ is common } (equiangular)

(ii) $\frac{TF}{TA} = \frac{TA}{TE}$ from the similarity result.

$TE.TF = TA^2$

but $TA = TB$ (tangents from a common external point T)

$\therefore TE \cdot TF = TB^2$.

(iii) rearranging, $\frac{TE}{TB} = \frac{TB}{TF}$. Also, angle T is common.

Hence, $\Delta EBT ||| \Delta BFT$ (ratios of sides are equal, the included angles of those sides are equal)

(iv) let $\angle TBF = \theta = BDA$ (alternate segment theorem)

$\angle TEB = \angle TBF$ (similarity). $\angle DBE = \angle TEB$ (alt ∠'s, ∥ lines) $= \theta$ } isosceles ΔDEB

Question 7.

(a) $z = \frac{1}{2}\cos\theta + \frac{1}{2}i\sin\theta$

(i) $|z| = \sqrt{(\frac{1}{2}\cos\theta)^2 + (\frac{1}{2}\sin\theta)^2}$

$= \sqrt{\frac{1}{4}(\cos^2\theta + \sin^2\theta)}$

$= \frac{1}{2}$.

alternatively, $\vec{z} = \frac{1}{2}\operatorname{cis}\theta$ has length of $\frac{1}{2}$ from $(0,0)$ on the Argand diagram, hence $|z| = \frac{1}{2}$.

(ii) i.e. $\operatorname{Im}\left(\frac{1}{1-z}\right) = \frac{2\sin\theta}{5-4\cos\theta}$ ⇐ prove it.

$\frac{1}{1-z} \times \frac{1-\bar{z}}{1-\bar{z}} = \frac{1-\bar{z}}{1-(z+\bar{z})+z\bar{z}}$, $z\bar{z} = |z|^2$ by inspection.

$= \frac{1-\frac{1}{2}\cos\theta + \frac{1}{2}i\sin\theta}{1-\cos\theta + \frac{1}{4}}$

the imaginary part is $\frac{\frac{1}{2}\sin\theta}{\frac{5}{4}-\cos\theta} = \frac{2\sin\theta}{5-4\cos\theta}$ as required.

(iii) This is the real part of $1+z+z^2+z^3+\ldots$

i.e. the real part of $\frac{1}{1-z}$, which is

$\frac{1-\frac{1}{2}\cos\theta}{\frac{5}{4}-\cos\theta} = \frac{4-2\cos\theta}{5-4\cos\theta}$

Note if you think $1+\frac{1}{2}\cos\theta + \frac{1}{2^2}\cos 2\theta + \ldots$ is $1+z+z^2+\ldots$ with $z = \frac{1}{2}\cos\theta$:

it is not, because if $z = \frac{1}{2}\cos\theta = \frac{1}{2}\cos\theta \operatorname{cis} 0$,

$1+z+z^2+\ldots = 1+\frac{1}{2}\cos\theta\operatorname{cis}0 + (\frac{1}{2}\cos\theta)^2\operatorname{cis}2(0)+\ldots$

(b)(i) since $x = p/q$ is a root,

$a(p/q)^3 - 3(p/q) + b = 0$.

$\frac{ap^3}{q^3} - \frac{3p}{q} + b = 0$

$\times q^3$: $ap^3 - 3pq^2 + bq^3 = 0$

now, factoring p out: $p(ap^2 - 3q^2) = -bq^3$.

The $(\ldots)$ is an integer. Also, p and q^3 have no common factor. Hence, p is a divisor of b.

similarly, factoring q out: $q(-3pq+bq^2) = -ap^3$

The $(\ldots)$ is an integer, and the q, p^3 pair has no common factor. So, q is a divisor of a.

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Question 7

(b)(i) (continued)

Having established that, the only possible rational roots of $x^3 - 3x - 1 = 0$ are: 1 and -1.

let $P(x) = x^3 - 3x - 1$.

$\left.\begin{array}{l} P(1) \neq 0. \\ P(-1) \neq 0 \end{array}\right\}$ $P(x)$ has no rational roots.

(ii) put $x = r + s\sqrt{d}$ into $P(x)$:

$(r+s\sqrt{d})^3 - 3(r+s\sqrt{d}) - 1 = 0$

$r^3 + 3r^2 s\sqrt{d} + 3rs^2 d + s^3 d\sqrt{d} - 3r - 3s\sqrt{d} - 1 = 0$

$r^3 + 3rs^2 d - 3r - 1 + \underbrace{\sqrt{d}}(3r^2 s + s^3 d - 3s) = 0$

$\neq 0$ since if it is it's rational.

for LHS to equal 0, the $\sqrt{d}(3r^2s + s^3 d - 3s)$ must equal 0, because if not we'll have $c + k\sqrt{d}$, $k \neq 0$

hence $3r^2 s + s^3 d - 3s = 0$.

putting $x = r - s\sqrt{d}$ into $P(x)$ we'll get:

$\underbrace{r^2 + 3rs^2 d - 3r - 1}_{= 0 \text{ if } r+s\sqrt{d} \text{ is to be a root}} - \sqrt{d}\,(\underbrace{3r^2 s + s^3 d - 3s}_{= 0 \text{ from before}}) = 0$ } So this statement is true, hence $r - s\sqrt{d}$ is a root.

Now, let the third root of $x^3 - 3x - 1 = 0$ be α.

sum of roots: $(r + s\sqrt{d}) + (r - s\sqrt{d}) + \alpha = 0$

$\alpha = -2r$, which is rational.

But $P(x)$ has no rational zero, so $\alpha = -2r$ can't be a root of $P(x) = 0$. But if $r + s\sqrt{d}$ is a root, $-2r$ must also be a root. Hence, no roots can be expressed in $r + s\sqrt{d}$ form, r, s, d rational.

(iii) $2\cos\frac{\pi}{9} = ?$ using the identity, $\cos 3\left(\frac{\pi}{9}\right) = 4\cos^3\frac{\pi}{9} - 3\cos\frac{\pi}{9}$

$0.5 = 4\cos^3 \pi/9 - 3\cos \pi/9 \Rightarrow 1 = 8y^3 - 6y$, where $y = \cos \pi/9$.

Apparently, by inspection, we don't have to find $\cos \pi/9$.

putting $x = 2\cos\frac{\pi}{9}$ into $P(x)$:

$P(2\cos \pi/9) = 8\cos^3 \pi/9 - 6\cos \pi/9 - 1$

$= (8y^3 - 6y) - 1 = 0$

Question 8.

(a)(i) $a^2+b^2 = (a-b)^2 + 2ab \geqslant 2ab$ — ①

since $(a-b)^2 \geqslant 0$.

similarly, $b^2+c^2 \geqslant 2bc$ — ②

$a^2+c^2 \geqslant 2ac$ — ③

①+②+③: $a^2+b^2+c^2 \geqslant ab+bc+ac$ — ④

RHS in the question

$= (a+b+c)^2 = (a+b)^2 + 2(a+b)c + c^2$

$= a^2+b^2+c^2+2(ab+bc+ac)$

$\geqslant ab+bc+ac+2(ab+bc+ac)$ from ④

$= 3(ab+bc+ca)$

$=$ LHS.

(ii) sides of triangle: $a, b, c > 0$.

① — $(b-c)^2 \leqslant a^2$ because of the triangle inequality, $|b-c| \leqslant a$. (also, $b+c \geqslant a$)

② — $(a-c)^2 \leqslant b^2$

③ — $(a-b)^2 \leqslant c^2$

①+②+③: $(b-c)^2+(a-c)^2+(a-b)^2 \leqslant a^2+b^2+c^2$

expanding, $2(a^2+b^2+c^2) - 2(ab+bc+ca) \leqslant a^2+b^2+c^2$

$a^2+b^2+c^2 \leqslant 2(ab+bc+ca)$.

Now, $(a+b+c)^2 = [a^2+b^2+c^2] + 2(ab+bc+ca)$

$\leqslant [2(ab+bc+ca)] + 2(ab+bc+ca)$

$= 4(ab+bc+ca)$ as required.

(b)(i) for $0 \leqslant x \leqslant 1$, $1 \leqslant e^x \leqslant e$, i.e. $1 \leqslant e^x < 3$.

Visualising the curve of $y = x^\alpha e^x$ for $0 \leqslant x \leqslant 1$, you can see that it lies under $y = 3x^\alpha$.

Hence, $\int_0^1 x^\alpha e^x\,dx < \int_0^1 3x^\alpha\,dx = \dfrac{3}{\alpha+1}$

(ii) for $n=0$, $\int_0^1 x^0 e^x\,dx = e-1 = -1+1e$ ✓

suppose $\int_0^1 x^k e^x\,dx = a_k + b_k e$ — ①

$\int_0^1 x^{k+1} e^x\,dx$ let $u = x^{k+1}$ and $\frac{dv}{dx} = e^x$, $\frac{du}{dx} = (k+1)x^k$, $v = e^x$

$= [e^x x^{k+1}]_0^1 - (k+1)\int_0^1 x^k e^k\,dx$

$= e - (k+1)(a_k + b_k e)$ from ①

$= \underbrace{-(k+1)a_k}_{\text{integer}} - \underbrace{[(k+1)b_k+1]}_{\text{integer}} e = a_{k+1} + b_{k+1} e$

Hence $\int_0^1 x^n e^x\,dx = a_n + b_n e$ for $n = 0, 1, 2, \ldots$

Question 8.

(b)(iii) when $r = -\frac{a}{b}$ (with either a or b being -ve),

$a + br = 0 \Rightarrow |a + br| = 0$

(note: you work this out by letting $a+br=0$, but it's better not to write this down).

Now let's work with the general situation and consider several cases.

$$|a + br| = \left|a + b\frac{p}{q}\right|$$

case 3: b is -ve, a +ve

case 1: a +ve, b +ve

case 2: a -ve, b -ve

case 4, a is -ve, b +ve

$$= \left|\frac{aq + bp}{q}\right|$$

$$= \left|\frac{|a|q + |b|p}{q}\right|, \left|\frac{-|a|q - |b|p}{q}\right|, \left|\frac{|a|q - |b|p}{q}\right| \text{ or } \left|\frac{-|a|q + |b|p}{q}\right|$$

but both $|a|q$ and $|b|p$ are integers.

It's obvious that the |numerator| in the first 2 cases is greater than 1. In the last 2, $|\text{numerator}| \geq 1$ since $|x - y| \geq 1$ when x and y are integers, $x \neq y$.

Hence, $|a + br| \geq \frac{1}{q}$ if $|a|q \neq |b|p$

(if it is then $r = -\frac{a}{b}$)

(iv) from (ii) we have $\int_0^1 x^n e^x \, dx$

$= a_n + b_n e$

$\geq \frac{1}{q}$ from (iii), assuming $e = \frac{p}{q}$ (rational)

but from (i) we have

$\int_0^1 x^n e^x \, dx < \frac{3}{n+1} \to 0$ as $n \to \infty$.

Since 0 is not $\geq \frac{1}{q}$ (where q doesn't approach ∞)

e cannot be expressed as $\frac{p}{q}$ and is irrational.

(if $q \to \infty$ then $e \div 0$, which is not true)