(a) $\int \sec^3 x \tan x \, dx$ let $u = \sec x$

$= \frac{\sec^3 x}{3} + C$ $du = \sec x \tan x \, dx$

(b) $\int \frac{dx}{x^2+2x+2}$ $x^2+2x+2 = (x+1)^2+1$

$= \ln\left((x+1) + \sqrt{(x+1)^2+1}\right)$

(c) let $\frac{x}{(x+3)(x-1)} \equiv \frac{a}{x+3} + \frac{b}{x-1}$

$x \equiv a(x-1) + b(x+3)$

$\equiv ax - a + bx + 3b$

$3b - a = 0 \Rightarrow a = 3b$

$a + b = 1 \Rightarrow b = 1/4, a = 3/4$

$\int \frac{x\,dx}{(x+3)(x-1)} = \frac{3}{4}\ln(x+3) + \frac{1}{4}\ln(x-1) + C$

or $\frac{1}{4}\ln(x-1)(x+3)^3 + C$

(d) $\int_0^{\pi/2} e^x \cos x \, dx$ let $u = \cos x$ $\frac{dv}{dx} = e^x$

$\frac{du}{dx} = -\sin x$ $v = e^x$

$= [e^x \cos x]_0^{\pi/2} - \int (e^x)(-\sin x)\,dx$

$= -1 + [e^x \sin x]_0^{\pi/2} - \int e^x \cos x \, dx$ let $u_1 = \sin x$ $\frac{dv_1}{dx} = e^x$

$\frac{du_1}{dx} = \cos x$ $v_1 = e^x$

$= -1 + e^{\pi/2} - \int e^x \cos x \, dx$

$\therefore \int_0^{\pi/2} e^x \cos x \, dx = \frac{1}{2}\left(e^{\pi/2} - 1\right)$

(e) $\int_0^{\pi/2} \frac{d\theta}{2+\cos\theta}$ let $t = \tan\frac{\theta}{2}$

$\cos\theta = \frac{1-t^2}{1+t^2}$

$= \int_0^1 \frac{1}{2 + \frac{1-t^2}{1+t^2}} \times \frac{2}{1+t^2}\,dt$

$\theta = 2\tan^{-1} t$

$d\theta = \ldots dt$

$= \int_0^1 \frac{2}{3+t^2}$

$\theta = \pi/2, t = 1$

$\theta = 0, t = 0$

$= \left[\frac{2}{\sqrt{3}} \tan^{-1}\frac{t}{\sqrt{3}}\right]_0^1$

$2(1+t^2) + 1 - t^2 = 3 + t^2$

$= \frac{2}{\sqrt{3}} \times \frac{\pi}{6} = \frac{1}{3\sqrt{3}}\pi$

(a) $z = 1 + 2i$, $w = 1 + i$.

(i) $z\bar{w} = (1+2i)(1-i)$
$= 1 - i + 2i + 2$ since $i^2 = -1$
$= 3 + i$

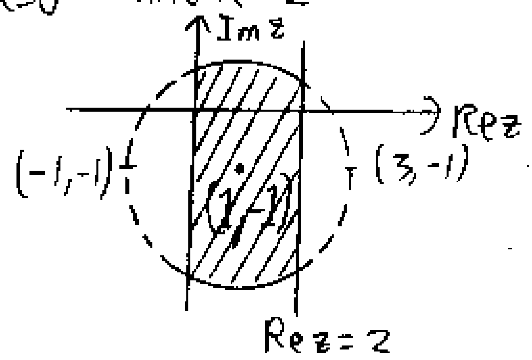
(ii) $\frac{1}{w} = \frac{1}{1+i} \times \frac{1-i}{1-i}$
$= \frac{1-i}{1+1} = \frac{1}{2} - \frac{1}{2}i$

(b) $0 \leq \text{Re}\, z \leq 2$

line $x=0$, line $x=2$

$|z - (1-i)| \leq 2$

circle, radius 2, centre $(1,-1)$



In most cases you don't need to give coordinates of points of intersection.

(c) (i) $2-i$ is a conjugate of $2+i$ and the coefficients of $P(x)$ are all real numbers.

(ii) $P(z) = (z-(2+i))(z-(2-i))(z-a)$
$= [z^2 - (2-i)z - (2+i)z + (2+i)(2-i)](z-a)$
$= [z^2 - 4z + 5](z-a)$

equating constants, $-5a = 20$
$a = -4$.

$P(z) = (z^2 - 4z + 5)(z+4)$

Alternative approach: $\alpha\beta\gamma = -20$
$\gamma = -4$.

then $\alpha + \beta + \gamma = -r \Rightarrow r = 0$
$\alpha\beta + \alpha\gamma + \beta\gamma = s \Rightarrow s = -11$

$P(z) = z^3 - 11z + 20$ then you can divide it by $(z+4)$.

Question 2

(d) let $S(n)$ be the statement
$$(\cos\theta - i\sin\theta)^n = \cos(n\theta) - i\sin(n\theta).$$

clearly, $S(1)$ is true.

If $S(k)$ is true, $S(k+1)$:
$$(\cos\theta - i\sin\theta)^{k+1} = (\cos\theta - i\sin\theta)^k(\cos\theta - i\sin\theta)$$
$$= (\cos(k\theta) - i\sin(k\theta))(\cos\theta - i\sin\theta)$$
$$= \cos\theta\cos(k\theta) - i\sin\theta\cos(k\theta) - i\cos\theta\sin(k\theta) - \sin\theta\sin(k\theta)$$
$$= \cos(\theta + k\theta) - i\sin(\theta + k\theta)$$ using compound angle formulae
$$= \cos[(k+1)\theta] - i\sin[(k+1)\theta]$$

$\therefore$ $S(k)$ true implies $S(k+1)$ true.

But $S(1)$ is true.

Hence $S(n)$ is true for $n \geq 1$, n integers

(e) (i) $1 - z = 1 - 2\cos\theta - 2i\sin\theta$

$\overline{1-z} = 1 - 2\cos\theta + 2i\sin\theta$

(ii) $$\frac{1}{1-z} = \frac{1}{1-z} \times \frac{\overline{1-z}}{\overline{1-z}}$$

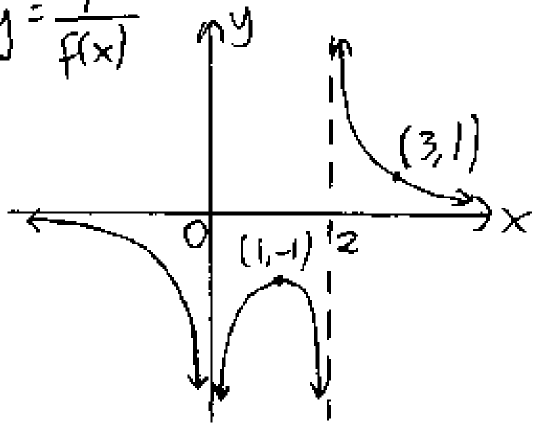
$$\mathrm{Re}\left(\frac{1}{1-z}\right) = \frac{1}{(1-2\cos\theta)^2 + (2\sin\theta)^2} \times (1 - 2\cos\theta)$$

$$= \frac{1-2\cos\theta}{1 - 4\cos\theta + (4\cos^2\theta + 4\sin^2\theta)} = \frac{1-2\cos\theta}{5-4\cos\theta}$$

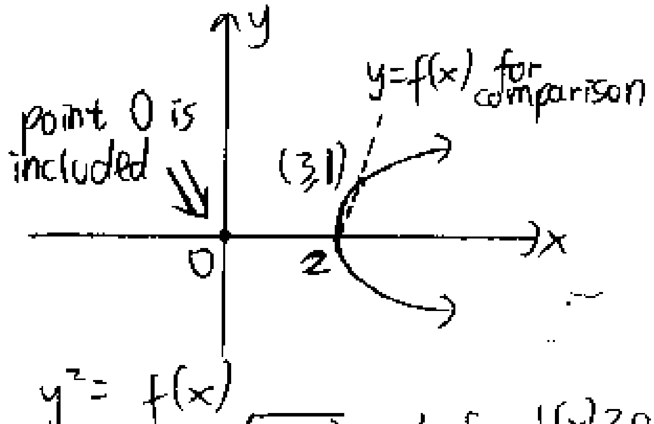
(iii) $$\mathrm{Im}\left(\frac{1}{1-z}\right) = \frac{1}{5-4\cos\theta} \times 2\sin\theta = \frac{2\sin\theta}{5-4\cos\theta}$$

Question 3

(a)(i) $y = \dfrac{1}{f(x)}$



(ii)

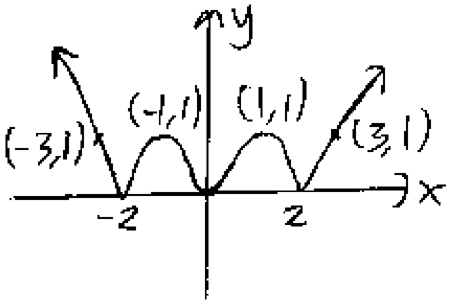


$y^2 = f(x)$

$y = \pm\sqrt{f(x)}$, only for $f(x) \geq 0$

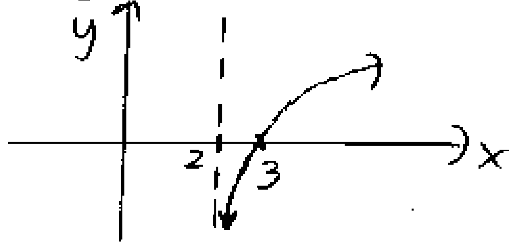
Question 3 2002 HSC ME2 p4

(a)(iii) $y = |f(|x|)|$



(iv) $y = \ln(f(x))$

only defined when $f(x) > 0$.



(b)(i) $y = \frac{c^2}{x} \Rightarrow \frac{dy}{dx} = c^2 \times -\frac{1}{x^2}$

at P, $\frac{dy}{dx} = \frac{-c^2}{c^2p^2} = \frac{-1}{p^2}$

tangent: $y - y_1 = m(x - x_1)$

$y - \frac{c}{p} = -\frac{1}{p^2}(x - cp)$

$p^2y - cp + x - cp = 0$

$x + p^2y = 2cp$ — ①

(ii) tangent from Q: $x + q^2y = 2cq$ — ②

solving ① and ②: $(2cp - p^2y) + q^2y = 2cq$

$y = \frac{2c(q-p)}{q^2 - p^2}$

$= \frac{2c}{q+p}$ using $q^2 - p^2 = (q-p)(q+p)$

$x = 2cp - p^2y$

$= 2cp - \frac{2p^2c}{p+q} = \frac{2cp^2 + 2cpq - 2cp^2}{p+q} = \frac{2cpq}{p+q}$

(iii) point $(cq, 0)$ satisfies ①:

$cq + p^2(0) = 2cp \Rightarrow q = 2p$.

Hence point T: $x_T = \frac{2cpq}{p+q} = \frac{4cp^2}{3p}$

$y_T = \frac{2c}{p+q} = \frac{2c}{3p} \Rightarrow p = \frac{2c}{3y}$

locus of T: $x = \frac{4cp^2}{3p} = \frac{4c}{\frac{2c}{y}} \times \frac{4c^2}{9y^2} = \frac{8c^2}{9y}$

$xy = \frac{8}{9}c^2$, eccentricity $= \sqrt{2}$ (rectangular)

Question 4

(a) (i) let $3-x^2 = x+x^2$, $x \geq 0$.

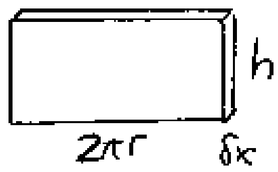
$$2x^2+x-3=0$$

$$x = \frac{-1 \pm \sqrt{1+4(2)(3)}}{2(2)} = \frac{-1\pm 5}{4}$$

$x = 1$ for $x \geq 0$. | $P(1,2)$

$y = 3-x^2 = 2$.

(ii)



$$h = (3-x^2)-(x+x^2) = 3-x-2x^2$$

$$r = x+1$$

$$V = \sum_{x=-1}^{1} 2\pi(x+1)(3-x-x^2)\,\delta x$$

$$= 2\pi\int_{-1}^{1}(-x^3-2x^2+2x+3)\,dx$$

(iii) $V = 2\pi\left[-\frac{x^4}{4}-\frac{2x^3}{3}+\frac{2x^2}{2}+3x\right]_{-1}^{1}$

$$= 2\pi\left(6-\frac{4}{3}\right) = \frac{28}{3}\pi \text{ units}^3$$

(b)(i) $\angle DSA = \pi - \angle DRA \ (=\pi/2)$

$\therefore$ DSAR cyclic quadrilateral.

$\therefore \angle DSR = \angle DAR$ (subtend same chord DR)

(ii) $\angle DSC = \angle DTC \ (=\pi/2)$

$\therefore$ CTSD is a cyclic quadrilateral (equal $\angle$'s on CD)

$\therefore \angle DST = \pi - \angle DCT$ (opposite angles in cyclic quadrilateral)

(iii) Let $\angle DCT = \theta$

$= \angle DAR$ (exterior $\angle$ of cyclic quad ABCD)

$= \angle DSR$ (from (i))

but $\angle DST = \pi - \theta$ (from (ii))

Hence $\angle DST + \angle DSR = \pi$

R, S, T are collinear.

(c) (i) This happens when the first digit ≥ 4.

There are 6 favourable first digits from the 9 digits available.

$P = 6/9 = 2/3$.

(ii) fav. outcomes $= {}^9C_3$ (pick the digits in any order, then order them ourselves)

$P = \frac{{}^9C_3}{{}^9P_3} = \frac{1}{6}$. Alternatively, since 3 cards can be arranged in $3!$ ways, $P(\text{it's descending}) = \frac{1}{3!}$

Question 5

(a) $4x^3-27x+k=0$ has roots α, α, β.

sum of roots: $2\alpha+\beta=0 \Rightarrow \beta=-2\alpha$.

$\alpha\alpha+\alpha\beta+\alpha\beta=\alpha^2-2\alpha^2-2\alpha^2$ (sum of products of roots taken 2 at a time)

$= -3\alpha^2 = -\frac{27}{4}$

$\alpha^2=\frac{27}{12} \Rightarrow \alpha=\pm\sqrt{\frac{27}{12}}=\pm 3/2$ *

$-\frac{k}{4}=\alpha\alpha\beta=-2\alpha^3$

$\alpha^3=\frac{k}{8} \Rightarrow k=8\alpha^3$

Alternatively, put $x=\pm 3/2$ into $P(x)=4x^3-27x+k=0$.

* Alternatively, α being a double root means

$P'(\alpha)=0 \Rightarrow 12\alpha^2-27=0$

(b) $x^3-5x^2+5=0$ has roots α, β, γ.

(i) let the new polynomial be in y. ⟸ you need this technique for more complex questions

$y=x-1 \Rightarrow x=y+1$

$(y+1)^3-5(y+1)^2+5=0$

$y^3+3y^2+3y+1-5(y^2+2y+1)+5=0$

$y^3-2y^2-7y+1=0$

(ii) let $y=x^2 \Rightarrow x=\sqrt{y}$, or $-\sqrt{y}$ (you'll get the same polynomial)

$(\sqrt{y})^3-5(\sqrt{y})^2+5=0$

$y\sqrt{y}-5y+5=0$

$y\sqrt{y}=5(y-1)$ ⟸ group the $\sqrt{y}$ on one side so you can square and get rid of

$y^3=25(y^2-2y+1)$

$y^3-25y^2+50y-25=0$

(iii) using $P(x)=x^3-5x^2+5=0$,

$\alpha^3-5\alpha^2+5=0$

$\beta^3-5\beta^2+5=0$

$\gamma^3-5\gamma^2+5=0$ +

$\alpha^3+\beta^3+\gamma^3=5(\alpha^2+\beta^2+\gamma^2)-15$ ↙ sum of roots of polynomial in (ii)

$=110$

(c) (i) $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

differentiating: $\dfrac{2x}{a^2}+\dfrac{2y}{b^2}\dfrac{dy}{dx}=0$

$$\frac{dy}{dx}=-\frac{b^2}{a^2}\frac{x}{y}$$

at $P(x_1, y_1)$, $\dfrac{dy}{dx}=\dfrac{-b^2}{a^2}\dfrac{x_1}{y_1}$

tangent: $y-y_1=m(x-x_1)$

$$y-y_1=-\frac{b^2x_1}{a^2y_1}(x-x_1)$$

$$a^2y_1y-a^2y_1^2=-b^2x_1x+b^2x_1^2$$

$\div a^2b^2$: $\dfrac{x_1x}{a^2}+\dfrac{y_1y}{b^2}=\dfrac{x_1^2}{a^2}+\dfrac{y_1^2}{b^2}$

(1) — $\dfrac{x_1x}{a^2}+\dfrac{y_1y}{b^2}=1$ since (x_1, y_1) is on the ellipse.

(ii) tangent from $Q(x_2, y_2)$:

(2) — $\dfrac{x_2x}{a^2}+\dfrac{y_2y}{b^2}=1$

Point $T(x_0, y_0)$ lies on both tangents, hence satisfies (1) and (2).

from (1): $\dfrac{x_1x_0}{a^2}+\dfrac{y_1y_0}{b^2}=1$ — (3)

from (2): $\dfrac{x_2x_0}{a^2}+\dfrac{y_2y_0}{b^2}=1$ — (4)

The equation of line PQ is an equation which is satisfied by P and Q (remember that a line is completely determined by 2 points). From (3) and (4), we can see that this line is $\dfrac{x_0x}{a^2}+\dfrac{y_0y}{b^2}=1$ — (5)

(iii) $x_S=ae$ and $x_R=a/e$. ⇐ best to leave the e until you find out it's necessary to write it in terms of a and b.

$$m_{TS}=\frac{y_0-0}{x_0-ae}$$

$$m_{SR}=\frac{0-y_R}{ae-a/e}=\frac{ey_R}{a(e^2-1)}$$

to find y_R, put $x=a/e$ into (5):

$$\frac{x_0}{ae}+\frac{y_0y_R}{b^2}=1$$

$$y_R=\frac{b^2(ae-x_0)}{ae\,y_0}=\frac{-b^2(x_0-ae)}{ae\,y_0}$$

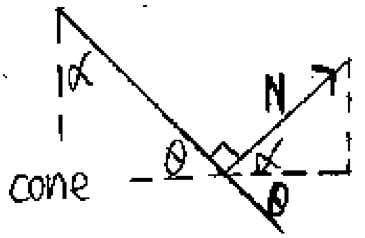
$$m_{TS}\times m_{SR}=m_{TS}\,y_R\cdot\frac{e}{a(e^2-1)}$$

$$=\frac{-b^2}{ae}\cdot\frac{e}{a(e^2-1)}=-\frac{b^2}{a^2(e^2-1)}=-1, \text{ since } b^2=a^2(e^2-1)$$

$\therefore TS\perp SR$

Question 6

(a) (i)



$\theta = 90° - \alpha$

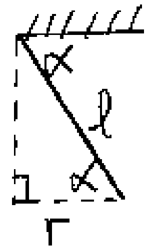
vertical component of N

$= N \sin\alpha$.

(ii) horizontally:

$-N\cos\alpha + T\cos\alpha = mrw^2$

$\cos\alpha(T-N) = mrw^2 \quad — ①$



vertically:

$N\sin\alpha + T\sin\alpha = mg$

$T + N = \dfrac{mg}{\sin\alpha} \quad — ②$

from ①: $\cos\alpha(T-N) = mrw^2$

$= m(l\cos\alpha)w^2$

$T - N = mlw^2 \quad — ③$

(iii) we can put $N = 0$ in ② and ③

②: $T = \dfrac{mg}{\sin\alpha}$

③: $T = mlw^2$

$\dfrac{mg}{\sin\alpha} = mlw^2$

$w^2 = \dfrac{g}{l\sin\alpha}$

$w = \sqrt{\dfrac{g}{l\sin\alpha}}$

(b) (i) $I_1 = \int_0^{\pi/4} \tan\theta \, d\theta$

$= -\int_0^{\pi/4} -\dfrac{\sin\theta}{\cos\theta} d\theta = -[\ln\cos\theta]_0^{\pi/4}$

$= -\ln\frac{1}{\sqrt{2}} + \ln 1 = \ln\sqrt{2} = \frac{1}{2}\ln 2$

(ii) After doing some inspection, apparently we can start from the given result. This seems to be the case for

$I_n + I_{n-2} = \int_0^{\pi/4} \tan^n\theta \, d\theta + \int_0^{\pi/4} \tan^{n-2}\theta \, d\theta$

$= \int_0^{\pi/4} \tan^{n-2}\theta \tan^2\theta \, d\theta + \int_0^{\pi/4} \tan^{n-2}\theta \, d\theta$

$= \int_0^{\pi/4} \tan^{n-2}\theta (1 + \tan^2\theta) \, d\theta$

$= \int_0^{\pi/4} \tan^{n-2}\theta \sec^2\theta \, d\theta$

$= \left[\dfrac{\tan^{n-1}\theta}{n-1}\right]_0^{\pi/4} = \dfrac{1}{n-1}$ as required

Question 6

(b)(iii) $I_n < I_{n-2}$ because $\tan^n\theta < \tan^{n-2}\theta$ for $0<\theta<\pi/4$.
(since $0<\tan\theta<1$)

$\therefore\ 2I_n < I_n + I_{n-2} = \frac{1}{(n-1)}$

$I_n < \frac{1}{2(n-1)}$

also, $2I_n > I_n + I_{n+2} = \frac{1}{(n+2)-1}$

$I_n > \frac{1}{2(n+1)}$

(iv) $I_5 = \frac{1}{4} - I_3$
$= \frac{1}{4} - \left(\frac{1}{2} - I_1\right)$
$= \frac{1}{4} - \frac{1}{2} + \frac{1}{2}\ln 2$ from (i)
$= \frac{1}{2}\ln 2 - \frac{1}{4}$

using (iii), $\frac{1}{12} < I_5 < \frac{1}{8}$
$1/6 < \ln 2 - 1/2 < 1/4$
$2/3 < \ln 2 < 3/4$.

Question 7

(a)(i) $\frac{dy}{dt} = \frac{dy}{dV} \times \frac{dV}{dt}$
$= -\frac{k}{A}\sqrt{y}$

$V = Ay$ since A = area of cross-section.
$\frac{dy}{dV} = \frac{1}{A}$

(ii) $\frac{dt}{dy} = -\frac{A}{k\sqrt{y}} = -\frac{A}{k}y^{-1/2}$

$t = -\frac{A}{k}2\sqrt{y} + C$, $C = \frac{A}{k}2\sqrt{y_0}$ or T (since $y=y_0$ when $t=0$, and $y=0$ when $t=T$)

$\frac{-k(t-T)}{2A} = \sqrt{y} \Rightarrow y = \frac{k^2}{4A^2}(t-T)^2$
$= \frac{k^2}{4A^2}T^2\left(1-\frac{t}{T}\right)^2$

but $C = T = \frac{A}{k}2\sqrt{y_0}$ $\therefore$ $y = \frac{k^2}{4A^2}\frac{A^2 4y_0}{k^2}\left(1-\frac{t}{T}\right)^2$
$= y_0\left(1-\frac{t}{T}\right)^2$, $0 \le t \le T$ obviously.

(iii) i.e. $y = \frac{1}{2}y_0$ at $t=10$.
$\frac{1}{2}y_0 = y_0\left(1-\frac{10}{T}\right)^2 \Rightarrow 1-\frac{10}{T} = \frac{1}{\sqrt{2}}$ * $\Rightarrow T = \frac{10\sqrt{2}}{\sqrt{2}-1}$

at $t=T$, $y=0$ so it takes $\frac{10\sqrt{2}}{\sqrt{2}-1}$ seconds to empty it.
* +ve root since $t<T$.

(b)(i) $\arg z_n = \alpha + n\beta$.

$\arg z_n$ is the angle z_n makes with the line parallel to the x-axis, or the "horizontal" in the diagram, measured counterclockwise from the +ve direction of that line.

From the diagram, $\theta_n = \arg z_{n+1} - \arg z_n$
$= \alpha + (n+1)\beta - (\alpha + n\beta)$
$= \beta$ as required

(ii) looking at ΔP_0OP_1,
$\angle P_0OP_1 + \angle P_0P_1O = \theta_0$ (exterior angle θ_0)
$\angle P_0OP_1 = \frac{\beta}{2}$ (ΔP_0OP_1 isosceles)

looking at $\Delta P_0P_2P_1$,
$\angle P_0P_2P_1 = \frac{\beta}{2}$ ($\Delta P_0P_2P_1$ isosceles)

$OP_0P_1P_2$ is cyclic quad because the 2 angles standing on common chord P_0P_1 are equal.

(iii) so, from (ii) we should have drawn lines OP_1 and P_0P_2 on the diagram (whether it's in the question paper or exam booklet). The next line to draw is P_1P_3.
from (ii) we have $\angle P_0P_2P_1 = \frac{\beta}{2}$
$= \angle P_2P_0P_1$ (isosceles Δ)

using $\Delta P_1P_3P_2$, $\angle P_1P_3P_2 = \frac{\beta}{2}$
$\therefore \angle P_1P_3P_2 = \angle P_2P_0P_1 \Rightarrow P_0P_1P_2P_3$ cyclic quad (equal angles on P_1P_2)

Now consider the angles on chord P_1P_2.
$\angle P_1OP_2 = \angle P_1P_0P_2$ (using cyclic quad $OP_0P_1P_2$)
$= \angle P_1P_3P_2$ (using cyclic quad $P_0P_1P_2P_3$)
$\therefore$ All angles on P_1P_2 from the other vertices are equal, hence O, P_0, P_1, P_2, P_3 are concyclic.

Alternatively, the 2 cyclic quads share 3 common vertices. But a circle is determined completely by 3 points, so the 2 circles are the same circle.

(iv) This means P_4 comes back to $(0,0)$ so we have a regular pentagon $P_0P_1P_2P_3P_4$. ($P_4 = 0$)
$\beta = \theta_n$ which is the exterior angle of the pentagon
$= \frac{2\pi}{5}$, 2π being $360°$ or full rotation

Question 8

(a) (i) $\cos(2m+1)\theta + i\sin(2m+1)\theta = (\cos\theta + i\sin\theta)^{2m+1}$

$RHS = {}^{2m+1}C_0 \cos^{2m+1}\theta (i\sin\theta)^0 + {}^{2m+1}C_1 \cos^{2m}\theta (i\sin\theta)^1 + \ldots + {}^{2m+1}C_{2m} \cos\theta (i\sin\theta)^{2m} + {}^{2m+1}C_{2m+1} (\cos\theta)^0 (i\sin\theta)^{2m+1}$, $2m+1$ is odd.

$i^k, k=0,1,2,\ldots$ alternates from $1, i, -1, -i, 1, i$ etc.

equating imaginary parts in LHS and RHS,

$$\sin(2m+1)\theta = {}^{2m+1}C_1 \cos^{2m}\theta \sin\theta - {}^{2m+1}C_3 \cos^{2m-2}\theta \sin^3\theta + \ldots + (i)^{2m}\, {}^{2m+1}C_{2m+1} (\cos\theta)^0 \sin^{2m+1}\theta$$

$$(i)^{2m} = (i^2)^m = (-1)^m$$

as required.

(ii) Upon inspection of the RHS in $p(x)$ and the RHS in (i), particularly the last term, we get a vague impression that we can try to divide the whole thing in (i) by $\sin^{m+1}\theta$ to see what the result looks like (provided $\sin^{2m+1}\theta \neq 0$).

$$① - \frac{\sin(2m+1)\theta}{\sin^{2m+1}\theta} = \binom{2m+1}{1}\cot^{2m}\theta - \binom{2m+1}{3}\cot^{2m-2}\theta + \ldots + (-1)^m$$

we write the RHS in cot probably because we see cot in the question ... trial and error.

comparing the RHS to $p(x)$, we see that we can put $x = \cot^2\theta$.
Roots or zeroes of $p(x)$ is when $p(x) = 0$, i.e. RHS in ① $= 0$.
This happens when the numerator on the LHS $= 0$

$$\sin(2m+1)\theta = 0$$
$$(2m+1)\theta = n\pi, \quad n \in \mathbb{Z}$$
$$\theta = \frac{n\pi}{2m+1}$$

$\therefore$ roots $= \cot^2\theta = \cot^2\left(\frac{n\pi}{2m+1}\right)$

but unique solutions can be obtained by restricting θ to the first quadrant, $0 < \theta \leq \pi/2$
(we exclude $\theta = 0$ because then $\sin^{2m+1}\theta = 0$)

$\therefore$ $0 < n \leq m + \frac{1}{2}$, but n integer.

$n = k = 1, 2, 3, \ldots, m$.

$\therefore$ $\alpha_k = \cot^2\left(\frac{k\pi}{2m+1}\right)$. α_k's are distinct because for each k the angle $\left(\frac{k\pi}{2m+1}\right)$ is unique and in the first quadrant.

Question 8

(a)(iii) The terms on the LHS are the roots α_k.

sum of roots $= -\frac{b}{a}$

$$= \frac{\binom{2m+1}{3}}{\binom{2m+1}{1}} = \frac{(2m+1)!}{(2m+1-3)!\,3!} \times \frac{(2m+1-1)!\,1!}{(2m+1)!}$$

$$= \frac{(2m)(2m-1)}{3!} = \frac{m(2m-1)}{3}$$

(iv)

$$\cot\theta < \frac{1}{\theta}, \quad \cot\theta > 0, \quad \frac{1}{\theta} > 0$$

$$\cot^2\theta < \left(\frac{1}{\theta}\right)^2$$

$$\sum \cot^2\theta < \sum \left(\frac{1}{\theta}\right)^2$$

$$\frac{m(2m-1)}{3} < \sum_{k=1}^{m} \left(\frac{2m+1}{k\pi}\right)^2$$

$$= \frac{(2m+1)^2}{1^2\pi^2} + \frac{(2m+1)^2}{2^2\pi^2} + \dots + \frac{(2m+1)^2}{m^2\pi^2}$$

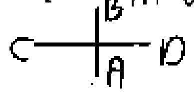
$$= \frac{(2m+1)^2}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{m^2}\right)$$

rearranging, $\frac{\pi^2}{3} < \left(\frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{m^2}\right) \frac{(2m+1)^2}{m(2m-1)}$

$$\frac{\pi^2}{6} < \left(\frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{m^2}\right) \frac{(2m+1)^2}{2m(2m-1)}$$

(b) Let's decode the given information in the first paragraph:

#1 : AB lies in South-North direction, while CD lies in West-East direction, i.e. from the top they look like



#2 : AB = 2a, CD = 2a

#3 : those lines are in horizontal planes, meaning they are horizontal, not that they're in the same 2-dimentional plane.

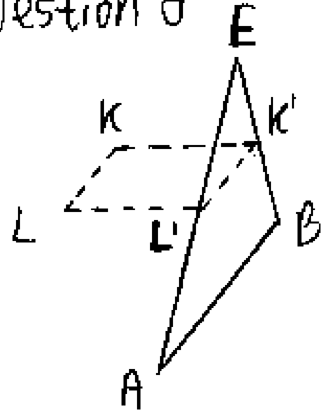
#4 : the horizontal planes are 2a apart, meaning EF = 2a, E is above F three dimentionally.

Now the second paragraph might be a bit confusing. A tetrahedron is a solid with four vertices and four plane faces, i.e. a triangular pyramid. I find it easiest to think of A as the vertex. B is right behind A.

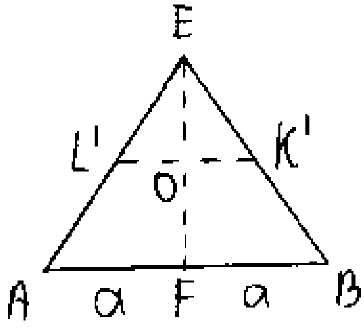
And yes you'll have rectangular cross-sections if you slice it that way.

Question 8

(b)(i)



$\Rightarrow$



using similarity,

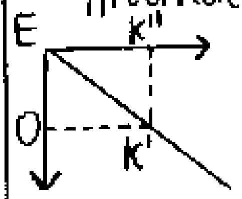
$$\frac{OK'}{FB} = \frac{OE}{FE}$$

$$OK' = \frac{a(a-x)}{2a} = \frac{a-x}{2}$$

Similarly, $OL' = \dfrac{a-x}{2}$

$\therefore L'K' = a-x = KL$

Alternatively, you can think of E being a fixed point and O and K' being moveable points, so that the gradient of line EK' is fixed.



$|EO| = m\,EK''$

$(a-x) = m\,OK'$

but when $x=-a$, $OK' = FB = a$

$m = 2$.

$OK' = \dfrac{a-x}{2}$

More generally, whenever you have two perpendicular lines, you can express displacement along one line in terms of displacement along the other line (the displacement doesn't need to be relative to the endpoint of the line) in a linear function $y = mx + b$.

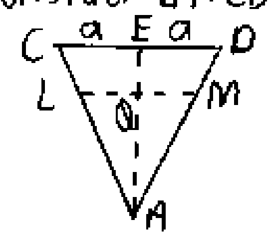
In this question, $KL = mx + b$

at $x=-a$, $KL = 2a$; $2a = -am + b$

at $x = a$, $KL = 0$; $0 = am + b$

Solving simultaneously, $m = -1$ and $b = a$

Now, consider $\triangle ACD$ to find LM :



$$LM = 2QM = 2\,\frac{ED \times QA}{EA} = a+x$$

$\therefore$ area $= KL \times LM = (a-x)(a+x) = (a^2 - x^2)$ units^2

(ii) $$V = \lim_{\Delta x \to 0} \sum_{x=-a}^{a} (a^2 - x^2)\Delta x$$

$$= \int_{-a}^{a} (a^2 - x^2)\,dx$$

$$= \left[a^2 x - \frac{x^3}{3}\right]_{-a}^{a}$$

$$= \left(a^3 - \frac{a^3}{3}\right) - \left(-a^3 + \frac{a^3}{3}\right)$$

$$= 2a^3 - \frac{2}{3}a^3 = \frac{4}{3}a^3 \text{ units}^3$$