2009 Maths Extension 2

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October 27, 2009

Q1

a)

Using $u = \ln x$

$$\begin{aligned}\int \frac{\ln x}{x}\ \mathrm{d}x &= \int u\ \mathrm{d}u \\ &= \frac{1}{2}u^2 \\ &= \frac{1}{2}(\ln x)^2 + C\end{aligned}$$

b)

Let $u = x$, $\mathrm{d}u = \mathrm{d}x$
$\mathrm{d}v = e^{2x}\ \mathrm{d}x$, $v = \frac{1}{2}e^{2x}$

$$\begin{aligned}\int xe^{2x}\ \mathrm{d}x &= \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}\ \mathrm{d}x \\ &= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C\end{aligned}$$

c)

$$\begin{aligned}\int \frac{x^2}{1+4x^2}\ \mathrm{d}x &= \frac{1}{4}\int \frac{4x^2+1-1}{1+4x^2}\ \mathrm{d}x \\ &= \frac{1}{4}\int 1 - \frac{1}{4}\frac{1}{\frac{1}{4}+x^2}\ \mathrm{d}x \\ &= \frac{1}{4}\left\{x - \frac{1}{4}\cdot 2\tan^{-1}(2x)\right\} \\ &= \frac{1}{4}x - \frac{1}{8}\tan^{-1}(2x) + C\end{aligned}$$

d)

We have $\frac{x-6}{x^2+3x-4} = \frac{x-6}{(x+4)(x-1)} = \frac{2}{x+4} - \frac{1}{x-1}$, so

$$\begin{aligned}\int_2^5 \frac{x-6}{x^2+3x-4}\ \mathrm{d}x &= \int_2^5 \frac{2}{x+4} - \frac{1}{x-1}\ \mathrm{d}x \\ &= \left[2\ln(x+4) - \ln(x-1)\right]_2^5 \\ &= 2\ln\frac{9}{6} - \ln\frac{4}{1} = 2\ln\frac{3}{4}\end{aligned}$$

e)

Let $x = \tan\theta$, $\mathrm{d}x = \sec^2\theta\ \mathrm{d}\theta$. When $x = 1$, $\theta = \frac{\pi}{4}$. When $x = \sqrt{3}$, $\theta = \frac{\pi}{3}$

$$\begin{aligned}
\int_1^{\sqrt{3}} \frac{1}{x^2\sqrt{1+x^2}}\ \mathrm{d}x &= \int_{\pi/4}^{\pi/3} \frac{1}{\tan^2\theta\sqrt{1+\tan^2\theta}} \sec^2\theta\ \mathrm{d}\theta \\
&= \int_{\pi/4}^{\pi/3} \frac{1}{\tan^2\theta\sec\theta} \sec^2\theta\ \mathrm{d}\theta \\
&= \int_{\pi/4}^{\pi/3} \frac{\cos^2\theta}{\sin^2\theta}\frac{1}{\cos\theta}\ \mathrm{d}\theta \\
&= \int_{\pi/4}^{\pi/3} \frac{\cos\theta}{\sin^2\theta}\ \mathrm{d}\theta \\
&= \left[-\frac{1}{\sin\theta}\right]_{\pi/4}^{\pi/3} = \sqrt{2} - \frac{2}{\sqrt{3}}
\end{aligned}$$

Q2

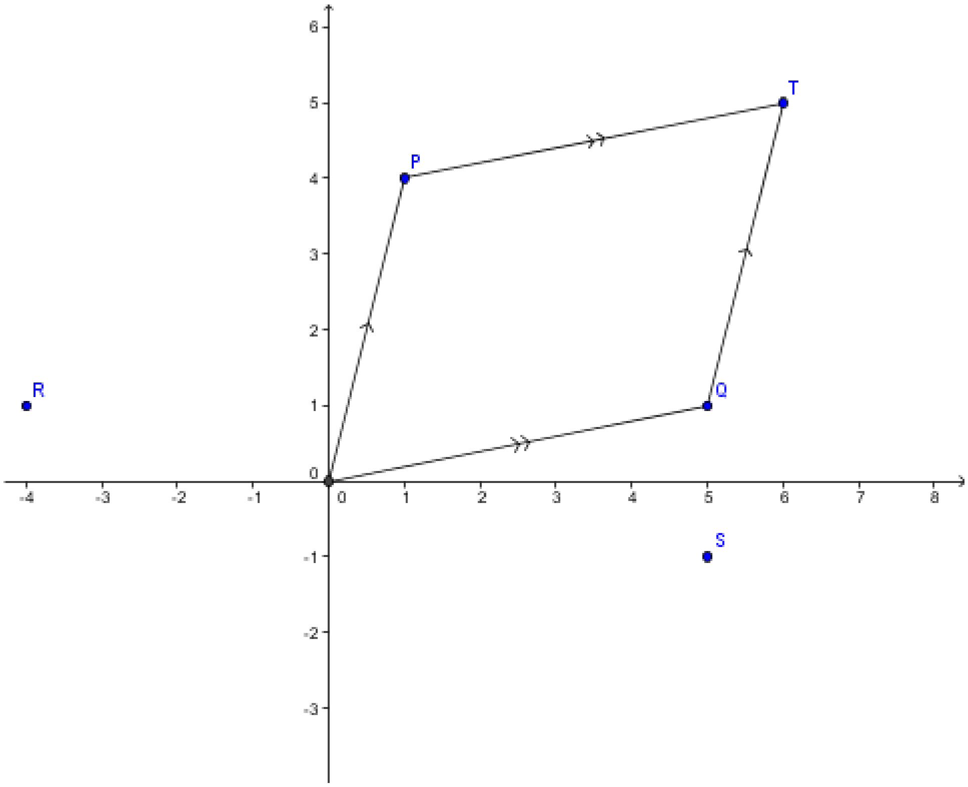
a)

$i^9 = \left(i^4\right)^2 i = 1^2 i = i$

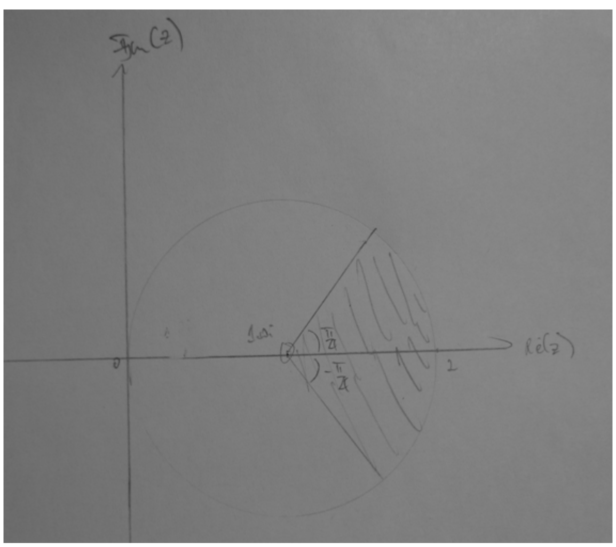
b)

$$\begin{aligned}
\frac{-2+3i}{2+i} &= \frac{-2+3i}{2+i}\cdot\frac{2-i}{2-i} \\
&= \frac{-4+2i+6i+3}{2^2+1^2} = \frac{1}{5}\left(-1+8i\right)
\end{aligned}$$

c)



d)



e)

i)

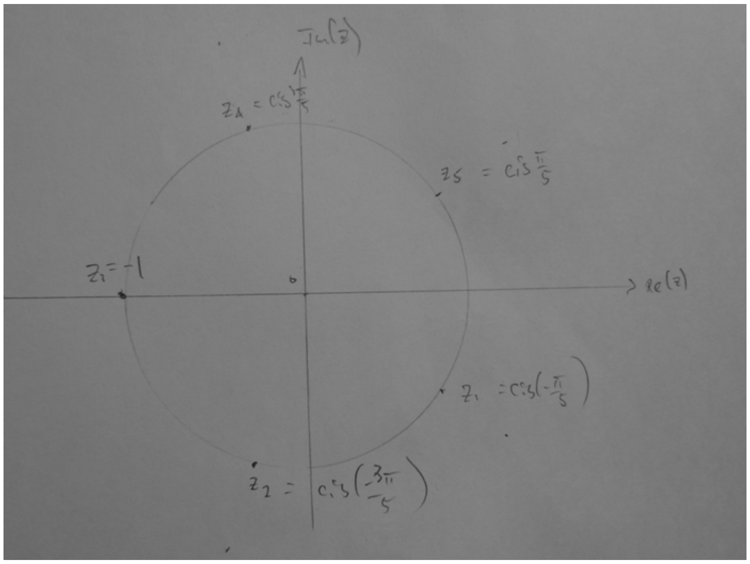
Let $z = r(\cos\theta + i\sin\theta)$, $r, \theta \in \mathbb{R}$ be a root. We are required to solve

$$\begin{aligned} z^5 &= -1 \\ r^5(\cos 5\theta + i\sin 5\theta) &= (\cos\pi + i\sin\pi) \end{aligned}$$

Hence, $r^5 = 1$, $r = 1$

and $5\theta = \pi + 2k\pi$, ie. $\theta = \frac{\pi}{5} + \frac{2}{5}k\pi$, $k \in \mathbb{Z}$. Letting $k = -2, -1, 0, 1, 2$ the five roots are $z = \left(\cos -\frac{3\pi}{5} + i\sin -\frac{3\pi}{5}\right)$, $\left(\cos -\frac{\pi}{5} + i\sin -\frac{\pi}{5}\right)$, $\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)$, $\left(\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}\right)$, -1

ii)



f)

i)

Let $z = x + iy$, $x, y \in \mathbb{R}$

$$\begin{aligned} z^2 &= 3 + 4i \\ x^2 - y^2 + 2ixy &= 3 + 4i \end{aligned}$$

Hence we have $x^2 - y^2 = 3$ and $xy = 2$. Substituting $y = \frac{2}{x}$ into the first equation,

$$\begin{aligned} x^2 - \frac{4}{x^2} &= 3 \\ x^4 - 4 &= 3x^2 \\ x^4 - 3x^2 - 4 &= 0 \\ \left(x^2 - 4\right)\left(x^2 + 1\right) &= 0 \end{aligned}$$

$x^2 + 1 = 0$ gives us no real solution for x. So, $x^2 - 4$ will give us our required solution. $x = \pm 2$, substituting into $xy = 2$ for the y values, we have our roots:

$$z = \pm (2 + i)$$

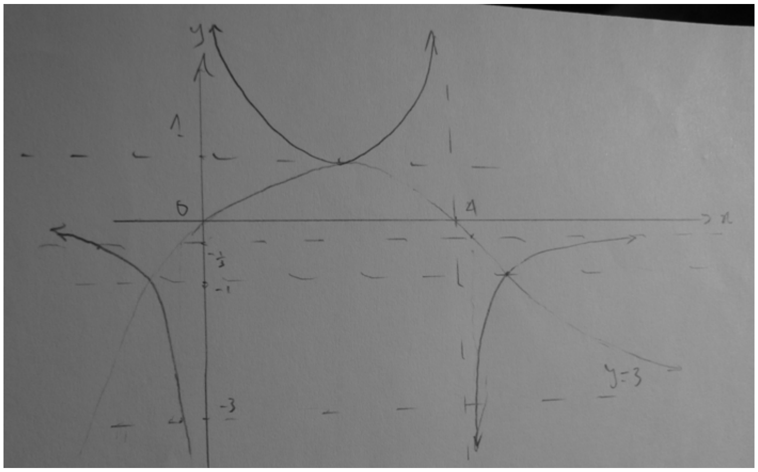
ii)

$$\begin{aligned} z^2 + iz - 1 - i &= 0 \\ z &= \frac{-i \pm \sqrt{-1 - 4(1)(-1 - i)}}{2} \\ &= \frac{-i \pm \sqrt{3 + 4i}}{2} \\ &= \frac{-i \pm (2 + i)}{2} \\ z &= 1 \\ &\text{or} \\ z &= -1 - i \end{aligned}$$

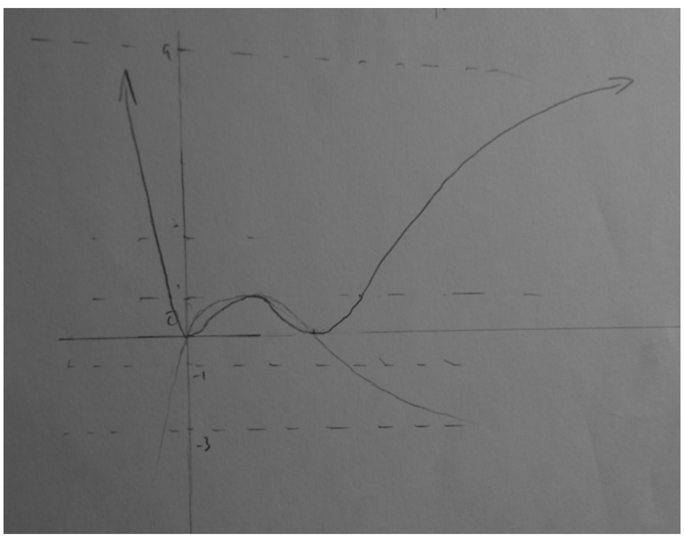
Q3

a)

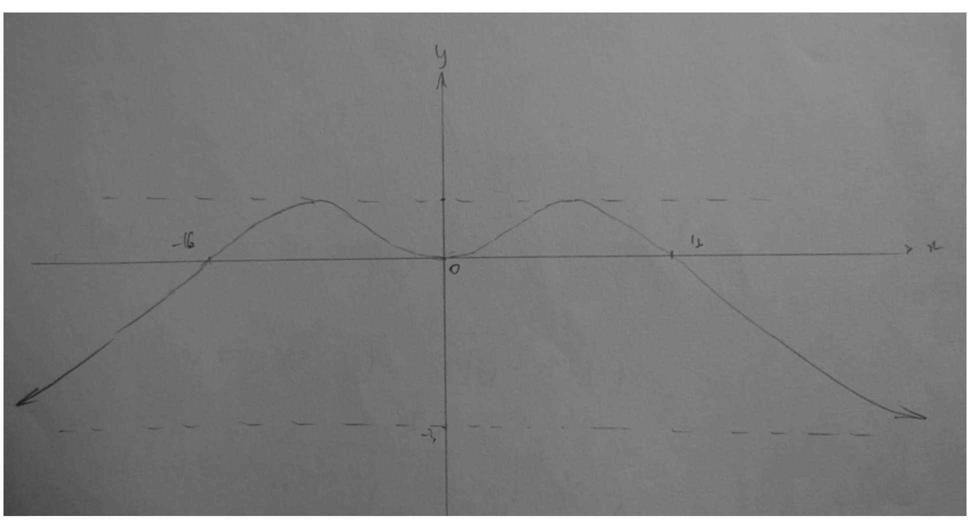
i)



ii)



iii)



b)

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{x^2+2xy+3y^2\right\} = \frac{\mathrm{d}}{\mathrm{d}x}\left\{18\right\}$$

$$
\begin{aligned}
2x+2y+2x\frac{\mathrm{d}y}{\mathrm{d}x}+6y\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\
\frac{\mathrm{d}y}{\mathrm{d}x}(2x+6y) &= -2(x+y)\\
\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{x+y}{x+3y}
\end{aligned}
$$

Tangent to the curve is statinary, whenever

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\
x+y &= 0\\
y &= -x
\end{aligned}
$$

substituting into the curve to obtain the required points

$$
\begin{aligned}
x^2+2x(-x)+3(-x)^2 &= 18\\
2x^2 &= 18\\
x^2 &= 9\\
x &= \pm 3
\end{aligned}
$$

so the points are $(3,-3)$ and $(-3,3)$.

c)

$P'(x)=3x^2+2ax+b$. Since $(x-1)^2$ is a factor, we have

$$
\begin{aligned}
P(1) &= 0\\
P'(1) &= 0
\end{aligned}
$$

$$
\begin{aligned}
1+a+b+5 &= 0\\
a+b &= -6\\
a &= -6-b
\end{aligned}
$$

and

$$
\begin{aligned}
3+2a+b &= 0\\
b &= -3-2a\\
b &= -3+2(6+b)\\
b &= 9+2b\\
-9 &= b\\
\therefore a &= 3
\end{aligned}
$$

d)

For the intersection points:

$$
\begin{aligned}
x+1 &= (x-1)^2\\
x+1 &= x^2-2x+1\\
0 &= x^2-3x\\
x &= 0 \text{ or } 3
\end{aligned}
$$

For the volume:

$$
\begin{aligned}
\delta V &= 2\pi x^2 \left\{(x+1)-(x-1)^2\right\} \delta x \\
&= 2\pi x^2 \left\{(x+1)-(x-1)^2\right\} \delta x \\
&= 2\pi x^2 \left\{(x+1)-x^2+2x-1\right\} \delta x \\
&= 2\pi x^2 \left\{-x^2+3x\right\} \delta x \\
&= 2\pi \left\{-x^4+3x^3\right\} \delta x \\
V &= \lim_{\delta x \to 0} \sum_{x=0}^{3} 2\pi \left\{-x^4+3x^3\right\} \delta x \\
&= \int_0^3 2\pi \left\{-x^4+3x^3\right\} \, \mathrm{d}x \\
&= 2\pi \left[-\frac{x^5}{5}+\frac{3}{4}x^4\right]_0^3 = \frac{243}{10}\pi
\end{aligned}
$$

Q4

a)

i)

m_T, m_N denotes gradient of tangent and normal respectively

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{x^2}{a^2}+\frac{y^2}{b^2}\right\} &= \frac{\mathrm{d}}{\mathrm{d}x}1 \\
\frac{2x}{a^2}+\frac{2y}{b^2}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\frac{2y}{b^2}\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{2x}{a^2} \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{b^2}{a^2}\frac{x}{y} \\
\therefore m_T &= -\frac{b^2}{a^2}\frac{x_0}{y_0} \\
m_N &= \frac{a^2 y_0}{b^2 x_0}
\end{aligned}
$$

Equation of normal:

$$
\begin{aligned}
y-y_0 &= m_N (x-x_0) \\
y-y_0 &= \frac{a^2 y_0}{b^2 x_0}(x-x_0)
\end{aligned}
$$

ii)

For the x-intercept of the normal, substitute, $(0, x_N)$:

$$
\begin{aligned}
-y_0 &= \frac{a^2 y_0}{b^2 x_0}(x_N-x_0) \\
-x_0 \tfrac{b^2}{a^2} &= x_N-x_0 \\
x_N &= x_0-x_0\frac{b^2}{a^2} = x_0 e^2
\end{aligned}
$$

$\therefore N\left(e^2 x_0, 0\right)$

iii)

$NS = ae - e^2 x_0 = e\left(a - ex_0\right)$ and similarly $NS' = e\left(a + ex_0\right)$, so

$$\frac{NS}{NS'} = \frac{a - ex_0}{a + ex_0}$$

From the foci-directrix definition of ellipse, we have $PS = ePM = e\left(\frac{a}{e} - x_0\right) = a - ex_0$ and similarly, $PS' = ePM' = a + ex_0$, so

$$\begin{aligned}\frac{PS}{PS'} &= \frac{a - ex_0}{a + ex_0}\\ &= \frac{NS}{NS'}\end{aligned}$$

iv)

Note that $\angle PNS + \angle PNS' = \pi$, so $\sin\angle PNS = \sin\angle PNS'$. Also,

$$\begin{aligned}\frac{PS}{PS'} &= \frac{NS}{NS'}\\ \therefore \frac{NS'}{PS'} &= \frac{NS}{PS}\end{aligned}$$

Using sine rule on $\triangle PNS$ we have

$$\begin{aligned}\frac{NS}{\sin\beta} &= \frac{PS}{\sin\angle PNS}\\ \therefore \sin\beta &= \frac{NS}{PS}\sin\angle PNS\end{aligned}$$

Likewise, we have

$$\begin{aligned}\sin\alpha &= \frac{NS'}{PS'}\sin\angle PNS'\\ &= \frac{NS}{PS}\sin\angle PNS\\ &= \sin\beta\end{aligned}$$

Since $0 < \alpha < \frac{\pi}{2}$ and $0 < \beta < \frac{\pi}{2}$, we have $\alpha = \beta$.

b)

i)

Let us assign up as the vertical positive direction and the centre of the circle as the positive horizontal direction.

Vertical direction:

$$\sum F_y = T\cos\alpha + N\sin\alpha - mg$$

and Horizontal direction

$$\sum F_x = T\sin\alpha - N\cos\alpha$$

ii)

Applying Newton's law vertically:

$$\begin{aligned}\sum F_y &= m\ddot{y}\\ T\cos\alpha + N\sin\alpha - mg &= 0\\ T\cos^2\alpha + N\sin\alpha\cos\alpha - mg\cos\alpha &= 0\end{aligned}$$

Applying Newton's law horizontally:

$$\begin{aligned}\sum F_x &= m\ddot{x}\\ T\sin\alpha - N\cos\alpha &= mr\omega^2\\ T\sin^2\alpha - N\cos\alpha\sin\alpha &= mr\omega^2\sin\alpha\end{aligned}$$

Adding those two final equations we have

$$\begin{aligned}T\left(\sin^2\alpha + \cos^2\alpha\right) - mg\cos\alpha &= mr\omega^2\sin\alpha\\ T &= mg\cos\alpha + mr\omega^2\sin\alpha\\ &= m\left(g\cos\alpha + r\omega^2\sin\alpha\right)\end{aligned}$$

Substituting back into the first equation, we have

$$\begin{aligned}m\cos\alpha\left(g\cos\alpha + r\omega^2\sin\alpha\right) + N\sin\alpha - mg &= 0\\ N\sin\alpha &= mg - mg\cos^2\alpha - mr\omega^2\sin\alpha\cos\alpha\\ N\sin\alpha &= mg\sin^2\alpha - mr\omega^2\sin\alpha\cos\alpha\\ N &= mg\sin\alpha - mr\omega^2\cos\alpha\end{aligned}$$

iii)

If $T = N$, we have (noting that $m \neq 0$)

$$\begin{aligned}m\left(g\cos\alpha + r\omega^2\sin\alpha\right) &= mg\sin\alpha - mr\omega^2\cos\alpha\\ g\cos\alpha + r\omega^2\sin\alpha &= g\sin\alpha - r\omega^2\cos\alpha\\ \omega^2 r\left(\sin\alpha + \cos\alpha\right) &= g\left(\sin\alpha - \cos\alpha\right)\\ \omega^2 &= \frac{g\left(\sin\alpha - \cos\alpha\right)}{r\left(\sin\alpha + \cos\alpha\right)}\\ &= \frac{g\left(\tan\alpha - 1\right)}{r\left(\tan\alpha + 1\right)}\end{aligned}$$

iv)

Since we have $\omega^2 > 0$ and r, g are positive, then

$$\begin{aligned}\tan\alpha - 1 &> 0\\ \tan\alpha &> 1\end{aligned}$$

So $\frac{\pi}{4} < \alpha < \frac{\pi}{2}$

Q5

a)

i)

$$\begin{aligned}\angle DAB &= \angle YAK \text{ (common angle)}\\ \angle ADB &= \angle AYK = \frac{\pi}{2} \text{ (given, AB is a diameter)}\\ \therefore \triangle ADB &\sim \triangle AYK \text{ (two common angles)}\\ \therefore \angle AKY &= \angle ABD\end{aligned}$$

Corresponding angles in similar triangles are equal.

ii)

$$
\begin{aligned}
\angle ABD &= \angle ACD \text{ (angles subtended by the same arc at the circumference of a circle are equal)}\\
\therefore \angle DKX &= \angle AKY \text{ (common angle)}\\
&= \angle ABD \text{ (previous question)}\\
&= \angle ACD
\end{aligned}
$$

Hence $CKDX$ is a cyclic quadrilateral (if a pair of angles at a vertex of a quadrilateral subtended by the same side are equal, then the quadrilateral is a cyclic quadrilateral).

iii)

$$
\begin{aligned}
\angle KCD &= \angle DAB \text{ (exterior angle of cyclic quad is equal to interior opposite angle)}\\
&= \pi - \angle DCB \text{ (interior opposite angles in a cyclic quad are supplementary)}\\
\therefore \pi &= \angle DCB + \angle KCD
\end{aligned}
$$

so B, C, K are collinear.

b)

i)

Let $u = x^{2n}$ and $\mathrm{d}v = xe^{x^2}\ \mathrm{d}x$, hence $\mathrm{d}u = 2nx^{2n-1}\ \mathrm{d}x$ and $v = \frac{1}{2}e^{x^2}$, so for $n \geq 1$

$$
\begin{aligned}
I_n &= \frac{1}{2}\left[e^{x^2}x^{2n}\right]_0^1 - \int \frac{2n}{2}x^{2n-1}e^{x^2}\ \mathrm{d}x\\
&= \frac{1}{2}e - n\int x^{2(n-1)+1}e^{x^2}\ \mathrm{d}x\\
&= \frac{1}{2}e - nI_{n-1}
\end{aligned}
$$

ii)

$$
\begin{aligned}
I_0 &= \int_0^1 xe^{x^2}\ \mathrm{d}x\\
&= \frac{1}{2}\left[e^{x^2}\right]_0^1 = \frac{e-1}{2}
\end{aligned}
$$

so using the recurrence formula

$$
\begin{aligned}
I_2 &= \frac{e}{2} - 2I_1\\
&= \frac{e}{2} - 2\left\{\frac{e}{2} - I_0\right\}\\
&= -\frac{e}{2} + (e-1) = \frac{e}{2} - 1
\end{aligned}
$$

c)

i)

$$
\begin{aligned}
f'(x) &= \frac{1}{2}\left(e^x + e^{-x}\right) - 1\\
f''(x) &= \frac{1}{2}\left(e^x - e^{-x}\right)
\end{aligned}
$$

For all $x > 0$, we have $e^x > 1$ and $e^{-x} < 1$ (ie. $-e^{-x} > 1$), so

$$\begin{aligned} \frac{1}{2}\left(e^x - e^{-x}\right) &> 0 \\ f''(x) &> 0 \end{aligned}$$

ii)

For all $x > 0$, we have $e^x > 1$ and $e^{-x} > 0$ so

$$\begin{aligned} \frac{1}{2}\left(e^x + e^{-x}\right) - 1 &> 0 \\ f'(x) &> 0 \end{aligned}$$

iii)

$f(0) = \frac{1-1}{2} - 0 = 0$. Since $f'(x) > 0$ and $f(x)$ is clearly continuous, then $f(x) > f(0)\ \forall x > 0$, hence

$$\begin{aligned} \frac{1}{2}\left(e^x - e^{-x}\right) - x &> 0 \\ \frac{1}{2}\left(e^x - e^{-x}\right) &> x \end{aligned}$$

Q6

a)

For the volume of the slice:

$$\begin{aligned} \delta V &= (2y)(y)\,\delta x \\ &= 2y^2\delta x \\ &= 2(4-x)\,\delta x \\ V &= \lim_{\delta x \to 0}\sum_{x=0}^{4} 2(4-x)\,\delta x \\ &= 2\int_0^4 (4-x)\ \mathrm{d}x \\ &= 2\left[4x - \frac{x^2}{2}\right]_0^4 = 2(16-8) = 16 \end{aligned}$$

b)

i)

Since α is a root

$$\begin{aligned} P(\alpha) &= 0 \\ \alpha^3 + q\alpha^2 + q\alpha + 1 &= 0 \end{aligned}$$

Its clear that $\alpha \neq 0$ from above.
Substituting $\frac{1}{\alpha}$ into the polynomial, we have

$$\begin{aligned} P\left(\frac{1}{\alpha}\right) &= \frac{1}{\alpha^3} + q\frac{1}{\alpha^2} + q\frac{1}{\alpha} + 1 \\ &= \frac{1}{\alpha^3}\left(1 + q\alpha + q\alpha^2 + \alpha^3\right) \\ &= 0 \end{aligned}$$

ii)

1) Since the coefficients of the cubic are real, then if α is a root, then so should its conjugate. Since α is not real, then $\alpha \neq \overline{\alpha}$. Since $\alpha \neq 1$, then $\alpha \neq \frac{1}{\alpha}$. Hence

$$\begin{aligned} \overline{\alpha} &= \frac{1}{\alpha} \\ \alpha\overline{\alpha} &= 1 \\ |\alpha|^2 &= 1 \\ |\alpha| &= 1 \end{aligned}$$

2) We have three roots: -1, α and $\overline{\alpha}$. Note that $\mathrm{Re}(\alpha) = \frac{1}{2}(\alpha + \overline{\alpha})$. Sum of roots is

$$\begin{aligned} -1 + \alpha + \overline{\alpha} &= -q \\ 2\mathrm{Re}(\alpha) &= 1 - q \\ \mathrm{Re}(\alpha) &= \frac{1-q}{2} \end{aligned}$$

c)

i)

Using Pythagoras' theorem, we have

$$\begin{aligned} PQ^2 &= PO^2 - QO^2 \\ &= x^2 + y^2 - r^2 \end{aligned}$$

ii)

$$\begin{aligned} PQ^2 &= PR^2 \\ x^2 + y^2 - r^2 &= (c-x)^2 \\ x^2 + y^2 - r^2 &= x^2 - 2cx + c^2 \\ y^2 &= r^2 - 2cx + c^2 \end{aligned}$$

iii)

$$\begin{aligned} y^2 &= r^2 - 2cx + c^2 \\ y^2 &= -2c\left(x - \frac{r^2+c^2}{2c}\right) \\ &= -4\frac{c}{2}\left(x - \frac{r^2+c^2}{2c}\right) \end{aligned}$$

So $a = \frac{c}{2}$ hence the focus is $\left(\frac{r^2+c^2}{2c} - \frac{c}{2}, 0\right) = \left(\frac{r^2}{2c}, 0\right)$

iv)

From the definition of a parabola, we have

$$\begin{aligned} PS &= PR \\ &= PQ \end{aligned}$$

from part (ii), hence $PS - PQ = 0$ which is independent of x

Q7

a)

i)

1)

$$\begin{aligned}
\frac{v\,\mathrm{d}v}{\mathrm{d}x} &= g - rv \\
\int_0^v \frac{v}{g-rv}\,\mathrm{d}v &= \int_0^x \mathrm{d}x \\
\frac{1}{r}\int_0^v \frac{rv-g}{g-rv} + \frac{g}{g-rv}\,\mathrm{d}v &= x \\
x &= \frac{1}{r}\left[-v - \frac{g}{r}\ln|g-rv|\right]_0^v \\
&= \frac{1}{r}\left\{-v - \frac{g}{r}\ln\left|\frac{g-rv}{g}\right|\right\} \\
&= \frac{g}{r^2}\ln\left(\frac{g}{g-rv}\right) - \frac{v}{r}
\end{aligned}$$

for $0 \le x \le L$

2)

$$\begin{aligned}
L &= \frac{9.8}{0.2^2}\ln\left(\frac{9.8}{9.8 - 0.2\cdot 30}\right) - \frac{30}{0.2} \\
&\approx 82\text{m}
\end{aligned}$$

to 2 significant figures.

ii)

The jumper's head stays out of the water if $x < 125$, $\forall t > 0$. Note that $29\sin t - 10\cos t = \sqrt{29^2+10^2}\sin(t+\alpha)$. Hence $x = e^{-\frac{t}{10}}\sqrt{29^2+10^2}\sin(t+\alpha) + 92$. Since $\sin(t+\alpha) \le 1$, $e^{-t/10} < 1\ \forall t > 0$, we have

$$\begin{aligned}
e^{-\frac{t}{10}}\sqrt{29^2+10^2}\sin(t+\alpha) &< \sqrt{29^2+10^2} \\
x &< \sqrt{29^2+10^2} + 92 \approx 122.68
\end{aligned}$$

So the jumper's head stays out of the water.

b)

i)

De Moivre's theorem states that $z^n = \cos n\theta + i\sin n\theta$, so

$$\begin{aligned}
z^n + z^{-n} &= \cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta) \\
&= \cos n\theta + i\sin n\theta + \cos(n\theta) - i\sin(n\theta) \\
&= 2\cos n\theta
\end{aligned}$$

ii)

$$\begin{aligned}
\left[z + \frac{1}{z}\right]^{2m} &= \frac{1}{z^{2m}}\left[z^2+1\right]^{2m} \\
(2\cos\theta)^{2m} &= \frac{1}{z^{2m}}\left(\binom{2m}{0}z^{4m} + \binom{2m}{1}z^{4m-2} + \cdots + \binom{2m}{m}z^{4m-2m} + \cdots + \binom{2m}{2m}z^0\right)
\end{aligned}$$

$$
\begin{aligned}
&= \binom{2m}{0} z^{2m} + \binom{2m}{1} z^{2m-2} + \cdots + \binom{2m}{m} z^0 + \binom{2m}{m+1} z^2 + \cdots + \binom{2m}{2m} z^{-2m} \\
&= \binom{2m}{0} \left(z^{2m} + z^{2m}\right) + \binom{2m}{1} \left(z^{2m-2} + z^{2-2m}\right) + \cdots + \binom{2m}{m-1} \left(z^2 + z^{-2}\right) + \binom{2m}{m} z^0 \\
&= \binom{2m}{0} 2\cos 2m\theta + \binom{2m}{1} 2\cos 2\left(m-1\right)\theta + \cdots + \binom{2m}{m-1} 2\cos 2\theta + \binom{2m}{m} \\
= 2 &\left[\binom{2m}{0} \cos 2m\theta + \binom{2m}{1} \cos 2\left(m-1\right)\theta + \cdots + \binom{2m}{m-1} \cos 2\theta\right] + \binom{2m}{m}
\end{aligned}
$$

iii)

$$
\begin{aligned}
\int_0^{\pi/2} \cos^{2m}\theta \ \mathrm{d}\theta &= \frac{1}{2^{2m}} \int_0^{\pi/2} 2\left[\binom{2m}{0} \cos 2m\theta + \binom{2m}{1} \cos 2\left(m-1\right)\theta + \cdots + \binom{2m}{m-1} \cos 2\theta\right] + \binom{2m}{m} \ \mathrm{d}\theta \\
&= \frac{1}{2^{2m}} \cdot \binom{2m}{m} \frac{\pi}{2} = \frac{\pi}{2^{2m+1}} \binom{2m}{m}
\end{aligned}
$$

since, $\forall m \geq n,\ m, n \in \mathbb{N}$, we have (let $u = 2\theta$)

$$
\begin{aligned}
\int_0^{\pi/2} \cos 2\left(m-n\right)\theta \ \mathrm{d}\theta &= \frac{1}{2\left(m-n\right)} \left[\sin 2\left(m-n\right)\theta\right]_0^{\pi/2} \\
&= \frac{1}{2\left(m-n\right)} \left[\sin\left(m-n\right)\pi - 0\right] = 0
\end{aligned}
$$

since $m - n \in \mathbb{Z}$.

Q8

a)

i)

$$
\begin{aligned}
\cot\theta + \frac{1}{2}\tan\frac{\theta}{2} &= \frac{1}{\tan\theta} + \frac{1}{2}\tan\frac{\theta}{2} \\
&= \frac{1 - \tan^2\frac{\theta}{2}}{2\tan\frac{\theta}{2}} + \frac{\tan\frac{\theta}{2}}{2} \\
&= \frac{1 - \tan^2\frac{\theta}{2} + \tan^2\frac{\theta}{2}}{2\tan\frac{\theta}{2}} = \frac{1}{2\tan\frac{\theta}{2}} = \frac{1}{2}\cot\frac{\theta}{2}
\end{aligned}
$$

ii)

For $n = 1$, we have

$$
\begin{aligned}
\sum_{r=1}^{1} \frac{1}{2^0} \tan\frac{x}{2^0} &= \tan\frac{x}{2} \\
&= \cot\frac{x}{2} - 2\cot x
\end{aligned}
$$

From the last question. So the statement is true for $n = 1$. Assume it is true for $n = k$, ie.

$$
\sum_{r=1}^{k} \frac{1}{2^{k-1}} \tan\frac{x}{2^k} = \frac{1}{2^{k-1}} \cot\frac{x}{2^k} - 2\cot x
$$

We now have

$$\begin{aligned}
\sum_{r=1}^{k+1} \frac{1}{2^{k-1}} \tan \frac{x}{2^k} &= \sum_{r=1}^{k} \frac{1}{2^{k-1}} \tan \frac{x}{2^k} + \frac{1}{2^k} \tan \frac{x}{2^{k+1}} \\
&= \frac{1}{2^{k-1}} \cot \frac{x}{2^k} - 2 \cot x + \frac{1}{2^k} \tan \frac{x}{2^{k+1}} \\
&= \frac{1}{2^{k-1}} \left(\cot \frac{x}{2^k} + \frac{1}{2} \tan \frac{x}{2^{k+1}} \right) - 2 \cot x \\
&= \frac{1}{2^{k-1}} \left(\cot \frac{x}{2^k} + \frac{1}{2} \tan \frac{x}{2^k} \frac{1}{2} \right) - 2 \cot x \\
&= \frac{1}{2^{k-1}} \left(\frac{1}{2} \cot \frac{x}{2^{k+1}} \right) - 2 \cot x \text{ (from part (i))} \\
&= \frac{1}{2^k} \cot \frac{x}{2^{k+1}} - 2 \cot x
\end{aligned}$$

Hence the statement is true for all $n \geq 1$ by the principle of mathematical induction.

iii)

$$\begin{aligned}
\lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{2^{r-1}} \tan \frac{x}{2^k} &= \lim_{n\to\infty} \left\{ \frac{1}{2^{n-1}} \cot \frac{x}{2^n} - 2 \cot x \right\} \\
&= \lim_{n\to\infty} \left\{ \frac{1}{2^{n-1}} \frac{\cos \frac{x}{2^n}}{\sin \frac{x}{2^n}} - 2 \cot x \right\} \\
&= \lim_{n\to\infty} \left\{ \cos \frac{x}{2^n} \frac{\frac{x}{2^n}}{\sin \frac{x}{2^n}} \cdot \frac{2}{x} - 2 \cot x \right\} \\
&= \frac{2}{x} - 2 \cot x
\end{aligned}$$

iv)

Let $x = \frac{\pi}{2}$, we have

$$\begin{aligned}
\tan \frac{\pi}{4} + \frac{1}{2} \tan \frac{\pi}{8} + \frac{1}{4} \tan \frac{\pi}{16} &= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{2^{r-1}} \tan \frac{\pi}{2^{k+1}} \\
&= \frac{2}{\pi/2} - 2 \cot \frac{\pi}{2} = \frac{4}{\pi}
\end{aligned}$$

b)

We can observe that area of lower rectangle < area under curve < area of upper rectangle, so

$$\begin{aligned}
\frac{1}{n} < &\quad \textstyle\int_{n-1}^{n} \frac{1}{x} \, \mathrm{d}x &< \frac{1}{n-1} \\
\frac{1}{n} < &\quad [\ln x]_{n-1}^{n} &< \frac{1}{n-1} \\
\frac{1}{n} < &\quad \ln \tfrac{n}{n-1} &< \frac{1}{n-1} \\
-\frac{1}{n-1} < &\quad \ln \tfrac{n-1}{n} &< -\frac{1}{n} \\
e^{-\frac{1}{n-1}} < &\quad 1 - \tfrac{1}{n} &< e^{-\frac{1}{n}} \\
e^{-\frac{n}{n-1}} < &\quad \left(1 - \tfrac{1}{n}\right)^n &< e^{-1}
\end{aligned}$$

c)

i)

$$
\begin{aligned}
P(A_1 \text{ wins the game}) &= P(A_1 \text{ wins} \mid A_1 \text{ wins first draw})\, P(A_1 \text{ wins first draw}) \\
&\quad + P(A_1 \text{ wins} \mid A_1, A_2, \ldots, A_n \text{lose their first draw})\, P(A_1, A_2, \ldots, A_n \text{lose their first draw}) \\
W &= 1 \times p + q^n W
\end{aligned}
$$

since the game has returned to the original scenario.

ii)

It is easy to note that $W_1 = p$ since this refers to the probability of A_1 winning on the first attempt. $W_2 - W_1 = q^n p$ which is the probability that A_1 wins on the 2nd attempt (everyone lose on their first attempt, with A_1 drawing the correct card on the 2nd).

So we have

$$
\begin{aligned}
W_1 &= p \\
W_2 - W_1 &= q^n p \\
W_3 - W_2 &= q^{2n} p \\
&\vdots \\
W_m - W_{m-1} &= q^{(m-1)n} p \\
\therefore W_m &= p \frac{1 - q^{nm}}{1 - q^n}
\end{aligned}
$$

And from the previous question we have $W = \frac{p}{1-q^n}$. Hence

$$
\begin{aligned}
\frac{W_m}{W} &= 1 - q^{nm} \\
&= 1 - \left(1 - \frac{1}{n}\right)^{nm}
\end{aligned}
$$

From b) we have

$$
\begin{aligned}
e^{-\frac{mn}{n-1}} < & \quad \left(1 - \tfrac{1}{n}\right)^{nm} && < e^{-m} \\
1 - e^{-\frac{mn}{n-1}} > & \quad 1 - \left(1 - \tfrac{1}{n}\right)^{nm} && > 1 - e^{-m} \\
1 - e^{-\frac{mn}{n-1}} > & \quad \frac{W_m}{W} && > 1 - e^{-m}
\end{aligned}
$$

When n is large, $\frac{n}{n-1}$ converges to 1, so the left hand side converges to the right hand side, so by the sandwich principle $\lim_{n\to\infty} \frac{W_m}{W} = 1 - e^{-m}$