HSC 2008 MATHEMATICS EXTENSION 2 (4 unit) EXAM : ANSWERS/SOLUTIONS.

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Question1

Q1.a Use substitution method.

I= $\displaystyle\int \frac{x^2}{(5+x^3)^2}dx$. Let $u = 5+x^3$. Then $du = 3x^2dx$. So, $I = \displaystyle\frac{1}{3}\int u^{-2}du = \frac{1}{3}-u^{-1}+c = \frac{-1}{3(5+x^3)}+c$

Q1.b Use integral table. I= $\displaystyle\int \frac{dx}{\sqrt{4x^2+1}} = \int \frac{1}{2}\frac{dx}{\sqrt{x^2+\left(\frac{1}{2}\right)^2}}dx$

$$= \frac{1}{2}\ln\left(x+\sqrt{x^2+0.25}\right)+c = \frac{1}{2}\ln\left(2x+\sqrt{4x^2+1}\right)+c'$$

Q1.c Use integration by parts. Recall $\int uv'dx = uv - \int u'vdx$. I $= \int_0^1 1.\tan^{-1}x\,dx$. So choose $u = \tan^{-1}x, v' = 1$. Then $u' = \dfrac{1}{1+x^2}, v = x$ and so,

I= $\displaystyle\left[x\tan^{-1}x\right]_0^1 - \int_0^1 \frac{x}{1+x^2}dx = \frac{\pi}{4} - \left[\frac{\ln(1+x^2)}{2}\right]_0^1 = \frac{\pi}{4} - \frac{\ln 2}{2}$

Q1.d Use subsitution, $u = \sqrt{2x-1}$, $du = dx/\sqrt{2x-1}$. I $= \displaystyle\int_1^2 \frac{dx}{x\sqrt{2x-1}} = \int \frac{du}{\frac{u^2+1}{2}} = 2\int \frac{du}{u^2+1} = 2\tan^{-1}u = \left[2\tan^{-1}\sqrt{2x-1}\right]_1^2 = 2\tan^{-1}\sqrt{3} - 2\tan^{-1}1 = 2\times\frac{\pi}{3} - 2\times\frac{\pi}{4} = \frac{\pi}{6}$.

Q1.e

$$\begin{aligned}
I &= \int_0^1 \frac{8(1-x)}{(2-x^2)(2-2x+x^2)}dx \\
&= \int_0^1 \frac{4-2x}{2-2x+x^2}dx + \int_0^1 \frac{-2x}{2-x^2}dx \text{ (by given result)} \\
&= -\int_0^1 \frac{2x-2}{2-2x+x^2}dx + \int_0^1 \frac{2}{(x-1)^2+1}dx + \left[\ln(2-x^2)\right]_0^1 \\
&\quad \text{since } x^2-2x+2 = (x-1)^2+1 \\
&= -\ln(x^2-2x+2)_0^1 + 2\tan^{-1}(x-1)_0^1 + \ln 1 - \ln 2 \\
&= -\ln 1 + \ln 2 + 2(0) - 2(-\pi/4) + \ln 1 - \ln 2 \\
&= \frac{\pi}{2}
\end{aligned}$$

Question2

Q2.a

$$\begin{aligned}(1-2i)(1-3i) &= a+ib\\ 1+2i-3i+6 &= a+ib\\ 7-i &= a+ib\\ a=7,\ & b=-1\end{aligned}$$

Q2.b

Q2.b.i

$$\frac{1+i\sqrt{3}}{1+i}\times\frac{1-i}{1-i}=\frac{1+i\sqrt{3}-i+\sqrt{3}}{2}=\frac{1+\sqrt{3}}{2}+i\left(\frac{\sqrt{3}-1}{2}\right)$$

Q2.b.ii

We use the notation $\operatorname{cis}\theta=\cos\theta+i\sin\theta$. $1+i\sqrt{3}=2\ \operatorname{cis}\frac{\pi}{3}$, $1+i=\sqrt{2}cis\frac{\pi}{4}$. Using mod-arg laws we get,

$$\frac{1+i\sqrt{3}}{1+i}=\frac{2\ \operatorname{cis}\frac{\pi}{3}}{\sqrt{2}\ \operatorname{cis}\frac{\pi}{4}}=\frac{2}{\sqrt{2}}\ \operatorname{cis}(\frac{\pi}{3}-\frac{\pi}{4})=\sqrt{2}\ \operatorname{cis}\frac{\pi}{12}$$

Q2.b.iii

Hence $\sqrt{2}\cos\frac{\pi}{12}+i\sqrt{2}\sin\frac{\pi}{12}=\frac{1+\sqrt{3}}{2}+i\left(\frac{\sqrt{3}-1}{2}\right)$.

Hence $\cos\frac{\pi}{12}=\frac{1+\sqrt{3}}{2\sqrt{2}}$

Q2.b.iv

Using part (ii) and De Moivre's Law we have

$$\left(\frac{1+i\sqrt{3}}{1+i}\right)^{12}=(\sqrt{2}\ \operatorname{cis}\frac{\pi}{12})^{12}=2^{0.5\times12}(\ \operatorname{cis}\frac{\pi}{12})^{12}=2^6\ \operatorname{cis}(\frac{\pi}{12}\times12)=64\ \operatorname{cis}\pi=-64$$

Q2.c Substitute $z=x+iy$ and we have $(x+iy)^2+(x-iy)^2=8$. Simplifying, $x^2-y^2+2xyi+x^2-y^2-2xyi=8$. Finally we get, $x^2-y^2=4$, a rectangular hyperbola.

Q2.d

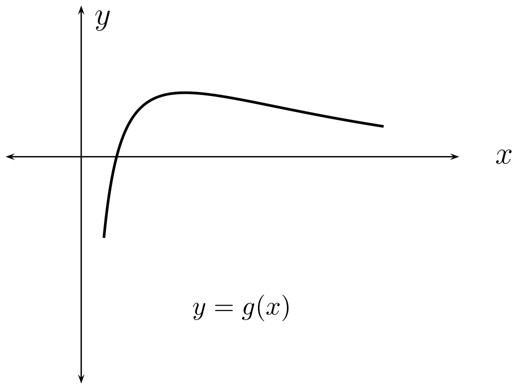
Q2.d.i The midpoint is $M=\frac{zw+z\bar{w}}{2}=\frac{z}{2}(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}+\cos\frac{2\pi}{3}-i\sin\frac{2\pi}{3})=\frac{z}{2}2\cos\frac{2\pi}{3}=-z\cos\frac{\pi}{3}=-\frac{z}{2}$ (since cos is negative in quadrant two)

Q2.d.ii Since the diagonals of any parallelogram bisect each other, then PS passes through M. So, $\vec{MS}=\vec{PO}+\vec{OM}=-z-z/2=-3z/2$.

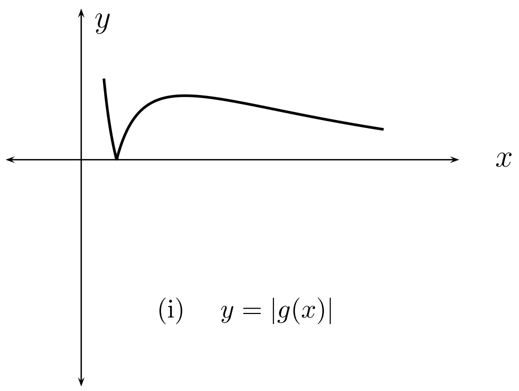
Hence, $\vec{OS}=\vec{OM}+\vec{MS}=-\frac{z}{2}+-\frac{3z}{2}=-2z$

Question3

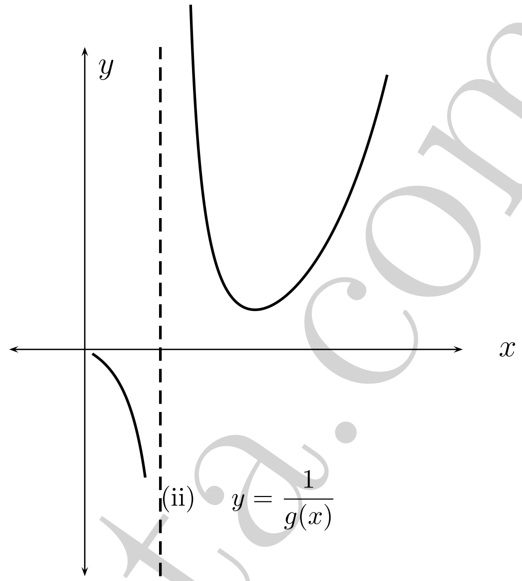
Q3.a



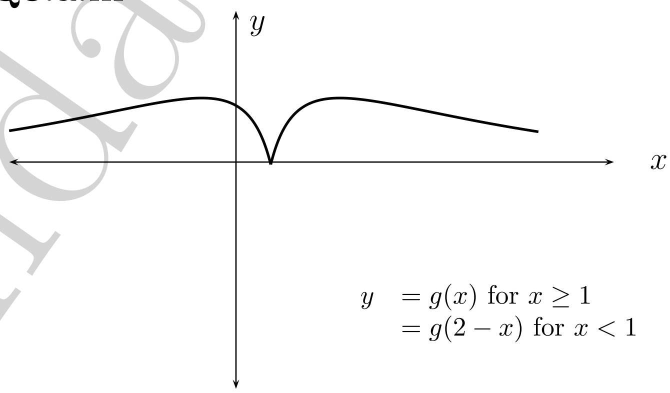
Q3.a.i



Q3.a.ii



Q3.a.iii



Q3.b

Q3.b.i

Method 1 Notice that if z were a real number then both z^2 and z^4 are non-negative and hence $p(z)$ cannot be zero. Hence there are no real zeros.

Method 2 Recall that $z^6 - 1 = (z^2)^3 - 1 = (z^2 - 1)(1 + z^2 + z^4)$.
Hence the roots of $1 + z^2 + z^4$ are those of $z^6 = 1$ that are not $z = \pm 1$. The 6th roots of unity are $\pm 1,\ \text{cis}\,\frac{\pi}{3},\ \text{cis}\,\frac{2\pi}{3},\ \text{cis}\,\frac{4\pi}{3},\ \text{cis}\,\frac{5\pi}{3}$
and so the zeros of $1 + z^2 + z^4$ are $z =\ \text{cis}\,\frac{\pi}{3},\ \text{cis}\,\frac{2\pi}{3},\ \text{cis}\,\frac{4\pi}{3},\ \text{cis}\,\frac{5\pi}{3}$, all four of which are non-real, so $p(z)$ has no real zeros.

Q3.b.ii

We showed this in part (i) solution already.

Q3.b.iii

$p(\alpha^2) = (\alpha^2)^4 + (\alpha^2)^2 + 1 = \alpha^8 + \alpha^4 + 1 = \alpha^2.\alpha^6 + \alpha^4 + 1 = \alpha^2 + \alpha^4 + 1 = 0$, since α is a zero of $p(z)$.

Q3.c

Q3.c.i

$$\begin{aligned} I_n &= \int_0^{\pi/4} \tan^{2n}\theta\,d\theta \\ &= \int_0^{\pi/4} \tan^{2n-2}\theta \times \tan^2\theta\,d\theta \\ &= \int_0^{\pi/4} \tan^{2n-2}\theta(\sec^2\theta - 1)\,d\theta \qquad \text{as required.} \\ &= \left[\frac{\tan^{2n-1}\theta}{2n-1}\right]_0^{\pi/4} - \int_0^{\pi/4} \tan^{2n-2}\theta\,d\theta \\ &= \frac{1}{2n-1} - I_{n-1} \end{aligned}$$

Q3.c.ii

$I_3 = 1/5 - I_2, I_2 = 1/3 - I_1, I_1 = \int_0^{\pi/4} \tan^2\theta d\theta = \int_0^{\pi/4} (\sec^2\theta - 1) d\theta = [\tan\theta - \theta]_0^{\pi/4} = 1 - \frac{\pi}{4}$

Hence $I_3 = 1/5 - 1/3 + (1 - \pi/4) = \frac{13}{15} - \frac{\pi}{4} = \frac{52 - 15\pi}{60}$

Q3.d

Resolve vertical forces: $T\cos\alpha = mg$

Resolve horizontal forces: $T\sin\alpha = m(l\sin\alpha)\omega^2$, and so $T = ml\omega^2$.

Hence $ml\omega^2\cos\alpha = mg$ and we have $\omega^2 = \frac{g}{l\cos\alpha}$.

Question4

Q4.a

Q4.a.i

Recall that the area of a triangle is half the base times the height!

Area $\Delta LOM = 1/2 \times k \times r = \frac{rk}{2}$

Q4.a.ii The perimeter of ΔKLM is $P = k + l + m$. Following the method of part (i) we have

Area $A = \frac{rk + rl + rm}{2} = \frac{r}{2}(k + l + m) = rP/2$ as required.

Q4.a.iii Let x be the distance from the foot of the board to the fence. Let h be the length of the board, which is the hypotenuse of a right angled triangle.

The perimeter is $P = 8 + x + h$.

Area $= 0.5 \times 8 \times x = 4x$. Hence $4x = x + h + 8$ and $h = 3x - 8$.

By Pythagoras $8^2 + x^2 = h^2 = (3x - 8)^2 = 9x^2 - 48x + 64$. Simplying gives, $8x^2 - 48 = 8x(x - 6) = 0$. Hence as x is non-zero, $x = 6$ units.

Q4.a.iv

The diagonal of the bigger triangle is $d = \sqrt{8^2 + 15^2} = 17$.

The area of the new smaller triangle is $0.5 \times (6+9) \times 8 - 0.5 \times 6 \times 8 = 36$.
Hence, $36 = \frac{1}{2}Pr = \frac{1}{2} \times r \times (9 + \sqrt{8^2+6^2} + 17)$. Hence $r = \dfrac{72}{9+10+17} = \dfrac{72}{36} = 2$.

Q4.b

Q4.b.i Rewrite as, $y^2 = b^2(1 - x^2/a^2)$. Differentiating with respect to x, $2yy' = b^2(-2x/a^2)$ which gives the derivative $y' = \dfrac{-b^2x}{a^2y}$. Evaluating this at (x_1, y_1) gives the tangent gradient as $m = \dfrac{-b^2x_1}{a^2y_1}$. The line through (x_1, y_1) with gradient m is

$y - y_1 = \frac{-b^2x_1}{a^2y_1}(x - x_1)$

$\dfrac{y_1y}{b^2} - \dfrac{y_1^2}{b^2} = -\dfrac{x_1x}{a^2} + \dfrac{x_1^2}{a^2}$

$\dfrac{x_1x}{a^2} + \dfrac{y_1y}{b^2} = \dfrac{x_1^2}{a^2} + \dfrac{y_1^2}{b^2}$

$\dfrac{x_1x}{a^2} + \dfrac{y_1y}{b^2} = 1$ since (x_1, y_1) lies on the ellipse

Q4.b.ii The tangent at Q is $\dfrac{x_2x}{a^2} + \dfrac{y_2y}{b^2} = 1$.
Subtracting the two equations,
$\dfrac{x_1x}{a^2} + \dfrac{y_1y}{b^2} = 1$
$\dfrac{x_2x}{a^2} + \dfrac{y_2y}{b^2} = 1$ gives

$$\frac{(x_1 - x_2)x}{a^2} + \frac{(y_1 - y_2)y}{b^2} = 1 \tag{1}$$

as required.

Q4.b.iii

O and T both lie on the line $\dfrac{(x_1 - x_2)x}{a^2} + \dfrac{(y_1 - y_2)y}{b^2} = 1$.
The midpoint of PQ is $M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.
Substituting M into the LHS of the equation (1) we have

$$\begin{aligned} LHS &= \frac{(x_1 - x_2)}{a^2}\frac{(x_1 + x_2)}{2} + \frac{(y_1 - y_2)}{b^2}\frac{(y_1 + y_2)}{2} \\ &= \frac{x_1^2 - x_2^2}{2a^2} + \frac{y_1^2 - y_2^2}{2b^2} \\ &= \frac{1}{2}\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}\right) - \left(\frac{x_2^2}{a^2} + \frac{y_2^2}{b^2}\right) \\ &= \frac{1}{2} \times 1 - \frac{1}{2} \times 1 \\ &= 0 \\ &= RHS \end{aligned}$$

Hence, M also lies on the line (1). Therefore as O, T, M are all on the same line, they are collinear.

Question5

Q5.a We have $P = \dfrac{21000}{7+3e^{-t/3}} = 21000\left(7+3e^{-t/3}\right)^{-1}$.

Q5.a.i Differentiating we have

$$\begin{aligned}\frac{dP}{dt} &= -21000(7+3e^{-t/3})^{-2}\times(-e^{-t/3})\\ &= \frac{21000e^{-t/3}}{(7+3e^{-t/3})^2}\\ &= 7000\frac{3e^{-t/3}}{(7+3e^{-t/3})^2}\\ &= 7000\left(\frac{3e^{-t/3}+7}{(7+3e^{-t/3})^2}-\frac{7}{(7+3e^{-t/3})^2}\right)\\ &= 7000\left(\frac{P}{21000}-\frac{7P^2}{21000\times 21000}\right)\\ &= \frac{P}{3}-\frac{P^2}{3000\times 3}\\ &= \frac{1}{3}\left(1-\frac{P}{3000}\right)P\end{aligned}$$

as required.

Q5.a.ii

$P(0) = \dfrac{21000}{7+3e^0} = 2100$

Q5.a.iii

As $t\to\infty$, $e^{-t/3}\to 0$. Thus $P \to \dfrac{21000}{7+0^+}\to 3000$.

Q5.a.iv

Today, $\dfrac{dP}{dt} = \dfrac{1}{3}\left(1-\dfrac{2100}{3000}\right)2100 = 210$ elephants per year

Total elephants in the year was 2100, thus as a percentage this is $\dfrac{210}{2100}\times 100\% = 10\%$

Q5.b $p(x) = x^{n+1}-(n+1)x+n$ where $n\in\mathbb{Z}$.

Q5.b.i The derivate is $p'(x) = (n+1)x^n-(n+1)$. When $x=1$, $p(1) = 1^{n+1}-(n+1).1+n = 1-n-1+n = 0$, and $p'(1) = (n+1).1^n-(n+1) = 0$. Hence, $p(1) = p'(1) = 0 \Longrightarrow p(x)$ has a double root at $x=1$, as required.

Q5.b.ii

$p''(x) = n(n+1)x^{n-1}$ and so $p''(x)\geq 0$ for $x\geq 0$ and $n\geq 1$.

The stationary points occur when $p'(x)=0$, so $(n+1)x^n-(n+1)=0 \to x^n = 1 \to x = 1 (x\geq 0)$. And $p''(1) = n(n+1)>0$ and hence the turning point at $(1,0)$ is a minimum. Since the curve is always concave up then $p(x)\geq 0$ for all $x\geq 0$.

Q5.b.iii

When $n=3$, $p(x) = x^4-4x+3$. But $(x-1)^2 = x^2-2x+1$ and so by inspection (or polynomial division) we have,

$$x^4 - 4x + 3 = (x^2 - 2x + 1)(x^2 + 2x + 3)$$

The zeros of $x^2 + 2x + 3$ are $x = \dfrac{-2 \pm \sqrt{4 - 4(3)(1)}}{2} = \dfrac{-2 \pm 2i\sqrt{2}}{2} = -1 \pm i\sqrt{2}$
Therefore,
$p(x) = (x-1)^2(x^2 + 2x + 3)$ over $\mathbb{R}$
$p(x) = (x-1)^2(x + 1 - i\sqrt{2})(x + 1 + i\sqrt{2})$ over $\mathbb{C}$.

Q5.c

Q5.c.i

$$\begin{aligned}(x-a)^2 &= b^2 - h^2\\ x - a &= \pm\sqrt{b^2 - h^2}\\ x &= a \pm \sqrt{b^2 - h^2}\end{aligned}$$

and denoting the smaller and larger solutions as x_1, x_2 resp. we have

$$\begin{aligned}x_1 &= a - \sqrt{b^2 - h^2}\\ x_2 &= a + \sqrt{b^2 - h^2}\end{aligned}$$

Q5.c.ii

The area of an annulus whose inner and outer radii are r, R resp. is $\pi(R^2 - r^2)$.
Hence, $A = \pi(x_2^2 - x_1^2)$
$A = \pi(a + \sqrt{b^2 - h^2})^2 - \pi(a - \sqrt{b^2 - h^2})^2$
$A = \pi(a^2 + 2a\sqrt{b^2 - h^2} + b^2 - h^2 - a^2 - (b^2 - h^2) + 2a\sqrt{b^2 - h^2})$
$A = 4\pi a\sqrt{b^2 - h^2}$.

Q5.c.iii

$$\begin{aligned}V &= 2\int_0^b 4\pi a\sqrt{b^2 - h^2}dh\\ &= 8\pi a\int_0^b \sqrt{b^2 - h^2}dh\\ &= 8\pi a\int b\cos\theta b\cos\theta d\theta \quad \text{using the substn } h = b\sin\theta,\ dh = b\cos\theta d\theta\\ &= 8\pi ab^2\int \cos^2\theta d\theta\\ &= 8\pi ab^2\int_0^{\pi/2}\frac{1+\cos 2\theta}{2}d\theta \text{ where when } h = b, \theta = \pi/2, \text{ and when } h = 0, \theta = 0\\ &= 8\pi ab^2\left[\theta/2 + \frac{\sin 2\theta}{4}\right]_0^{\pi/2}\\ &= 8\pi ab^2(\pi/4 + 0 - 0 - 0)\\ &= 2\pi^2 ab^2\end{aligned}$$

Question6

Q6.a

Now, $|\omega|^2 = \omega\bar{\omega}$, and since $\omega^3 = 1$, then $|\omega| = 1$ and , so $\omega\bar{\omega} = 1$ so $\omega^2 = 1/\omega = \bar{\omega}$.
Also $0 = \omega^3 - 1 = (\omega - 1)(1 + \omega + \omega^2) = (\omega - 1)(1 + \omega + \bar{\omega})$ Hence $\omega + \bar{\omega} = -1$ since $Im(\omega) > 0$ (and so $\omega \neq 1$).

Consider the sum and product of the roots, we have
$\Sigma\alpha = -B/A = -a$
$\Sigma\alpha\beta = C/A = b$
$\alpha\beta\gamma = -D/A = -c$
Hence for our roots $1, -\omega, -\bar{\omega}$ we have
$a = -1 + \omega + \bar{\omega} = -2$
$b = -\omega - \bar{\omega} + \omega\bar{\omega} = 1 - \omega - \bar{\omega} = 2$
$c = -1 \times -\omega \times -\bar{\omega} = -\omega\bar{\omega} = -1.$
Hence $P(z) = z^3 - 2z^2 + 2z - 1$

Q6.b

Q6.b.i

Tangent at P:
Differentiate to get the gradient of line l, $\dfrac{2x}{a^2} = \dfrac{2yy'}{b^2}$. Hence, $y' = \dfrac{b^2x}{a^2y}$ So the gradient is $m = y'|_{\text{at}P} = \dfrac{b\sec\theta}{a\tan\theta}$.
The tangent line at P has equation
$y - b\tan\theta = \dfrac{b\sec\theta}{a\tan\theta}(x - a\sec\theta)$
$ya\tan\theta - ba\tan^2\theta = bx\sec\theta - ab\sec^2\theta$
$bx\sec\theta - ay\tan\theta = ab(\sec^2\theta - \tan^2\theta)$
$bx\sec\theta - ay\tan\theta - ab = 0$ since $\tan^2\theta + 1 = \sec^2\theta$ and so $\sec^2\theta - \tan^2\theta = 1$.

Q6.b.ii

$$\begin{aligned} SR &= \left|\frac{b\sec\theta.(ae) - a\tan\theta \times 0 - ab}{\sqrt{b^2\sec^2\theta + a^2\tan^2\theta}}\right| \\ &= \frac{|ab(e\sec\theta - 1)|}{\sqrt{b^2\sec^2\theta + a^2\tan^2\theta}} \\ &= \frac{ab(e\sec\theta - 1)}{\sqrt{b^2\sec^2\theta + a^2\tan^2\theta}} \end{aligned}$$

since for a hyperbola $e > 1$ and also, $\sec\theta \geq 1$ for all θ, and therefore $e\sec\theta - 1 \geq 0$. That's why we dropped the absolute value sign.

Q6.b.iii

$$\begin{aligned} SR \times S'R' &= \frac{ab(e\sec\theta - 1)}{\sqrt{b^2\sec^2\theta + a^2\tan^2\theta}} \times \frac{ab(e\sec\theta + 1)}{\sqrt{b^2\sec^2\theta + a^2\tan^2\theta}} \\ &= \frac{a^2b^2(e^2\sec^2\theta - 1)}{a^2\tan^2\theta + b^2\sec^2\theta} \\ &= \frac{a^2\,a^2(e^2-1)\,(e^2\sec^2\theta - 1)}{a^2\tan^2\theta + a^2(e^2-1)\,\sec^2\theta} \text{ since } b^2 = a^2(e^2-1) \text{ for a hyperbola} \\ &= \frac{a^4(e^2-1)\,(e^2\sec^2\theta - 1)}{-a^2 + a^2e^2\sec^2\theta} \\ &= a^2(e^2-1) \\ &= b^2 \end{aligned}$$

as required.

Q6.c

Q6.c.i

RTP: $\dfrac{1}{\binom{n}{r}} = \dfrac{r}{r-1}\left[\dfrac{1}{\binom{n-1}{r-1}} - \dfrac{1}{\binom{n}{r-1}}\right]$

Recall $\dbinom{n}{r} = \dfrac{n!}{r!(n-r)!}$

$$\begin{aligned}
LHS &= \frac{1}{\binom{n}{r}} = \frac{r!(n-r)!}{n!}
\end{aligned}$$

$$\begin{aligned}
RHS &= \frac{r}{r-1}\left[\frac{(r-1)!(n-1-r+1)!}{(n-1)!} - \frac{(r-1)!(n-r+1)!}{n!}\right] \\
&= \frac{r}{r-1}\left[\frac{n(r-1)!(n-r)! - (r-1)!(n-r+1)!}{n!}\right] \\
&= \frac{r}{r-1} \times \frac{(r-1)!(n-r)!}{n!}\,(n-(n-r+1)) \\
&= \frac{1}{r-1} \times \frac{r!(n-r)!}{n!}\,(r-1) \\
&= \frac{r!(n-r)!}{n!} = LHS
\end{aligned}$$

as required.

Q6.c.ii

$$\begin{aligned}
LHS &= \frac{1}{\binom{r}{r}} + \frac{1}{\binom{r+1}{r}} + \cdots + \frac{1}{\binom{m}{r}} \\
&= \frac{r}{r-1}\left[\frac{1}{\binom{r-1}{r-1}} - \frac{1}{\binom{r}{r-1}} + \frac{1}{\binom{r}{r-1}} - \frac{1}{\binom{r+1}{r-1}}\right. \\
&\quad \left. + \frac{1}{\binom{r+1}{r-1}} - \frac{1}{\binom{r+2}{r-1}} + \cdots + \frac{1}{\binom{m-1}{r}} - \frac{1}{\binom{m}{r-1}}\right] \\
&= \frac{r}{r-1}\left(1 - \frac{1}{\binom{m}{r-1}}\right) = RHS
\end{aligned}$$

as required **Q6.c.iii**

As $m \to \infty$, $\binom{m}{r-1} \to \infty$, so $\dfrac{1}{\binom{m}{r-1}} \to 0$.

Hence the limiting value as m increases without bound is $\dfrac{r}{r-1}$

Question7

Q7.a The sample space consists of all the combinations of three balls, so it has size $\binom{3n}{3}$.

Q7.a.i

$p_s = 3 \times \frac{\binom{n}{3}}{\binom{3n}{3}} = 3 \times \frac{n!3!(3n-3)!}{3!(n-3)!(3n)!} = 3 \times \frac{n(n-1)(n-2)}{3n(3n-1)(3n-2)} = \frac{(n-1)(n-2)}{(3n-1)(3n-2)}$

Q7.a.ii

$p_d = \frac{\binom{n}{1} \times \binom{n}{1} \times \binom{n}{1}}{\binom{3n}{3}} = \frac{n^3 3!(3n-3)!}{(3n)!} = \frac{6n^3}{3n(3n-1)(3n-2)} = \frac{2n^2}{(3n-1)(3n-2)}$

Q7.a.iii

$p_m = 6 \times \frac{\binom{n}{2} \times \binom{n}{1}}{\binom{3n}{3}} = 6 \times \frac{n(n-1)n.3.2.1}{2.3n(3n-1)(3n-2)} = 6 \times \frac{n(n-1)}{(3n-1)(3n-2)}$

Q7.a.iv

$p_s : p_d : p_m = \frac{(n-1)(n-2)}{(3n-1)(3n-2)} : \frac{2n^2}{(3n-1)(3n-2)} : 6 \times \frac{n(n-1)}{(3n-1)(3n-2)}$

$= (n-1)(n-2) : 2n^2 : 6n(n-1)$

$= (1-\frac{1}{n})(1-\frac{2}{n}) : 2 : 6(1-\frac{1}{n})$by dividing each ratio by n^2 and for large n this equals approximately $1 : 2 : 6$ as required.

Q7.b

Q7.b.i

$\angle RPT = \angle PQT$

(since angle between tangent and chord is equal to angle opposite segment)

$\therefore \angle T = \pi - 2\angle RPT - 2\theta$

(angle sum in Δ equals π radians)

$\angle TSP = \pi - \angle T - \theta - \angle RPT$

(angle sum in Δ equals π radians)

$= \pi - \pi + 2\angle RPT + 2\theta - \theta - \angle RPT$

$= \angle RPT + \theta$

$\angle TSP = \angle TPS$

as required.

Q7.b.ii

$ST = c$ since ΔSTP is isosceles as two base angles are equal.

By the intersecting chords distance product rule, we have $QT \times RT = PT^2$. Hence $(b+c) \times (c-a) = c^2$

$\therefore bc + c^2 - ab - ac = c^2$

$\therefore bc = ab + ac$

$\therefore \frac{1}{a} = \frac{1}{b} + \frac{1}{c}$ as required. (By dividing thoughout by abc.)

Q7.c

$$v = b - (b - v_0)e^{-\alpha t} \tag{2}$$

Q7.c.i

$LHS = \dfrac{dv}{dt} = \alpha(b - v_0)e^{-\alpha t} = \alpha(b - v) = RHS$, as required.

Q7.c.ii

b represents the velocity of the water.

Q7.c.iii

Firstly, $x(0) = 0$ is given. Also we manipulate the equation (2) to get $e^{-\alpha t}$ and t as the subjects.

$e^{-\alpha t} = \dfrac{b - v}{b - v_0}$

$\therefore -\alpha t = \ln\left(\dfrac{b - v}{b - v_0}\right)$

$\therefore t = -\dfrac{1}{\alpha}\ln\left(\dfrac{b - v}{b - v_0}\right)$

Using $v = \dfrac{dx}{dt}$ in equation (2) we have,

$$\begin{aligned}
\frac{dx}{dt} &= b - (b - v_0)e^{-\alpha t} \\
x &= \textstyle\int b\,dt - (b - v_0)\int e^{-\alpha t}dt \\
x &= bt + \frac{b - v_0}{\alpha}e^{-\alpha t} + c \\
\text{but } 0 &= 0 + \frac{b - v_0}{\alpha} + c \\
\therefore c &= \frac{v_0 - b}{\alpha} \\
\therefore x &= bt + \frac{b - v_0}{\alpha}e^{-\alpha t} + \frac{v_0 - b}{\alpha}
\end{aligned}$$

now substituting for t and $e^{-\alpha t}$ in terms of v etc.

$$\begin{aligned}
x &= \frac{-b}{\alpha}\ln\left(\frac{b - v}{b - v_0}\right) + \frac{(b - v)}{\alpha} + \frac{(v_0 - b)}{\alpha} \\
x &= \frac{b}{\alpha}\ln\left(\frac{b - v_0}{b - v}\right) + \frac{(v_0 - v)}{\alpha}
\end{aligned}$$

as required.

Q7.c.iv

On substitution of $v_0 = b/10$ and $v = b/2$ we obtain,

$$\begin{aligned}
x &= \frac{b}{\alpha}\ln\left(\frac{b - b/10}{b - b/2}\right) + \frac{(b/10 - b/2)}{\alpha} \\
x &= \frac{b}{\alpha}\ln\left(\frac{9}{5}\right) - \frac{2b}{5\alpha} \\
x &= \frac{b}{\alpha}(\ln 1.8 - 0.4)
\end{aligned}$$

Question8

Q8.a

Step 1 For $n = 1$ $lhs = \cos\theta$.
$rhs = \dfrac{\sin 2\theta}{2\sin\theta} = \dfrac{2\sin\theta\cos\theta}{2\sin\theta} = \cos\theta$. So its true when $n = 1$.

Step 2 Assume the result is true when $n = k$, so assume

$$\cos\theta + \cos 3\theta + \cdots + \cos(2k-1)\theta = \frac{\sin 2k\theta}{2\sin\theta} \tag{3}$$

Step 3 We must prove it true for the case $n = k + 1$ under the assumption (3), so we must prove

$$\cos\theta + \cos 3\theta + \cdots + + \cos(2k-1)\theta + \cos(2k+1)\theta = \frac{\sin(2k+2)\theta}{2\sin\theta} \tag{4}$$

We try subtracting the right hand sides of the equations (3, 4) and show that the difference is $\cos(2k\theta + \theta)$ (as by inspection of the left hand sides of those equations)

$$\begin{aligned} RHS \text{ of } (\ 4) - RHS \text{ of } (\ 3) &= \frac{\sin(2k+2)\theta}{2\sin\theta} - \frac{\sin 2k\theta}{2\sin\theta} \\ &= \frac{\sin(2k\theta+\theta+\theta) - \sin(2k\theta+\theta-\theta)}{2\sin\theta} \\ &= \frac{2\cos(2k\theta+\theta)\sin\theta}{2\sin\theta} \quad \text{by the result given in line one of the question} \\ &= \cos(2k\theta+\theta) \end{aligned}$$

which is the required difference and hence we have shown the result true for $n = k + 1$.

Step 4 Finally, we have shown that the result holds true when $n = k + 1$ whenever it is true for $n = k$. We also showed it true when $n = 1$. Hence inductively it is true for all integers $n \geq 1$.

Q8.b

Q8.b.i

$$
\begin{aligned}
A &= \sum_{k=1}^{n} A_k \\
&= 2\pi R^2 \sin\delta \sum_{k=1}^{n} \cos\frac{(2k-1)\delta}{2} \\
&= 2\pi R^2 \sin\delta \left(\cos\frac{\delta}{2} + \cos 3\frac{\delta}{2} + \cdots \cos(2n-1)\frac{\delta}{2}\right) \\
&= 2\pi R^2 \sin\delta \times \frac{\sin 2n\frac{\delta}{2}}{2\sin\frac{\delta}{2}} \text{ using part (a)} \\
&= 2\pi R^2 \sin\delta \times \frac{\sin n\delta}{2\sin\frac{\delta}{2}} \\
&= 2\pi R^2 2\sin\frac{\delta}{2}\cos\frac{\delta}{2} \times \frac{\sin\frac{\pi}{2}}{2\sin\frac{\delta}{2}} \\
&= 2\pi R^2 \cos\frac{\delta}{2} \\
A &= 2\pi R^2 \cos\frac{\pi}{4n}
\end{aligned}
$$

where we twice used the fact that $n\delta = \pi/2$.

Q8.b.ii

$\lim_{n\to\infty} A = 2\pi R^2 \cos 0 = 2\pi R^2.$

Q8.c

Q8.c.i We are given the function $f(t) = \sin(a+nt)\sin b - \sin a \sin(b-nt)$

$a+b<\pi, \quad a,b>0. \quad n\neq 0$

The $f(0) = \sin(a)\sin b - \sin a\sin(b) = 0$ as required.

Finding the derivatives we have

$$
\begin{aligned}
f'(t) &= n\cos(a+nt)\sin b + n\sin a\cos(b-nt) \\
f''(t) &= -n^2\sin(a+nt)\sin b + n^2\sin a\sin(b-nt) \\
\\
&= -n^2\left(\sin(a+nt)\sin b - \sin a\sin(b-nt)\right) \\
&= -n^2 f(t)
\end{aligned}
$$

as required.

Q8.c.ii

Method 1: expand both sides and show they are equal.

$RHS = \sin(a+b)\sin nt = \sin a\cos b\sin nt + \sin b\cos a\sin nt$

$LHS = \sin b(\sin a\cos nt + \sin nt\cos a) - \sin a(\sin b\cos nt - \sin a\sin nt)$

$= \sin b\cos a\sin nt + \sin a\cos b\sin nt$

$= RHS$

as required.

Method 2: Use the solution for SHM with phase 0.

Q8.c.iii

Manipulating this expression we have $\sin(a+nt)\sin b - \sin a \sin(b-nt) = 0$ and hence we solve $f(t) = 0$.
So we solve $\sin(a+b)\sin nt = 0$ (by part (ii)).
But since $0 < a+b < \pi$ then we know that $\sin(a+b) \neq 0$, and hence we may cancel it to get
$\sin nt = 0$ which has solution $nt = k\pi$ where $k \in \mathbb{Z}$. Hence $t = k\pi/n$ where $k \in \mathbb{Z}$.

The End